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Research Article

Energy- and Regularity-Dependent Stability Estimates for Near-Field Inverse Scattering in Multidimensions

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We prove new global Hölder-logarithmic stability estimates for the near-field inverse scattering problem in dimension $d \geq 3$. Our estimates are given in uniform norm for coefficient difference and related stability efficiently increases with increasing energy and/or coefficient regularity. In addition, a global logarithmic stability estimate for this inverse problem in dimension $d = 2$ is also given.

1. Introduction

We consider the Schrödinger equation:

$$L\psi = E\psi, \qquad L = -\Delta + v(x), \quad x \in \mathbb{R}^d, \ d \geq 2, \tag{1}$$

where

$$\begin{gathered} v \text{ is real-valued}, \quad v \in \mathbb{L}^\infty\left(\mathbb{R}^d\right), \\ v(x) = O\left(|x|^{-d-\varepsilon}\right), \quad |x| \longrightarrow \infty, \text{ for some } \varepsilon > 0. \end{gathered} \tag{2}$$

We consider the resolvent $R(E)$ of the Schrödinger operator L in $\mathbb{L}^2(\mathbb{R}^d)$:

$$R(E) = (L - E)^{-1}, \quad E \in \mathbb{C} \setminus \sigma(L), \tag{3}$$

where $\sigma(L)$ is the spectrum of L in $\mathbb{L}^2(\mathbb{R}^d)$. We assume that $R(x, y, E)$ denotes the Schwartz kernel of $R(E)$ as of an integral operator. We consider also

$$R^+(x, y, E) = R(x, y, E + i0), \quad x, y \in \mathbb{R}^d, \ E \in \mathbb{R}_+. \tag{4}$$

We recall that in the framework of (1) the function $R^+(x, y, E)$ describes scattering of the spherical waves,

$$R_0^+(x, y, E) = -\frac{i}{4}\left(\frac{\sqrt{E}}{2\pi|x-y|}\right)^{(d-2)/2} H_{(d-2)/2}^{(1)}\left(\sqrt{E}|x-y|\right), \tag{5}$$

generated by a source at y (where $H_\mu^{(1)}$ is the Hankel function of the first kind of order μ). We recall also that $R^+(x, y, E)$ is the Green function for $L - E$, $E \in \mathbb{R}_+$, with the Sommerfeld radiation condition at infinity.

In addition, the function

$$\begin{gathered} S^+(x, y, E) = R^+(x, y, E) - R_0^+(x, y, E), \\ x, y \in \partial B_r, \quad E \in \mathbb{R}_+, \quad r \in \mathbb{R}_+ \end{gathered} \tag{6}$$

is considered as near-field scattering data for (1), where B_r is the open ball of radius r centered at 0.

We consider, in particular, the following near-field inverse scattering problem for (1).

Problem 1. Given S^+ on $\partial B_r \times \partial B_r$ for some fixed r, $E \in \mathbb{R}_+$, find v on B_r.

This problem can be considered under the assumption that v is a priori known on $\mathbb{R}^d \setminus B_r$. Actually, in the present paper we consider Problem 1 under the assumption that $v \equiv 0$ on $\mathbb{R}^d \setminus B_r$ for some fixed $r \in \mathbb{R}_+$. Below in this paper we always assume that this additional condition is fulfilled.

It is well known that the near-field scattering data of Problem 1 uniquely and efficiently determine the scattering amplitude f for (1) at fixed energy E, see [1]. Therefore, approaches of [2–12] can be applied to Problem 1 via this reduction.

In addition, it is also known that the near-field data of Problem 1 uniquely determine the Dirichlet-to-Neumann map in the case when E is not a Dirichlet eigenvalue for operator L in B_r, see [8, 13]. Therefore, approaches of [3, 8, 14–24] can be also applied to Problem 1 via this reduction.

However, in some case it is much more optimal to deal with Problem 1 directly, see, for example, logarithmic stability results of [25] for Problem 1 in dimension $d = 3$. A principal improvement of estimates of [25] was given recently in [26]: stability of [26] efficiently increases with increasing regularity of v.

Problem 1 can be also considered as an example of ill-posed problem: see [27, 28] for an introduction to this theory.

In the present paper we continue studies of [25, 26]. We give new global Hölder-logarithmic stability estimates for Problem 1 in dimension $d \geq 3$, see Theorem 1. Our estimates are given in uniform norm for coefficient difference and related stability efficiently increases with increasing energy and/or coefficient regularity. Results of such a type for the Gel'fand inverse problem were obtained recently in [15] for $d \geq 3$ and in [29] for $d = 2$.

The main feature of our new estimates is the explicit dependence on the energy E. These estimates consist of two parts, the first is Hölder and the second is logarithmic; when E increases, the logarithmic part decreases and the Hölder part becomes dominant.

In addition, we give also global logarithmic stability estimates for Problem 1 in dimension $d = 2$, see Theorem 2.

2. Stability Estimates

We recall that if v satisfies (2) and $\operatorname{supp} v \subset B_{r_1}$ for some $r_1 > 0$, then

$$S^+(E) \text{ is bounded in } \mathbb{L}^2(\partial B_r \times \partial B_r) \quad \text{for any } r > r_1, \tag{7}$$

where $S^+(E)$ is the near-field scattering data of v for (1) with $E > 0$, for more details see, for example, Section 2 of [25].

2.1. Estimates for $d \geq 3$. In this subsection we assume for simplicity that

$$\begin{gathered} v \in \mathbb{W}^{m,1}(\mathbb{R}^d) \quad \text{for some } m > d,\ v \text{ is real-valued}, \\ \operatorname{supp} v \subset B_{r_1} \quad \text{for some } r_1 > 0, \end{gathered} \tag{8}$$

where

$$\mathbb{W}^{m,1}(\mathbb{R}^d) = \left\{ v : \partial^J v \in \mathbb{L}^1(\mathbb{R}^d),\ |J| \leq m \right\}, \quad m \in \mathbb{N} \cup 0, \tag{9}$$

where

$$\begin{gathered} J \in (\mathbb{N} \cup 0)^d, \quad |J| = \sum_{i=1}^{d} J_i, \\ \partial^J v(x) = \frac{\partial^{|J|} v(x)}{\partial x_1^{J_1} \cdots \partial x_d^{J_d}}. \end{gathered} \tag{10}$$

Let

$$\|v\|_{m,1} = \max_{|J| \leq m} \left\| \partial^J v \right\|_{\mathbb{L}^1(\mathbb{R}^d)}. \tag{11}$$

Note that (8)⇒(2).

Theorem 1. *Let $E > 0$ and $r > r_1$ be given constants. Let dimension $d \geq 3$ and potentials v_1, v_2 satisfy* (8). *Let $\|v_j\|_{m,1} \leq N$, $j = 1, 2$, for some $N > 0$. Let $S_1^+(E)$ and $S_2^+(E)$ denote the near-field scattering data for v_1 and v_2, respectively. Then for $\tau \in (0, 1)$ and any $s \in [0, s^*]$ the following estimate holds:*

$$\begin{aligned} &\|v_2 - v_1\|_{L^\infty(B_r)} \\ &\quad \leq C_1 (1 + E)^{5/2} \delta^\tau + C_2 (1 + E)^{(s - s^*)/2} \left(\ln\left(3 + \delta^{-1}\right) \right)^{-s}, \end{aligned} \tag{12}$$

where $s^* = (m - d)/d$, $\delta = \|S_1^+(E) - S_2^+(E)\|_{\mathbb{L}^2(\partial B_r \times \partial B_r)}$, and constants $C_1, C_2 > 0$ depend only on N, m, d, r, τ.

Proof of Theorem 1 is given in Section 5. This proof is based on results presented in Sections 3 and 4.

2.2. Estimates for d=2. In this subsection we assume for simplicity that

$$\begin{gathered} v \text{ is real-valued}, \quad v \in C^2(\overline{B}_{r_1}), \\ \operatorname{supp} v \subset B_{r_1} \quad \text{for some } r_1 > 0. \end{gathered} \tag{13}$$

Note also that (13)⇒(2).

Theorem 2. *Let $E > 0$ and $r > r_1$ be given constants. Let dimension $d = 2$ and potentials v_1, v_2 satisfy* (13). *Let $\|v_j\|_{C^2(B_r)} \leq N$, $j = 1, 2$, for some $N > 0$. Let $S_1^+(E)$ and $S_2^+(E)$ denote the near-field scattering data for v_1 and v_2, respectively. Then*

$$\|v_1 - v_2\|_{L^\infty(B_r)} \leq C_3 \left(\ln\left(3 + \delta^{-1}\right) \right)^{-3/4} \left(\ln\left(3 \ln\left(3 + \delta^{-1}\right)\right) \right)^2, \tag{14}$$

where $\delta = \|S_1^+(E) - S_2^+(E)\|_{\mathbb{L}^2(\partial B_r \times \partial B_r)}$ and constant $C_3 > 0$ depends only on N, m, r.

Proof of Theorem 2 is given in Section 7. This proof is based on results presented in Sections 3 and 6.

2.3. Concluding Remarks

Remark 3. The logarithmic stability estimates for Problem 1.1 of [25, 26] follow from estimate (12) for $d = 3$ and $s = s^*$. Apparently, using the methods of [19, 20] it is possible to improve estimate (12) for $s^* = m - d$.

Remark 4. In the same way as in [25, 26] for dimesnsion $d = 3$, using estimates (12) and (14), one can obtain logarithmic stability estimates for the reconstruction of a potential v from the inverse scattering amplitude f for any $d \geq 2$.

Remark 5. Actually, in the proof of Theorem 1 we obtain the following estimate (see formula (57)):

$$\|v_1 - v_2\|_{\mathbb{L}^\infty(B_r)} \le C_4(1+E)^2\sqrt{E+\rho^2}\, e^{2\rho(r+1)}\delta + C_5\left(E+\rho^2\right)^{-(m-d)/2d}, \tag{15}$$

where constants $C_4, C_5 > 0$ depend only on N, m, d, r and the parameter $\rho > 0$ is such that $E + \rho^2$ is sufficiently large: $E + \rho^2 \ge C_6(N, r, m)$. Estimate of Theorem 1 follows from estimate (15).

3. Alessandrini-Type Identity for Near-Field Scattering

In this section we always assume that assumptions of Theorems 1 and 2 are fulfilled (in the cases of dimension $d \ge 3$ and $d = 2$, resp.).

Consider the operators $\widehat{R}_j$, $j = 1, 2$, defined as follows

$$\left(\widehat{R}_j\phi\right)(x) = \int_{\partial B_r} R_j^+(x, y, E)\,\phi(y)\,dy, \quad x \in \partial B_r,\ j = 1, 2. \tag{16}$$

Note that

$$\left\|\widehat{R}_1 - \widehat{R}_2\right\|_{\mathbb{L}^2(\partial B_r)} \le \|S_1^+(E) - S_2^+(E)\|_{\mathbb{L}^2(\partial B_r)\times\mathbb{L}^2(\partial B_r)}. \tag{17}$$

We recall that (see [25]) for any functions $\phi_1, \phi_2 \in C(\mathbb{R}^d)$, sufficiently regular in $\mathbb{R}^d \setminus \partial B_r$ and satisfying

$$\begin{gathered} -\Delta\phi + v(x)\phi = E\phi, \quad \text{in } \mathbb{R}^d \setminus \partial B_r, \\ \lim_{|x|\to+\infty} |x|^{(d-1)/2}\left(\frac{\partial}{\partial|x|}\phi - i\sqrt{E}\phi\right) = 0, \end{gathered} \tag{18}$$

with $v = v_1$ and $v = v_2$, respectively, the following identity holds:

$$\begin{aligned} &\int_{B_r} (v_2 - v_1)\,\phi_1\phi_2\,dx \\ &\quad = \int_{\partial B_r}\left(\frac{\partial\phi_1}{\partial\nu_+} - \frac{\partial\phi_1}{\partial\nu_-}\right)\left[\left(\widehat{R}_1 - \widehat{R}_2\right)\left(\frac{\partial\phi_2}{\partial\nu_+} - \frac{\partial\phi_2}{\partial\nu_-}\right)\right]dx, \end{aligned} \tag{19}$$

where where ν_+ and ν_- are the outward and inward normals to ∂B_r, respectively.

Remark 6. The identity (19) is similar to the Alessandrini identity (see Lemma 1 of [14]), where the Dirichlet-to-Neumann maps are considered instead of operators $\widehat{R}_j$.

To apply identity (19) to our considerations, we use also the following lemma.

Lemma 7. *Let $E, r > 0$ and $d \ge 2$. Then, there is a positive constant C_7 (depending only on r and d) such that for any $\phi \in C(\mathbb{R}^d \setminus B_r)$ satisfying*

$$\begin{gathered} -\Delta\phi = E\phi, \quad \text{in } \mathbb{R}^d \setminus \overline{B}_r, \\ \lim_{|x|\to+\infty} |x|^{(d-1)/2}\left(\frac{\partial}{\partial|x|}\phi - i\sqrt{E}\phi\right) = 0, \\ \phi|_{\partial B_r} \in \mathbb{H}^1(\partial B_r), \end{gathered} \tag{20}$$

the following inequality holds:

$$\left\|\frac{\partial\phi}{\partial\nu_+}\bigg|_{\partial B_r}\right\|_{\mathbb{L}^2(\partial B_r)} \le C_7(1+E)\left\|\phi|_{\partial B_r}\right\|_{\mathbb{H}^1(\partial B_r)}, \tag{21}$$

where $\mathbb{H}^1(\partial B_r)$ denotes the standard Sobolev space on ∂B_r.

The proof of Lemma 7 is given in Section 8.

4. Faddeev Functions

In dimension $d \ge 3$, we consider the Faddeev functions h, ψ, G (see [6, 8, 30, 31]):

$$h(k, l) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-ilx} v(x)\,\psi(x, k)\,dx, \tag{22}$$

where $k, l \in \mathbb{C}^d$, $k^2 = l^2$, $\operatorname{Im} k = \operatorname{Im} l \ne 0$,

$$\psi(x, k) = e^{ikx} + \int_{\mathbb{R}^d} G(x - y, k)\,v(y)\,\psi(y, k)\,dy, \tag{23}$$

$$\begin{gathered} G(x, k) = e^{ikx} g(x, k), \\ g(x, k) = -(2\pi)^{-d}\int_{\mathbb{R}^d}\frac{e^{i\xi x}d\xi}{\xi^2 + 2k\xi}, \end{gathered} \tag{24}$$

where $x \in \mathbb{R}^d$, $k \in \mathbb{C}^d$, $\operatorname{Im} k \ne 0$, $d \ge 3$,

One can consider (22), (23) assuming that

$$\begin{gathered} v \text{ is a sufficiently regular function on } \mathbb{R}^d \\ \text{with suffucient decay at infinity.} \end{gathered} \tag{25}$$

For example, in connection with Theorem 1, we consider (22), (23) assuming that

$$v \in \mathbb{L}^\infty(B_r), \quad v \equiv 0 \text{ on } \mathbb{R} \setminus B_r. \tag{26}$$

We recall that (see [6, 8, 30, 31])

$$\left(\Delta + k^2\right)G(x, k) = \delta(x), \quad x \in \mathbb{R}^d,\ k \in \mathbb{C}^d \setminus \mathbb{R}^d, \tag{27}$$

formula (23) at fixed k is considered as an equation for

$$\psi = e^{ikx}\mu(x, k), \tag{28}$$

where μ is sought in $\mathbb{L}^\infty(\mathbb{R}^d)$; as a corollary of (23), (24), and (27), ψ satisfies (1) for $E = k^2$; h of (22) is a generalized "scattering" amplitude.

In addition, h, ψ, G in their zero energy restriction, that is for $E = 0$ were considered for the first time in [32]. The Faddeev functions h, ψ, G were, actually, rediscovered in [32].

Let

$$\Sigma_E = \left\{k \in \mathbb{C}^d : k^2 = k_1^2 + \cdots + k_d^2 = E\right\},$$
$$\Theta_E = \left\{k \in \Sigma_E,\, l \in \Sigma_E : \operatorname{Im} k = \operatorname{Im} l\right\}, \tag{29}$$
$$|k| = \left(|\operatorname{Re} k|^2 + |\operatorname{Im} k|^2\right)^{1/2}.$$

Let

$$v \text{ satisfy } (8), \quad \|v\|_{m,1} \le N,$$
$$\widehat{v}(p) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ipx} v(x)\, dx, \quad p \in \mathbb{R}^d, \tag{30}$$

then we have that

$$\mu(x,k) \longrightarrow 1 \quad \text{as } |k| \longrightarrow \infty \tag{31}$$

and, for any $\sigma > 1$,

$$|\mu(x,k)| + |\nabla\mu(x,k)| \le \sigma \quad \text{for } |k| \ge \lambda_1(N,m,d,r,\sigma), \tag{32}$$

where $x \in \mathbb{R}^d$, $k \in \Sigma_E$;

$$\widehat{v}(p) = \lim_{\substack{(k,l)\in\Theta_E,\, k-l=p \\ |\operatorname{Im} k|=|\operatorname{Im} l| \to \infty}} h(k,l), \quad \text{for any } p \in \mathbb{R}^d, \tag{33}$$

$$\begin{aligned} &|\widehat{v}(p) - h(k,l)| \\ &\quad \le \frac{c_1(m,d,r)\, N^2}{(E+\rho^2)^{1/2}}, \quad \text{for } (k,l) \in \Theta_E,\ p = k - l, \\ &\quad |\operatorname{Im} k| = |\operatorname{Im} l| = \rho, \quad E + \rho^2 \ge \lambda_2(N,m,d,r), \\ &\quad p^2 \le 4\left(E + \rho^2\right). \end{aligned} \tag{34}$$

Results of the type (31), (32) go back to [32]. For more information concerning (32) see estimate (4.11) of [33]. Results of the type (33), (34) (with less precise right-hand side in (34)) go back to [6]. Estimate (34) follows, for example, from formulas (23), (22), and the estimate

$$\begin{aligned} &\left\|\Lambda^{-s} g(k) \Lambda^{-s}\right\|_{\mathbb{L}^2(\mathbb{R}^d) \to \mathbb{L}^2(\mathbb{R}^d)} \\ &\quad = O\left(|k|^{-1}\right) \quad \text{as } |k| \longrightarrow \infty,\ k \in \mathbb{C}^d \setminus \mathbb{R}^d, \end{aligned} \tag{35}$$

for $s > 1/2$, where $g(k)$ denotes the integral operator with the Schwartz kernel $g(x-y,k)$ and Λ denotes the multiplication operator by the function $(1 + |x|^2)^{1/2}$. Estimate (35) was formulated, first, in [34] for $d \ge 3$. Concerning proof of (35), see [35].

In addition, we have that

$$\begin{aligned} &h_2(k,l) - h_1(k,l) \\ &\quad = (2\pi)^{-d} \int_{\mathbb{R}^d} \psi_1(x,-l)\left(v_2(x) - v_1(x)\right)\psi_2(x,k)\, dx \\ &\qquad \text{for } (k,l) \in \Theta_E, \quad |\operatorname{Im} k| = |\operatorname{Im} l| \ne 0, \\ &\qquad \text{and } v_1,\, v_2 \text{ satisfying } (25), \end{aligned} \tag{36}$$

and, under assumptions of Theorem 1,

$$\begin{aligned} &\left|\widehat{v_1}(p) - \widehat{v_2}(p) - h_1(k,l) + h_2(k,l)\right| \\ &\quad \le \frac{c_2(m,d,r)\, N \|v_1 - v_2\|_{\mathbb{L}^\infty(B_r)}}{(E+\rho^2)^{1/2}} \\ &\quad \text{for } (k,l) \in \Theta_E, \quad p = k - l, \quad |\operatorname{Im} k| = |\operatorname{Im} l| = \rho, \\ &\quad E + \rho^2 \ge \lambda_3(N,m,d,r), \quad p^2 \le 4\left(E + \rho^2\right), \end{aligned} \tag{37}$$

where h_j, ψ_j denote h and ψ of (22) and (23) for $v = v_j$, $j = 1, 2$.

Formula (36) was given in [36]. Estimate (37) was given for example in [15].

5. Proof of Theorem 1

Let

$$\begin{aligned} &\mathbb{L}_\mu^\infty\left(\mathbb{R}^d\right) = \left\{u \in \mathbb{L}^\infty\left(\mathbb{R}^d\right) : \|u\|_\mu < +\infty\right\}, \\ &\|u\|_\mu = \operatorname{ess\,sup}_{p\in\mathbb{R}^d} (1 + |p|)^\mu |u(p)|, \quad \mu > 0. \end{aligned} \tag{38}$$

Note that

$$\begin{aligned} &w \in \mathbb{W}^{m,1}\left(\mathbb{R}^d\right) \Longrightarrow \widehat{w} \in \mathbb{L}_\mu^\infty\left(\mathbb{R}^d\right) \cap C\left(\mathbb{R}^d\right), \\ &\|\widehat{w}\|_\mu \le c_3(m,d)\, \|w\|_{m,1}, \quad \text{for } \mu = m, \end{aligned} \tag{39}$$

where $\mathbb{W}^{m,1}$, $\mathbb{L}_\mu^\infty$ are the spaces of (9), (38),

$$\widehat{w}(p) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ipx} w(x)\, dx, \quad p \in \mathbb{R}^d. \tag{40}$$

Using the inverse Fourier transform formula

$$w(x) = \int_{\mathbb{R}^d} e^{-ipx} \widehat{w}(p)\, dp, \quad x \in \mathbb{R}^d, \tag{41}$$

we have that

$$\begin{aligned} \|v_1 - v_2\|_{\mathbb{L}^\infty(B_r)} &\le \sup_{x\in\overline{B}_r} \left|\int_{\mathbb{R}^d} e^{-ipx}\left(\widehat{v_2}(p) - \widehat{v_1}(p)\right) dp\right| \\ &\le I_1(\kappa) + I_2(\kappa), \quad \text{for any } \kappa > 0, \end{aligned} \tag{42}$$

where

$$\begin{aligned} I_1(\kappa) &= \int_{|p|\le\kappa} \left|\widehat{v_2}(p) - \widehat{v_1}(p)\right| dp, \\ I_2(\kappa) &= \int_{|p|\ge\kappa} \left|\widehat{v_2}(p) - \widehat{v_1}(p)\right| dp. \end{aligned} \tag{43}$$

Using (39), we obtain that

$$\left|\widehat{v}_2(p)-\widehat{v}_1(p)\right| \le 2c_3(m,d)\,N(1+|p|)^{-m}, \quad p\in\mathbb{R}^d. \tag{44}$$

Let

$$c_4=\int_{p\in\mathbb{R}^d,\,|p|=1} dp. \tag{45}$$

Combining (43), (44), we find that, for any $\kappa>0$,

$$I_2(\kappa)\le 2c_3(m,d)\,Nc_4\int_{\kappa}^{+\infty}\frac{dt}{t^{m-d+1}}\le\frac{2c_3(m,d)\,Nc_4}{m-d}\frac{1}{\kappa^{m-d}}. \tag{46}$$

Due to (37), we have that

$$\begin{aligned}\left|\widehat{v}_2(p)-\widehat{v}_1(p)\right| \le{}& \left|h_2(k,l)-h_1(k,l)\right| \\ &+\frac{c_2(m,d,r)\,N\|v_1-v_2\|_{\mathbb{L}^\infty(B_r)}}{(E+\rho^2)^{1/2}},\end{aligned} \tag{47}$$

$$\text{for } (k,l)\in\Theta_E,\quad p=k-l,\quad |\operatorname{Im}k|=|\operatorname{Im}l|=\rho,$$

$$E+\rho^2\ge\lambda_3(N,m,d,r),\quad p^2\le 4\left(E+\rho^2\right).$$

Let

$$\delta=\|S_1^+(E)-S_2^+(E)\|_{\mathbb{L}^2(\partial B_r\times\partial B_r)}. \tag{48}$$

Combining (17), (19), and (36), we get that

$$\begin{aligned}&\left|h_2(k,l)-h_1(k,l)\right| \\ &\quad\le\delta\left\|\frac{\partial\phi_1}{\partial\nu_+}-\frac{\partial\phi_1}{\partial\nu_-}\right\|_{\mathbb{L}^2(B_r)}\left\|\frac{\partial\phi_2}{\partial\nu_+}-\frac{\partial\phi_2}{\partial\nu_-}\right\|_{\mathbb{L}^2(B_r)},\end{aligned} \tag{49}$$

$$(k,l)\in\Theta_E,\quad |\operatorname{Im}k|=|\operatorname{Im}l|\ne 0,$$

where ϕ_j, $j=1,2$, denotes the solution of (18) with $v=v_j$, satisfying

$$\phi_j(x)=\psi_j(x,k),\quad \text{for } x\in\overline{B}_r. \tag{50}$$

Using (21), (32), and the fact that $C^1(\partial B_r)\subset\mathbb{H}^1(\partial B_r)$, we find that

$$\begin{aligned}&\left\|\frac{\partial\phi_j}{\partial\nu_+}-\frac{\partial\phi_j}{\partial\nu_-}\right\|_{\mathbb{L}^2(B_r)} \\ &\quad\le\sigma c_5(r,d)(1+E)\ \exp\left(|\operatorname{Im}k|(r+1)\right),\end{aligned} \tag{51}$$

$$k\in\Sigma_E,\quad |k|\ge\lambda_1(N,m,d,r,\sigma),\quad j=1,2.$$

Here and bellow in this section the constant σ is the same that in (32).

Combining (49) and (51), we obtain that

$$\begin{aligned}&\left|h_2(k,l)-h_1(k,l)\right|\le c_5^2\sigma^2(1+E)^2e^{2\rho(r+1)}\delta, \\ &\quad\text{for } (k,l)\in\Theta_E,\quad \rho=|\operatorname{Im}k|=|\operatorname{Im}l|, \\ &\quad E+\rho^2\ge\lambda_1^2(N,m,d,r,\sigma).\end{aligned} \tag{52}$$

Using (47), (52), we get that

$$\begin{aligned}&\left|\widehat{v}_2(p)-\widehat{v}_1(p)\right|\le c_5^2\sigma^2(1+E)^2e^{2\rho(r+1)}\delta \\ &\quad+\frac{c_2(m,d,r)\,N\|v_1-v_2\|_{\mathbb{L}^\infty(B_1)}}{(E+\rho^2)^{1/2}}, \\ &p\in\mathbb{R}^d,\quad p^2\le 4\left(E+\rho^2\right),\quad E+\rho^2\ge\max\left\{\lambda_1^2,\lambda_3\right\}.\end{aligned} \tag{53}$$

Let

$$\varepsilon=\left(\frac{1}{2c_2(m,d,r)\,Nc_6}\right)^{1/d},\quad c_6=\int_{p\in\mathbb{R}^d,\,|p|\le 1}dp, \tag{54}$$

and $\lambda_4(N,m,d,r,\sigma)>0$ be such that

$$\begin{aligned}&E+\rho^2\ge\lambda_4(N,m,d,r,\sigma) \\ &\quad\Longrightarrow\begin{cases}E+\rho^2\ge\lambda_1^2(N,m,d,r,\sigma), \\ E+\rho^2\ge\lambda_3(N,m,d,r), \\ \left(\varepsilon\left(E+\rho^2\right)^{1/2d}\right)^2\le 4\left(E+\rho^2\right).\end{cases}\end{aligned} \tag{55}$$

Using (43), (53), we get that

$$\begin{aligned}I_1(\kappa)\le{}& c_6\kappa^d\Bigg(c_5^2\sigma^2(1+E)^2e^{2\rho(r+1)}\delta \\ &+\frac{c_2(m,d,r)\,N\|v_1-v_2\|_{\mathbb{L}^\infty(B_1)}}{(E+\rho^2)^{1/2}}\Bigg), \\ &\kappa>0,\ \kappa^2\le 4\left(E+\rho^2\right), \\ &E+\rho^2\ge\lambda_4(N,m,d,r,\sigma).\end{aligned} \tag{56}$$

Combining (42), (46), and (56) for $\kappa=\varepsilon(E+\rho^2)^{1/2d}$ and (55), we get that

$$\begin{aligned}&\|v_1-v_2\|_{\mathbb{L}^\infty(B_r)} \\ &\quad\le c_7(N,m,d,r,\sigma)(1+E)^2\sqrt{E+\rho^2}\ e^{2\rho(r+1)}\delta \\ &\quad+c_8(N,m,d)\left(E+\rho^2\right)^{-(m-d)/2d} \\ &\quad+\frac{1}{2}\|v_1-v_2\|_{\mathbb{L}^\infty(B_r)}, \\ &\quad E+\rho^2\ge\lambda_4(N,m,d,r,\sigma).\end{aligned} \tag{57}$$

Let $\tau'\in(0,1)$,

$$\beta=\frac{1-\tau'}{2(r+1)},\qquad \rho=\beta\ln\left(3+\delta^{-1}\right), \tag{58}$$

and $\delta_1 = \delta_1(N, m, d, \sigma, r, \tau') > 0$ be such that

$$\delta \in (0, \delta_1) \Longrightarrow \begin{cases} E + \left(\beta \ln\left(3 + \delta^{-1}\right)\right)^2 \\ \qquad \geq \lambda_4 (N, m, d, r, \sigma), \\ E + \left(\beta \ln\left(3 + \delta^{-1}\right)\right)^2 \\ \qquad \leq (1 + E)\left(\beta \ln\left(3 + \delta^{-1}\right)\right)^2. \end{cases} \tag{59}$$

Then for the case when $\delta \in (0, \delta_1)$, due to (57), we have that

$$\begin{aligned} &\frac{1}{2}\|v_1 - v_2\|_{\mathbb{L}^\infty(B_r)} \\ &\quad \leq c_7(1+E)^2\left(E + \left(\beta \ln\left(3+\delta^{-1}\right)\right)^2\right)^{1/2}\left(3+\delta^{-1}\right)^{2\beta(r+1)}\delta \\ &\qquad + c_8\left(E + \left(\beta \ln\left(3+\delta^{-1}\right)\right)^2\right)^{-(m-d)/2d} \\ &\quad = c_7(1+E)^2\left(E + \left(\beta \ln\left(3+\delta^{-1}\right)\right)^2\right)^{1/2}(1+3\delta)^{1-\tau'}\delta^{\tau'} \\ &\qquad + c_8\left(E + \left(\beta \ln\left(3+\delta^{-1}\right)\right)^2\right)^{-(m-d)/2d}. \end{aligned} \tag{60}$$

Combining (59) and (60), we obtain that for $s \in [0, s^*]$, $\tau \in (0, \tau')$ and $\delta \in (0, \delta_1)$ the following estimate holds:

$$\begin{aligned} \|v_2 - v_1\|_{L^\infty(B_r)} &\leq c_9(1+E)^{5/2}\delta^{\tau} \\ &\quad + c_{10}(1+E)^{(s-s^*)/2}\left(\ln\left(3+\delta^{-1}\right)\right)^{-s}, \end{aligned} \tag{61}$$

where $s^* = (m-d)/d$ and $c_9, c_{10} > 0$ depend only on N, m, d, r, σ, τ', and τ.

Estimate (61) in the general case (with modified c_9 and c_{10}) follows from (61) for $\delta \leq \delta_1(N, m, d, \sigma, r, \tau')$ and the property that

$$\|v_j\|_{\mathbb{L}^\infty(B_r)} \leq c_{11}(m, d)\, N. \tag{62}$$

This completes the proof of (12).

6. Buckhgeim-Type Analogs of the Faddeev Functions

Let us identify $\mathbb{R}^2$ with $\mathbb{C}$ and use coordinates $z = x_1 + ix_2$, $\overline{z} = x_1 - ix_2$, where $(x_1, x_2) \in \mathbb{R}^2$. Following [21–23, 37], we consider the functions G_{z_0}, ψ_{z_0}, $\widetilde{\psi}_{z_0}$, δh_{z_0} going back to Buckhgeim's paper [3] and being analogs of the Faddeev functions:

$$\begin{aligned} \psi_{z_0}(z, \lambda) &= e^{\lambda(z-z_0)^2} \\ &\quad + \int_{B_r} G_{z_0}(z, \zeta, \lambda)\, v(\zeta)\, \psi_{z_0}(\zeta, \lambda)\, d\operatorname{Re}\zeta\, d\operatorname{Im}\zeta, \\ \widetilde{\psi}_{z_0}(z, \lambda) &= e^{\overline{\lambda}(\overline{z}-\overline{z}_0)^2} \\ &\quad + \int_{B_r} \overline{G_{z_0}(z, \zeta, \lambda)}\, v(\zeta)\, \widetilde{\psi}_{z_0}(\zeta, \lambda)\, d\operatorname{Re}\zeta\, d\operatorname{Im}\zeta, \end{aligned} \tag{63}$$

$$\begin{aligned} G_{z_0}(z, \zeta, \lambda) &= \frac{1}{4\pi^2}\int_{B_r} \frac{e^{-\lambda(\eta-z_0)^2 + \overline{\lambda}(\overline{\eta}-\overline{z}_0)^2}\, d\operatorname{Re}\eta\, d\operatorname{Im}\eta}{(z-\eta)\left(\overline{\eta}-\overline{\zeta}\right)} \\ &\qquad \times e^{\lambda(z-z_0)^2 - \overline{\lambda}(\overline{\zeta}-\overline{z}_0)^2}, \\ &\qquad z = x_1 + ix_2, \quad z_0 \in B_r, \quad \lambda \in \mathbb{C}, \end{aligned} \tag{64}$$

where v satisfies (13),

$$\begin{aligned} \delta h_{z_0}(\lambda) &= \int_{B_r} \widetilde{\psi}_{z_0,1}(z, -\lambda)\left(v_2(z) - v_1(z)\right) \\ &\qquad \times \psi_{z_0,2}(z, \lambda)\, d\operatorname{Re} z\, d\operatorname{Im} z, \quad \lambda \in \mathbb{C}, \end{aligned} \tag{65}$$

where v_1, v_2 satisfy (13) and $\widetilde{\psi}_{z_0,1}$, $\psi_{z_0,2}$ denote $\widetilde{\psi}_{z_0}$, ψ_{z_0} of (63) for $v = v_1$ and $v = v_2$, respectively.

We recall that (see [21, 22]):

(i) the function G_{z_0} satisfies the equations

$$\begin{aligned} 4\frac{\partial^2}{\partial z \partial \overline{z}} G_{z_0}(z, \zeta, \lambda) &= \delta(z - \zeta), \\ 4\frac{\partial^2}{\partial \zeta \partial \overline{\zeta}} G_{z_0}(z, \zeta, \lambda) &= \delta(z - \zeta), \end{aligned} \tag{66}$$

where $z, z_0, \zeta \in B_r$, $\lambda \in \mathbb{C}$, and δ is the Dirac delta function;

(ii) formulas (63) at fixed z_0 and λ are considered as equations for ψ_{z_0}, $\widetilde{\psi}_{z_0}$ in $L^\infty(B_r)$;

(iii) as a corollary of (63), (64), (66), the functions ψ_{z_0}, $\widetilde{\psi}_{z_0}$ satisfy (1) in B_r for $E = 0$ and $d = 2$;

(iv) the function δh_{z_0} is similar to the right side of (36).

Let potentials $v, v_1, v_2 \in C^2(\overline{B}_r)$, and

$$\begin{aligned} &\|v\|_{C^2(\overline{B}_r)} \leq N, \qquad \|v_j\|_{C^2(\overline{B}_r)} \leq N, \quad j = 1, 2, \\ &(v_1 - v_2)|_{\partial B_r} = 0, \qquad \frac{\partial}{\partial \nu}(v_1 - v_2)\Big|_{\partial B_r} = 0, \end{aligned} \tag{67}$$

then we have that

$$\psi_{z_0}(z,\lambda) = e^{\lambda(z-z_0)^2}\mu_{z_0}(z,\lambda),$$
$$\tilde{\psi}_{z_0}(z,\lambda) = e^{\bar{\lambda}(\bar{z}-\bar{z}_0)^2}\tilde{\mu}_{z_0}(z,\lambda), \tag{68}$$

$$\mu_{z_0}(z,\lambda) \longrightarrow 1, \quad \tilde{\mu}_{z_0}(z,\lambda) \longrightarrow 1 \quad \text{as } |\lambda| \longrightarrow \infty \tag{69}$$

and, for any $\sigma > 1$,

$$\left|\mu_{z_0}(z,\lambda)\right| + \left|\nabla\mu_{z_0}(z,\lambda)\right| \le \sigma, \tag{70a}$$

$$\left|\tilde{\mu}_{z_0}(z,\lambda)\right| + \left|\nabla\tilde{\mu}_{z_0}(z,\lambda)\right| \le \sigma, \tag{70b}$$

where $\nabla = (\partial/\partial x_1, \partial/\partial x_2)$, $z = x_1 + ix_2$, $z_0 \in B_r$, $\lambda \in \mathbb{C}$, $|\lambda| \ge \rho_1(N,r,\sigma)$;

$$v_2(z_0) - v_1(z_0) = \lim_{\lambda\to\infty}\frac{2}{\pi}|\lambda|\,\delta h_{z_0}(\lambda) \quad \text{for any } z_0 \in B_r, \tag{71}$$

$$\left|v_2(z_0) - v_1(z_0) - \frac{2}{\pi}|\lambda|\,\delta h_{z_0}(\lambda)\right| \le \frac{c_{12}(N,r)(\ln(3|\lambda|))^2}{|\lambda|^{3/4}}, \quad \text{for } z_0 \in B_r, \quad |\lambda| \ge \rho_2(N,r). \tag{72}$$

Formulas (68) can be considered as definitions of μ_{z_0}, $\tilde{\mu}_{z_0}$. Formulas (69), (71) were given in [21, 22] and go back to [3]. Estimates (70a) and (70b) were proved in [33]. Estimate (72) was obtained in [21, 37].

7. Proof of Theorem 2

We suppose that $\tilde{\psi}_{z_0,1}(\cdot,-\lambda)$, $\psi_{z_0,2}(\cdot,\lambda)$, $\delta h_{z_0}(\lambda)$ are defined as in Section 6 but with $v_j - E$ in place of v_j, $j = 1,2$. Note that functions $\tilde{\psi}_{z_0,1}(\cdot,-\lambda)$, $\psi_{z_0,2}(\cdot,\lambda)$ satisfy (1) in B_r with $v = v_j$, $j = 1,2$, respectively. We also use the notation $N_E = N + E$. Then, using (72), we have that

$$\left|v_2(z_0) - v_1(z_0) - \frac{2}{\pi}|\lambda|\,\delta h_{z_0}(\lambda)\right| \le \frac{c_{12}(N_E,r)(\ln(3|\lambda|))^2}{|\lambda|^{3/4}}, \quad \text{for } z_0 \in B_r,\ |\lambda| \ge \rho_2(N_E,r). \tag{73}$$

Let

$$\delta = \left\|S_1^+(E) - S_2^+(E)\right\|_{\mathbb{L}^2(\partial B_r\times\partial B_r)}. \tag{74}$$

Combining (17), (19), and (65), we get that

$$\left|\delta h_{z_0}(\lambda)\right| \le \delta\left\|\frac{\partial\phi_1}{\partial\nu_+} - \frac{\partial\phi_1}{\partial\nu_-}\right\|_{\mathbb{L}^2(B_r)}\left\|\frac{\partial\phi_2}{\partial\nu_+} - \frac{\partial\phi_2}{\partial\nu_-}\right\|_{\mathbb{L}^2(B_r)}, \quad (k,l) \in \Theta_E, \quad |\operatorname{Im} k| = |\operatorname{Im} l| \ne 0, \tag{75}$$

where ϕ_j, $j = 1,2$, denotes the solution of (18) with $v = v_j$, satisfying

$$\phi_1(x) = \tilde{\psi}_{z_0,1}(x,-\lambda),$$
$$\phi_2(x) = \psi_{z_0,2}(x,\lambda), \quad \text{for } x \in \overline{B}_r. \tag{76}$$

Using (21), (70a), and (70b) and the fact that $C^1(\partial B_r) \subset \mathbb{H}^1(\partial B_r)$, we find that

$$\left\|\frac{\partial\phi_j}{\partial\nu_+} - \frac{\partial\phi_j}{\partial\nu_-}\right\|_{\mathbb{L}^2(B_r)} \le \sigma c_{13}(r)(1+E)\exp\left(|\lambda|\left(4r^2+4r\right)\right), \quad \lambda \in \mathbb{C}, \quad |\lambda| \ge \rho_1(N_E,r,\sigma), \quad j = 1,2. \tag{77}$$

Here and bellow in this section the constant σ is the same that in (70a) and (70b).

Combining (75), (77), we obtain that

$$\left|\delta h_{z_0}(\lambda)\right| \le c_{14}(E,r,\sigma)\exp\left(|\lambda|\left(8r^2+8r\right)\right)\delta, \quad \lambda \in \mathbb{C}, \quad |\lambda| \ge \rho_1(N_E,r,\sigma). \tag{78}$$

Using (73) and (78), we get that

$$|v_2(z_0) - v_1(z_0)| \le c_{14}(E,r,\sigma)\exp\left(|\lambda|\left(8r^2+8r\right)\right)\delta + \frac{c_{12}(N_E,r)(\ln(3|\lambda|))^2}{|\lambda|^{3/4}}, \quad z_0 \in B_r, \quad \lambda \in \mathbb{C}, \quad |\lambda| \ge \rho_3(N_E,r,\sigma) = \max\{\rho_1,\rho_2\}. \tag{79}$$

We fix some $\tau \in (0,1)$ and let

$$\beta = \frac{1-\tau}{8r^2+8r}, \qquad \lambda = \beta\ln\left(3+\delta^{-1}\right), \tag{80}$$

where δ is so small that $|\lambda| \ge \rho_3(N_E,r,\sigma)$. Then due to (79), we have that

$$\begin{aligned}\|v_1 - v_2\|_{\mathbb{L}^\infty(B_r)} &\le c_{14}(E,r,\sigma)\left(3+\delta^{-1}\right)^{\beta(8r^2+8r)}\delta + c_{12}(N_E,r)\frac{\left(\ln\left(3\beta\ln\left(3+\delta^{-1}\right)\right)\right)^2}{\left(\beta\ln\left(3+\delta^{-1}\right)\right)^{3/4}} \\ &= c_{14}(E,r,\sigma)(1+3\delta)^{1-\tau}\delta^\tau + c_{12}(N_E,r)\beta^{-3/4}\frac{\left(\ln\left(3\beta\ln\left(3+\delta^{-1}\right)\right)\right)^2}{\left(\ln\left(3+\delta^{-1}\right)\right)^{3/4}},\end{aligned} \tag{81}$$

where τ, β, and δ are the same as in (80).

Using (81), we obtain that

$$\|v_1 - v_2\|_{\mathbb{L}^\infty(B_r)} \le c_{15}(N,E,r,\sigma)\left(\ln\left(3+\delta^{-1}\right)\right)^{-3/4} \times \left(\ln\left(3\ln\left(3+\delta^{-1}\right)\right)\right)^2, \tag{82}$$

for $\delta = \|S_1^+(E) - S_2^+(E)\|_{\mathbb{L}^2(\partial B_r \times \partial B_r)} \le \delta_2(N_E, r, \sigma)$, where δ_2 is a sufficiently small positive constant. Estimate (82) in the general case (with modified c_{15}) follows from (82) for $\delta \le \delta_2(N_E, r, \sigma)$ and the property that $\|v_j\|_{\mathbb{L}^\infty(B_r)} \le N$.

This completes the proof of (14).

8. Proof of Lemma 7

In this section we assume for simplicity that $r = 1$ and therefore $\partial B_r = \mathbb{S}^{d-1}$.

We fix an orthonormal basis in $\mathbb{L}^2(\partial B_r)$:

$$\left\{f_{jp} : j \ge 0; 1 \le p \le p_j\right\}, \quad f_{jp} \text{ is a spherical harmonic of degree } j, \tag{83}$$

where p_j is the dimension of the space of spherical harmonics of order j,

$$p_j = \binom{j+d-1}{d-1} - \binom{j+d-3}{d-1}, \tag{84}$$

where

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}, \quad \text{for } n \ge 0, \qquad \binom{n}{k} = 0, \quad \text{for } n < 0. \tag{85}$$

The precise choice of f_{jp} is irrelevant for our purposes. Besides orthonormality, we only need f_{jp} to be the restriction of a homogeneous harmonic polynomial of degree j to the sphere ∂B_r and so $|x|^j f_{jp}(x \mid |x|)$ is harmonic pn $\mathbb{R}^d$. In the Sobolev spaces $\mathbb{H}^s(\partial B_r)$ the norm is defined by

$$\left\|\sum_{j,p} c_{jp} f_{jp}\right\|^2_{\mathbb{H}^s(\partial B_r)} = \sum_{j,p} (1+j)^{2s} \left|c_{jp}\right|^2. \tag{86}$$

The solution ϕ of the exterior Dirichlet problem

$$\begin{gathered} -\Delta\phi = E\phi, \quad \text{in } \mathbb{R}^d \setminus \overline{B}_r, \\ \lim_{|x| \to +\infty} |x|^{(d-1)/2} \left(\frac{\partial}{\partial |x|}\phi - i\sqrt{E}\phi\right) = 0, \\ \phi|_{\partial B_r} = u \in \mathbb{H}^1(\partial B_r), \end{gathered} \tag{87}$$

can be expressed in the following form (see, e.g., [1, 38]):

$$\phi = \sum_{j,p} c_{jp} \phi_{jp}, \tag{88}$$

where c_{jp} are expansion coefficients of u in the basis $\{f_{jp} : j \ge 0; 1 \le p \le p_j\}$, and

$$\begin{gathered} \phi_{jp} \text{ denotes the solution of (89) with } u = f_{jp}, \\ \phi_{jp}(x) = h_{jp}(|x|) f_{jp}\left(\frac{x}{|x|}\right), \\ h_{jp}(|x|) = |x|^{-(d-2)/2} \frac{H^{(1)}_{j+((d-2)/2)}\left(\sqrt{E}|x|\right)}{H^{(1)}_{j+((d-2)/2)}\left(\sqrt{E}\right)}, \end{gathered} \tag{89}$$

where $H^{(1)}_\mu$ is the Hankel function of the first kind. Let

$$\phi^0_{jp}(x) = |x|^{-j-d+2} f_{jp}\left(\frac{x}{|x|}\right). \tag{90}$$

Note that ϕ^0_{jp} is harmonic in $\mathbb{R}^d \setminus \{0\}$ and

$$\lim_{|x| \to +\infty} |x|^{(d-1)/2} \left(\frac{\partial}{\partial |x|}\phi^0_{jp} - i\sqrt{E}\phi^0_{jp}\right) = 0, \quad \text{for } j + \frac{d-3}{2} > 0. \tag{91}$$

Using the Green formula and the radiation condition for ϕ_{jp}, ϕ^0_{jp}, we get that

$$\begin{aligned} &\int_{\mathbb{R}^d \setminus B_r} E\phi_{jp}\phi^0_{jp} dx \\ &= \int_{\mathbb{R}^d \setminus B_r} \left(\Delta\phi^0_{jp}\phi_{jp} - \Delta\phi_{jp}\phi^0_{jp}\right) dx \\ &= \int_{\partial B_r} \left(\frac{\partial\phi^0_{jp}}{\partial\nu_+}\phi_{jp} - \frac{\partial\phi_{jp}}{\partial\nu_+}\phi^0_{jp}\right) dx, \quad \text{for } j + \frac{d-3}{2} > 0. \end{aligned} \tag{92}$$

Due to (89) and (90), we have that

$$\left|\int_{\partial B_r} \frac{\partial\phi^0_{jp}}{\partial\nu_+}\phi_{jp} dx\right| = (j+d-2)\int_{\partial B_r} f^2_{jp} dx = j+d-2. \tag{93}$$

Using also the following property of the Hankel function of the first kind (see, e.g., [39]):

$$\left|H^{(1)}_\mu(x)\right| \text{ is a decreasing function of } x, \text{ for } x \in \mathbb{R}_+, \mu \in \mathbb{R}, \tag{94}$$

we get that

$$\begin{aligned}\left|\int_{\mathbb{R}^d\setminus B_r}\phi_{jp}\phi_{jp}^0dx\right| &= \left|\int_1^{+\infty}t^{-j-d+2}h_{jp}(t)\,t^{d-1}dt\right|\\ &= \left|\int_1^{+\infty}t^{-j-(d/2)}\frac{H^{(1)}_{j+((d-2)/2)}\left(\sqrt{E}t\right)}{H^{(1)}_{j+((d-2)/2)}\left(\sqrt{E}\right)}dt\right|\\ &\le \int_1^{+\infty}t^{-j-(d/2)}dt = \frac{1}{j+(d/2)-1}\\ &\le 2,\quad \text{for } j+\frac{d-3}{2}>0.\end{aligned}\tag{95}$$

Combining (89), (90), (92), (93), and (95), we obtain that

$$\left|\int_{\partial B_r}\frac{\partial\phi_{jp}^0}{\partial\nu_+}\phi_{jp}dx\right| = \left|\frac{h_{jp}'(r)}{h_{jp}(r)}\right| \le j+d-2+2E,\quad \text{for } j+\frac{d-3}{2}>0.\tag{96}$$

Let consider the cases when $j+((d-3)/2)\le 0$.

Case 1 ($j=0$, $d=2$). Using the property $dH_0^{(1)}(t)/dt = -H_1^{(1)}(t)$, we get that

$$\frac{h_{jp}'(r)}{h_{jp}(r)} = \sqrt{E}\frac{H_1^{(1)}\left(\sqrt{E}\right)}{H_0^{(1)}\left(\sqrt{E}\right)}.\tag{97}$$

We recall that functions $H_0^{(1)}$ and $H_1^{(1)}$ have the following asymptotic forms (see, e.g. [39]):

$$\begin{aligned}H_0^{(1)}(t) &\sim \frac{2i}{\pi}\ln\left(\frac{t}{2}\right),\quad \text{as } t\longrightarrow +0,\\ H_0^{(1)}(t) &\sim \sqrt{\frac{2}{\pi t}}e^{i(t-\pi/4)},\quad \text{as } t\longrightarrow +\infty,\\ H_1^{(1)}(t) &\sim -\frac{i}{\pi}\left(\frac{2}{t}\right),\quad \text{as } t\longrightarrow +0,\\ H_1^{(1)}(t) &\sim \sqrt{\frac{2}{\pi t}}e^{i(t-3\pi/4)},\quad \text{as } t\longrightarrow +\infty.\end{aligned}\tag{98}$$

Using (94) and (98), we get that for some $c>0$

$$\frac{H_1^{(1)}(t)}{H_0^{(1)}(t)} \le c\left(1+\frac{1}{t}\right).\tag{99}$$

Combining (97) and (99), we obtain that for $j=0$, $d=2$

$$\left|\frac{h_{jp}'(r)}{h_{jp}(r)}\right| \le c\left(1+\sqrt{E}\right).\tag{100}$$

Case 2 ($j=0$, $d=3$). We have that

$$H^{(1)}_{j+((d-2)/2)}(t) = \sqrt{\frac{2}{\pi t}}e^{i(t-\pi/2)}.\tag{101}$$

Using (89) and (101), we get that for $j=0$, $d=3$

$$\frac{h_{jp}'(r)}{h_{jp}(r)} = -1+i\sqrt{E}.\tag{102}$$

Combining (86)–(89), (96), (100), and (102), we get that for some constant $c' = c'(d) > 0$:

$$\begin{aligned}\left\|\frac{\partial\phi}{\partial\nu_+}\bigg|_{\partial B_r}\right\|^2_{\mathbb{L}^2(\partial B_r)} &= \sum_{j,p}c_{jp}^2\left|\frac{h_{jp}'(r)}{h_{jp}(r)}\right|^2\\ &\le c'(1+E)^2\sum_{j,p}(1+j)^2c_{jp}^2.\end{aligned}\tag{103}$$

Using (86) and (103), we obtain (21).

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