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Research Article

Bessel Transform of (k, γ)-Bessel Lipschitz Functions

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Using a generalized translation operator, we obtain an analog of Theorem 5.2 in Younis (1986) for the Bessel transform for functions satisfying the (k,γ)-Bessel Lipschitz condition in $L_{2,\alpha}(\mathbb{R}_+)$.

1. Introduction and Preliminaries

Younis Theorem 5.2 [1] characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transforms; namely, we have the following.

Theorem 1 (see [1]). *Let $f \in L^2(\mathbb{R})$. Then the followings are equivalent:*

(1) $\|f(x+h) - f(x)\|_2 = O(h^\alpha/(\log(1/h))^\beta)$ *as* $h \to 0$, $0 < \alpha < 1$, $\beta > 0$,

(2) $\int_{|x|\geq r} |\mathcal{F}(f)(x)|^2 dx = O(r^{-2\alpha}(\log r)^{-2\beta})$ *as* $r \to +\infty$,

where $\mathcal{F}$ stands for the Fourier transform of f.

In this paper, we obtain a generalization of Theorem 1 for the Bessel transform. For this purpose, we use a generalized translation operator.

Assume that $L_{2,\alpha}(\mathbb{R}_+)$; $\alpha > -1/2$ is the Hilbert space of measurable functions $f(t)$ on $\mathbb{R}_+$ with finite norm

$$\|f\|_{2,\alpha} = \left(\int_0^\infty |f(x)|^2 x^{2\alpha+1} dx\right)^{1/2}. \tag{1}$$

Let

$$B = \frac{d^2}{dt^2} + \frac{(2\alpha+1)}{t}\frac{d}{dt} \tag{2}$$

be the Bessel differential operator.

For $\alpha \geq -1/2$, we introduce the Bessel normalized function of the first kind j_α defined by

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+\alpha+1)}\left(\frac{z}{2}\right)^{2n}, \tag{3}$$

where Γ is the gamma function (see [2]).

The function $y = j_\alpha(x)$ satisfies the differential equation

$$By + y = 0, \tag{4}$$

with the initial conditions $y(0) = 1$ and $y'(0) = 0$. $j_\alpha(z)$ is function infinitely differentiable, even, and, moreover, entirely analytic.

Lemma 2. *For $x \in \mathbb{R}_+$ the following inequality is fulfilled:*

$$|1 - j_\alpha(x)| \geq c, \tag{5}$$

with $x \geq 1$, where $c > 0$ is a certain constant which depends only on α.

Proof. Analog of Lemma 2.9 is in [3]. □

Lemma 3. *The following inequalities are valid for Bessel function j_α:*

(1) $|j_\alpha(x)| \leq 1$, *for all* $x \in \mathbb{R}^+$,

(2) $1 - j_\alpha(x) = O(x^2)$, $0 \leq x \leq 1$.

Proof. See [4]. □

The Bessel transform we call the integral transform from [2, 5, 6]

$$\widehat{f}(\lambda) = \int_0^{\infty} f(t)\, j_\alpha(\lambda t)\, t^{2\alpha+1} dt, \quad \lambda \in \mathbb{R}^+. \tag{6}$$

The inverse Bessel transform is given by the formula

$$f(t) = \left(2^{\alpha}\Gamma(\alpha+1)\right)^{-2} \int_0^{\infty} \widehat{f}(\lambda)\, j_\alpha(\lambda t)\, \lambda^{2\alpha+1} d\lambda. \tag{7}$$

We have the Parseval's identity

$$\|\widehat{f}\|_{2,\alpha} = 2^{\alpha}\Gamma(\alpha+1)\, \|f\|_{2,\alpha}. \tag{8}$$

In $L_{2,\alpha}(\mathbb{R}_+)$, consider the generalized translation operator T_h defined by

$$T_h f(t) = c_\alpha \int_0^{\pi} f\left(\sqrt{t^2 + h^2 - 2th\cos\varphi}\right) \sin^{2\alpha}\varphi\, d\varphi, \tag{9}$$

where

$$c_\alpha = \left(\int_0^{\pi} \sin^{2\alpha}\varphi\, d\varphi\right)^{-1} = \frac{\Gamma(\alpha+1)}{\Gamma(1/2)\,\Gamma(\alpha+(1/2))}. \tag{10}$$

The following relations connect the generalized translation operator and the Bessel transform; in [7] we have

$$\left(\widehat{T_h f}\right)(\lambda) = j_\alpha(\lambda h)\, \widehat{f}(\lambda). \tag{11}$$

2. Main Result

In this section we give the main result of this paper. We need first to define (k,γ)-Bessel Lipschitz class.

Definition 4. Let $0 < k < 1$ and $\gamma \geq 0$. A function $f \in L_{2,\alpha}(\mathbb{R}^+)$ is said to be in the (k,γ)-Bessel Lipschitz class, denoted by $\mathrm{Lip}(k,\gamma,2)$, if

$$\|T_h f(t) - f(t)\|_{2,\alpha} = O\left(\frac{h^k}{(\log(1/h))^{\gamma}}\right), \quad \text{as } h \longrightarrow 0. \tag{12}$$

Our main result is as follows.

Theorem 5. *Let $f \in L_{2,\alpha}(\mathbb{R}^+)$. Then the followings are equivalents*

(1) $f \in \mathrm{Lip}(k,\gamma,2)$.

(2) $\int_r^{\infty} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = O(r^{-2k}/(\log r)^{2\gamma})$, *as* $r \to +\infty$.

Proof. (1) ⇒ (2) Assume that $f \in \mathrm{Lip}(k,\gamma,2)$. Then we have

$$\begin{aligned} &\|T_h f(t) - f(t)\|_{2,\alpha}^2 \\ &\quad = \frac{1}{(2^{\alpha}\Gamma(\alpha+1))^2} \int_0^{\infty} |1 - j_\alpha(\lambda h)|^2 \left|\widehat{f}(\lambda)\right|^2 \lambda^{2\alpha+1} d\lambda. \end{aligned} \tag{13}$$

If $\lambda \in [1/h, 2/h]$ then $\lambda h \geq 1$ and Lemma 2 implies that

$$1 \leq \frac{1}{c^2} \left|1 - j_\alpha(\lambda h)\right|. \tag{14}$$

Then

$$\begin{aligned} &\int_{1/h}^{2/h} \left|\widehat{f}(\lambda)\right|^2 \lambda^{2\alpha+1} d\lambda \\ &\quad \leq \frac{1}{c^2} \int_{1/h}^{2/h} |1 - j_\alpha(\lambda h)|^2 \left|\widehat{f}(\lambda)\right|^2 \lambda^{2\alpha+1} d\lambda \\ &\quad \leq \frac{1}{c^2} \int_0^{\infty} |1 - j_\alpha(\lambda h)|^2 \left|\widehat{f}(\lambda)\right|^2 \lambda^{2\alpha+1} d\lambda \\ &\quad = O\left(\frac{h^{2k}}{(\log(1/h))^{2\gamma}}\right). \end{aligned} \tag{15}$$

We obtain

$$\int_r^{2r} \left|\widehat{f}(\lambda)\right|^2 \lambda^{2\alpha+1} d\lambda \leq C \frac{r^{-2k}}{(\log r)^{2\gamma}}, \tag{16}$$

where C is a positive constant.

So that

$$\begin{aligned} &\int_r^{\infty} \left|\widehat{f}(\lambda)\right|^2 \lambda^{2\alpha+1} d\lambda \\ &\quad = \left[\int_r^{2r} + \int_{2r}^{4r} + \int_{4r}^{8r} + \cdots\right] \left|\widehat{f}(\lambda)\right|^2 \lambda^{2\alpha+1} d\lambda \\ &\quad \leq C \frac{r^{-2k}}{(\log r)^{2\gamma}} + C \frac{(2r)^{-2k}}{(\log 2r)^{2\gamma}} + C \frac{(4r)^{-2k}}{(\log 4r)^{2\gamma}} + \cdots \\ &\quad \leq C \frac{r^{-2k}}{(\log r)^{2\gamma}} \left(1 + 2^{-2k} + \left(2^{-2k}\right)^2 + \left(2^{-2k}\right)^3 + \cdots\right) \\ &\quad \leq CK \frac{r^{-2k}}{(\log r)^{2\gamma}}, \end{aligned} \tag{17}$$

where $K = (1 - 2^{-2k})^{-1}$ since $2^{-2k} < 1$.

This proves that

$$\int_r^{\infty} \left|\widehat{f}(\lambda)\right|^2 \lambda^{2\alpha+1} d\lambda = O\left(\frac{r^{-2k}}{(\log r)^{2\gamma}}\right) \quad \text{as } r \longrightarrow +\infty. \tag{18}$$

(2) ⇒ (1) Suppose now that

$$\int_r^{\infty} \left|\widehat{f}(\lambda)\right|^2 \lambda^{2\alpha+1} d\lambda = O\left(\frac{r^{-2k}}{(\log r)^{2\gamma}}\right) \quad \text{as } r \longrightarrow +\infty. \tag{19}$$

We write

$$\int_0^{\infty} |1 - j_\alpha(\lambda h)|^2 \left|\widehat{f}(\lambda)\right|^2 \lambda^{2\alpha+1} d\lambda = I_1 + I_2, \tag{20}$$

where

$$I_1 = \int_0^{1/h} \left|1 - j_\alpha(\lambda h)\right|^2 \left|\widehat{f}(\lambda)\right|^2 \lambda^{2\alpha+1} d\lambda,$$
$$I_2 = \int_{1/h}^{\infty} \left|1 - j_\alpha(\lambda h)\right|^2 \left|\widehat{f}(\lambda)\right|^2 \lambda^{2\alpha+1} d\lambda. \tag{21}$$

Estimate the summands I_1 and I_2 from above. It follows from the inequality $|j_\alpha(\lambda h)| \leq 1$ that

$$\begin{aligned} I_2 &= \int_{1/h}^{\infty} \left|1 - j_\alpha(\lambda h)\right|^2 \left|\widehat{f}(\lambda)\right|^2 \lambda^{2\alpha+1} d\lambda \\ &\leq 4 \int_{1/h}^{\infty} \left|\widehat{f}(\lambda)\right|^2 \lambda^{2\alpha+1} d\lambda = O\left(\frac{h^{2k}}{(\log(1/h))^{2\gamma}}\right). \end{aligned} \tag{22}$$

To estimate I_1, we use the inequality (2) of Lemma 3. Set

$$\phi(x) = \int_x^{\infty} \left|\widehat{f}(\lambda)\right|^2 \lambda^{2\alpha+1} d\lambda. \tag{23}$$

Using integration by parts, we obtain

$$\begin{aligned} I_1 &\leq -C_1 h^2 \int_0^{1/h} s^2 \phi'(s)\, ds \\ &\leq -C_1 \phi\left(\frac{1}{h}\right) + 2C_1 h^2 \int_0^{1/h} s\phi(s)\, ds \\ &\leq C_2 h^2 \int_0^{1/h} s\phi(s)\, ds \\ &\leq C_2 h^2 \int_0^{1/h} s s^{-2k} (\log s)^{-2\gamma} ds \\ &\leq C_3 h^{2k} (\log(1/h))^{-2\gamma}, \end{aligned} \tag{24}$$

where C_1, C_2, and C_2 are positive constants and this ends the proof. □

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