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Research Article

The Spectrum of the Operator $D(r, 0, s, 0, t)$ over the Sequence Spaces c_0 and c

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We have examined the spectra of the operator $D(r, 0, s, 0, t)$ on the sequence spaces c_0 and c.

1. Introduction

Spectral theory is an important branch of mathematics due to its application in other branches of science. It has been proved to be a standard tool of mathematical sciences because of its usefulness and application-oriented scope in different fields. In numerical analysis, the spectral values may determine whether a discretization of a differential equation will get the right answer or how fast a conjugate gradient iteration will converge. In aeronautics, the spectral values may determine whether the flow over a wing is laminar or turbulent. In electrical engineering, it may determine the frequency response of an amplifier or the reliability of a power system. In quantum mechanics, it may determine atomic energy levels and thus the frequency of a laser or the spectral signature of a star. In structural mechanics, it may determine whether an automobile is too noisy or whether a building will collapse in an earthquake. In ecology, the spectral values may determine whether a food web will settle into a steady equilibrium. In probability theory, they may determine the rate of convergence of a Markov process.

In summability theory, different classes of matrices have been investigated. Characterization of matrix classes is found in Rath and Tripathy [1], Tripathy [2], Tripathy and Sen [3], and many others. There are particular types of summability methods like Nörlund, Riesz, Euler, and Abel. Matrix methods have been studied from different aspects recently by Altin et al. [4], Tripathy and Baruah [5], and others. Still there is a lot to be explored on spectra of some matrix operators transforming one class of sequences into another class of sequences. The spectra of the difference operator have also been investigated on some classes of sequences. Altay and Başar [6–8] studied the spectra of difference operator Δ and generalized difference operator on c_0, c, and ℓ_p. Recently, the fine spectrum of $B(r, s, t)$ over the sequence spaces c_0, c, ℓ_p, and bv_p has been studied by Furkan et al. [9, 10]. A detailed account of the development and initial works on spectra of some matrix classes are found in the monograph of Başar [11].

Throughout the paper w, ℓ_∞, c, and c_0 denote the space of all bounded, convergent, and null sequences $x = (x_k)$ with complex terms, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$. The zero sequence is denoted by $\theta = (0, 0, 0, \ldots)$. Kızmaz [12] defined the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ as follows:

$$Z(\Delta) = \{x = (x_k) : (\Delta x_k) \in Z\}, \quad \text{for } Z \in \{l_\infty, c, c_0\}, \quad (1)$$

where

$$\Delta x = (\Delta x_k) = (x_k - x_k + 1), \quad \forall k \in \mathbb{N} = \{0, 1, 2, \ldots\}. \quad (2)$$

The previous spaces are Banach spaces, normed by $\|x\|_\Delta = \|x_1\| + \sup_k \|\Delta x_k\|$.

Different classes of sequence spaces using the difference operator have been introduced and investigated in the recent past by Tripathy et al. [13, 14], Tripathy and Mahanta [15], Tripathy and Sarma [16], and many others. The idea of Kizmaz [12] was applied to introduce a new type of generalized difference operator on sequence spaces by Tripathy and Esi [17].

Let $m \in \mathbb{N}$ be fixed; then Tripathy and Esi [17] have introduced the following type of difference sequence spaces:

$$Z(\Delta_m) = \{x = (x_k) : (\Delta_m x_k) \in Z\}, \quad \text{for } Z \in \{\ell_\infty, c, c_0\}, \tag{3}$$

where

$$\Delta_m x = (\Delta_m x_k) = (x_k - x_{k+m}). \tag{4}$$

Let $m, n \geq 0$ be fixed integers; then Esi et al. [18] have introduced the following type of difference sequence spaces:

$$Z(\Delta_m^n) = \{x = (x_k) \in w : \Delta_m^n x = (\Delta_m^n x_k) \in Z\}, \quad \text{for } Z \in \{\ell_\infty, c, c_0\}, \tag{5}$$

where

$$\Delta_m^n x = (\Delta_m^n x_k) = (\Delta_m^{n-1} x_k - \Delta_m^{n-1} x_{k+m}), \quad \Delta_m^0 x_k = x_k \quad \forall k \in \mathbb{N}. \tag{6}$$

Taking $n = 1$, we have the sequence spaces $\ell_\infty(\Delta_m)$, $c(\Delta_m)$, and $c_0(\Delta_m)$ studied by Tripathy and Esi [17]. Taking $m = 1$, we have the sequence spaces $\ell_\infty(\Delta^n)$, $c(\Delta^n)$, and $c_0(\Delta^n)$ studied by Et and Çolak [19]. Taking $m = 1$ and $n = 1$, we have the sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$, and $c_0(\Delta)$ studied by Kızmaz [12].

2. Preliminaries and Definition

Let X be a linear space. By $B(X)$, we denote the set of all bounded linear operators on X into itself. If $T \in B(X)$, where X is a Banach space, then the adjoint operator T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*\varphi)(x) = \varphi(Tx)$ for all $\varphi \in X^*$ and $x \in X$.

Let $T : D(T) \to X$ be a linear operator, defined on $D(T) \subset X$, where $D(T)$ denote the domain of T and X is a complex normed linear space. For $T \in B(X)$ we associate a complex number α with the operator $(T - \alpha I)$ denoted by T_α defined on the same domain $D(T)$, where I is the identity operator. The inverse $(T - \alpha I)^{-1}$, denoted by T_α^{-1}, is known as the resolvent operator of T_α.

A regular value of α is a complex number α of T such that

(R1) T_α^{-1} exists,

(R2) T_α^{-1} is bounde,

(R3) T_α^{-1} is defined on a set which is dense in X.

The resolvent set of T is the set of all such regular values α of T, denoted by $\rho(T)$. Its complement is given by $\mathbb{C} \setminus \rho(T)$ in the complex plane $\mathbb{C}$ which is called the spectrum of T, denoted by $\sigma(T)$. Thus the spectrum $\sigma(T)$ consists of those values of $\alpha \in \mathbb{C}$, for which T_α is not invertible.

2.1. Classification of Spectrum. The spectrum $\sigma(T)$ is partitioned into three disjoint sets as follows.

(i) The point (discrete) spectrum $\sigma_p(T)$ is the set such that T_α^{-1} does not exist. Further $\alpha \in \sigma_p(T)$ is called the eigenvalue of T.

(ii) The continuous spectrum $\sigma_c(T)$ is the set such that T_α^{-1} exists and satisfies (R3) but not (R2); that is, T_α^{-1} is unbounded.

(iii) The residual spectrum $\sigma_r(T)$ is the set such that T_α^{-1} exists (and may be bounded or not) but does not satisfy (R3); that is, the domain of T_α^{-1} is not dense in X.

This is to note that in finite dimensional case, continuous spectrum coincides with the residual spectrum and are equal to the empty set and the spectrum consists of only the point spectrum.

Let E and F be two sequence spaces and let $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from E into F, denoted by $A : E \to F$, if for every sequence $x = (x_n) \in E$ the sequence $Ax = \{(Ax)_n\}$ is in F where $(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k$ ($n \in \mathbb{N}$ and $x \in E$), provided the right hand side converges for every $n \in \mathbb{N}$ and $x \in E$.

Our main focus in this paper is on the matrix $A = D(r, 0, s, 0, t)$, where

$$D(r, 0, s, 0, t) = \begin{bmatrix} r & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & r & 0 & 0 & 0 & 0 & \cdots \\ s & 0 & r & 0 & 0 & 0 & \cdots \\ 0 & s & 0 & r & 0 & 0 & \cdots \\ t & 0 & s & 0 & r & 0 & \cdots \\ 0 & t & 0 & s & 0 & r & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{7}$$

We assume here and hereafter that s and t are complex parameters which do not simultaneously vanish.

Lemma 1. *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(c)$ from c to itself if and only if*

(1) *the rows of A in ℓ_1 and their ℓ_1 norms are bounded,*

(2) *the columns of A are in c,*

(3) *the sequence of row sums of A is in c.*

The operator norm of T is the supremum of the ℓ_1 norms of the rows.

Lemma 2. *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(c_0)$ from c_0 to itself if and only if*

(1) *the rows of A in ℓ_1 and their ℓ_1 norms are bounded,*

(2) *the columns of A are in c_0.*

The operator norm of T is the supremum of the ℓ_1 norms of the rows.

From the previous two lemmas we have the following results.

Lemma 3. *$D(r,0,s,0,t) : c \to c$ is a bounded linear operator with*

$$\|D(r,0,s,0,t)\|_{(c,,\ c)} = |r| + |s| + |t|. \tag{8}$$

Lemma 4. *$D(r,0,s,0,t) : c_0 \to c_0$ is a bounded linear operator with*

$$\|D(r,0,s,0,t)\|_{(c,c)} = \|D(r,0,s,0,t)\|_{(c_0,c_0)}. \tag{9}$$

In this paper, our purpose is to determine the fine spectrum of the operator $D(r,0,s,0,t)$ over the sequence spaces c_0 and c.

3. The Fine Spectrum of the Operator $D(r,0,s,0,t)$ over the Sequence Space c_0

Theorem 5. *Let s be a complex number such that $\sqrt{s^2} = -s$ and define the set by*

$$S = \left\{\alpha \in C : \left|\frac{2(r-\alpha)}{-s+\sqrt{s^2-4t(r-\alpha)}}\right| \le 1\right\}. \tag{10}$$

Then

$$\sigma\left(D(r,0,s,0,t), c_0\right) = S. \tag{11}$$

Proof. At first we have to prove that $(D(r,0,s,0,t) - \alpha I)^{-1}$ exists and is in $B(c_0)$ for $\alpha \notin S$ and secondly we have to show that $(D(r,0,s,0,t) - \alpha I)$ is not invertible for $\alpha \in S$.

Without loss of generality we may assume that $\sqrt{s^2} = -s$. Let $\alpha \notin S$; then it is easy to see that $\alpha \neq r$ and so $(D(r,0,s,0,t) - \alpha I)$ is triangle and has an inverse.

Let

$$(D(r,0,s,0,t)-\alpha I)^{-1} = \begin{bmatrix} a_1 & 0 & 0 & 0 & 0 & \cdots \\ a_2 & a_1 & 0 & 0 & 0 & \cdots \\ a_3 & a_2 & a_1 & 0 & 0 & \cdots \\ a_4 & a_3 & a_2 & a_1 & 0 & \cdots \\ a_5 & a_4 & a_3 & a_2 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{12}$$

We have

$$\begin{bmatrix} r-\alpha & 0 & 0 & 0 & 0 & \cdots \\ 0 & r-\alpha & 0 & 0 & 0 & \cdots \\ s & 0 & r-\alpha & 0 & 0 & \cdots \\ 0 & s & 0 & r-\alpha & 0 & \cdots \\ t & 0 & s & 0 & r-\alpha & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \times \begin{bmatrix} a_1 & 0 & 0 & 0 & 0 & \cdots \\ a_2 & a_1 & 0 & 0 & 0 & \cdots \\ a_3 & a_2 & a_1 & 0 & 0 & \cdots \\ a_4 & a_3 & a_2 & a_1 & 0 & \cdots \\ a_5 & a_4 & a_3 & a_2 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{13}$$

On solving the previous system of equations we get

$$\begin{aligned} a_1 &= \frac{1}{r-\alpha}, \\ a_2 &= 0, \\ a_3 &= -\frac{s}{(r-\alpha)^2}, \\ a_4 &= 0, \\ a_5 &= \frac{s^2-(r-\alpha)t}{(r-\alpha)^3}, \\ &\vdots \end{aligned} \tag{14}$$

In general this sequence is obtained recursively by

$$a_n = -\frac{sa_{n-2}+ta_{n-4}}{r-\alpha}, \quad \text{for } n \ge 5. \tag{15}$$

It is easy to verify that

$$\begin{aligned} a_{2n+1} &= \frac{1}{\sqrt{s^2-4t(r-\alpha)}} \\ &\quad \times \left\{\left[\frac{-s+\sqrt{s^2-4t(r-\alpha)}}{2(r-\alpha)}\right]^{n+1} - \left[\frac{-s-\sqrt{s^2-4t(r-\alpha)}}{2(r-\alpha)}\right]^{n+1}\right\} \\ &\qquad \text{for } n \in Z^+, \\ a_{2n} &= 0, \quad \text{for } n \in \mathbb{N}. \end{aligned} \tag{16}$$

On taking

$$\begin{aligned} u_1 &= \left[\frac{-s+\sqrt{s^2-4t(r-\alpha)}}{2(r-\alpha)}\right], \\ u_2 &= \left[\frac{-s-\sqrt{s^2-4t(r-\alpha)}}{2(r-\alpha)}\right], \end{aligned} \tag{17}$$

we have

$$a_{2n+1} = \frac{1}{\sqrt{s^2 - 4t(r-\alpha)}}\left(u_1^{n+1} - u_2^{n+1}\right), \quad \text{for } n \in \mathbb{N}. \tag{18}$$

If one assumes $\sqrt{s^2} = s$, then we will get the same sequence as in the case of $\sqrt{s^2} = -s$.

If $y = \sqrt{s^2 - 4t(r-\alpha)}$, then

$$\begin{aligned} a_{2n+1} &= \frac{1}{y\{2(r-\alpha)\}^{n+1}}\left\{(-s+y)^{n+1} - (-s-y)^{n+1}\right\} \\ &= \frac{1}{\{2(r-\alpha)\}^{n+1}} \\ &\quad \times \left[2(n+1)(-s)^n + \frac{(n+1)n(n-1)}{6}(-s)^{n-2}y^2 + \cdots\right]. \end{aligned} \tag{19}$$

If we put $s^2 = 4t(r-\alpha)$, then $a_{2n+1} = (2(n+1)/-s)[-s/2(r-\alpha)]^{n+1}$, for $n \in \mathbb{N}$ and $a_{2n} = 0$, for $n \in \mathbb{N}$.

Thus, on simple calculation we get $(a_n) \in \ell_1$ if and only if $|-s/2(r-\alpha)| < 1$. Therefore $\alpha \notin S$ implies $a_n \to 0$, as $n \to \infty$.

Next, we assume that $s^2 \neq 4t(r-\alpha)$. Since $\alpha \notin S$, therefore $|u_1| < 1$. Now we have to show that $|u_2| < 1$.

We have that $|u_1| < 1$ implies $|1 + \sqrt{1 - (4t(r-\alpha))/s^2}| < |2(r-\alpha)/-s|$.

Since $|1 - \sqrt{z}| \le |1 + \sqrt{z}|$ for any $z \in \mathbb{C}$, we must have $|1 - \sqrt{1 + (4t(r-\alpha))/s^2}| < |2(r-\alpha)/-s|$ which leads us to conclude that $|u_2| < 1$ and hence in this case also $\alpha \notin S$ implies that $a_n \to 0$ as $n \to \infty$.

Now,

$$\begin{aligned} &\left\|(D(r,0,s,0,t-\alpha I)^{-1}\right\|_{(c_0,c_0)} \\ &\quad = \sup_n \sum_{k=1}^{n} |a_k| = \sum_{k=1}^{\infty} |a_k| \\ &\quad \le \frac{1}{\left|\sqrt{s^2 - 4t(r-\alpha)}\right|}\left(\sum_{n=0}^{\infty} |u_1|^{n+1} + \sum_{n=0}^{\infty} |u_2|^{n+1}\right) < \infty. \end{aligned} \tag{20}$$

Since $|u_1| < 1$ and $|u_2| < 1$, we have $\alpha \notin \sigma(D(r,0,s,0,t), c_0)$.

Thus, $\alpha \notin S \Rightarrow \alpha \notin \sigma(D(r,0,s,0,t), c_0)$ and hence $\sigma(D(r,0,s,0,t), c_0) \subseteq S$.

Next we have to show that $S \subseteq \sigma(D(r,0,s,0,t), c_0)$.

Let $\alpha \in S$. If $\alpha = r$, then $(D(r,0,s,0,t) - \alpha I)$ is represented by the matrix

$$D(0,0,s,0,t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ s & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & s & 0 & 0 & 0 & 0 & \cdots \\ t & 0 & s & 0 & 0 & 0 & \cdots \\ 0 & t & 0 & s & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{21}$$

Since $D(r,0,s,0,t) - rI = D(0,0,s,0,t)$ does not have a dense range, so it is not invertible.

Again if $s^2 = 4t(r-\alpha)$, then $a_{2n+1} = (2(n+1)/-s)[-s/2(r-\alpha)]^{n+1}$ for $n \in Z^+$ and $a_{2n} = 0$, for $n \in \mathbb{N}$.

Now $\alpha \in S \Rightarrow |-s/2(r-\alpha)| \ge 1$ and hence $(a_n) \notin c_0$; therefore, we have $(D(r,0,s,0,t) - \alpha I)^{-1} \notin B(c_0)$.

Next we assume that $\alpha \neq r$ and $s^2 \neq 4t(r-\alpha)$.

Since $\alpha \neq r$, we have that $(D(r,0,s,0,t) - \alpha I)$ is a triangle. Further $s^2 \neq 4t(r-\alpha)$, so we must have $|u_1| > |u_2|$. This implies $(a_n) \notin c_0$ and hence $(D(r,0,s,0,t) - \alpha I)^{-1} \notin B(c_0)$. This shows that $S \subseteq \sigma(D(r,0,s,0,t), c_0)$.

This completes the proof. □

Theorem 6. *One has $\sigma_p(D(r,0,s,0,t), c_0) = \emptyset$.*

Proof. Suppose $D(r,0,s,0,t)x = \alpha x$ for $x \neq \theta = (0,0,0,\ldots)$ in c_0. Then by solving the system of linear equations we have

$$\begin{aligned} rx_0 &= \alpha x_0, \\ sx_0 + rx_2 &= \alpha x_2, \\ tx_0 + sx_2 + rx_4 &= \alpha x_4, \\ &\vdots \end{aligned} \tag{22}$$

and

$$\begin{aligned} rx_1 &= \alpha x_1, \\ sx_1 + rx_3 &= \alpha x_3, \\ tx_1 + sx_3 + rx_5 &= \alpha x_5, \\ &\vdots \end{aligned} \tag{23}$$

If x_{n_0} is the first nonzero entry of the sequence $x = (x_n)$, then from the previous system of linear (22) and (23) we have $tx_{n_0-4} + sx_{n_0-2} + rx_{n_0} = \alpha x_{n_0}$ and we obtain that $\alpha = r$ and from the next of either (22) or (23) we get x_{n_0}=0 which is a contradiction. This completes the proof. □

If $T : c_0 \to c_0$ is a bounded linear operator with the matrix A, then it is known that the adjoint operator $T^* : c_0^* \to c_0^*$ is defined by the transpose A^t of the matrix A. It should be noted that the dual space c_0^* of c_0 is isometrically isomorphic to the Banach space ℓ_1 of absolutely summable sequences normed by $\|x\| = \sum |x_n|$.

Theorem 7. *One has $\sigma_p(D(r,0,s,0,t)^*, c_0^*)$=$S_1$, where*

$$S_1 = \left\{\alpha \in C : \left|\frac{2(r-\alpha)}{-s + \sqrt{s^2 - 4t(r-\alpha)}}\right| < 1\right\}. \tag{24}$$

Proof. Suppose $D(r,0,s,0,t)^* x = \alpha x$ for $x \neq \theta = (0,0,0,\ldots)$ in $c_0^* \cong \ell_1$; that is, considering the following system of linear equations we have

$$\begin{aligned} rx_0 + sx_2 + tx_4 &= \alpha x_0, \\ rx_1 + sx_3 + tx_5 &= \alpha x_1, \\ rx_2 + sx_4 + tx_6 &= \alpha x_2, \\ rx_3 + sx_5 + tx_7 &= \alpha x_3, \\ &\vdots \end{aligned} \tag{25}$$

If $\alpha = r$, then we may choose $x_0 \neq 0$ and so $x = (x_0, 0, 0, \ldots)$ is an eigenvector corresponding to $\alpha = r$ and hence $\alpha \in \sigma_p(D(r, 0, s, 0, t)^*, c_0^*)$.

Next we assume that $\alpha \neq r$; then from the previous system of equations we have

$$x_{2n+4} = \frac{a_{2n+1}(r-\alpha)^{n+1}}{t^{n+1}}(\alpha - r)\, x_0 + \frac{a_{2n+3}(r-\alpha)^{n+2}}{t^{n+1}} x_2, \quad \text{for } n \in \mathbb{N}, \tag{26}$$

$$x_{2n+5} = \frac{a_{2n+1}(r-\alpha)^{n+1}}{t^{n+1}}(\alpha - r)\, x_1 + \frac{a_{2n+3}(r-\alpha)^{n+2}}{t^{n+1}} x_3 \quad \text{for } n \in \mathbb{N}. \tag{27}$$

If α is a number such that $|2(r-\alpha)/-s+\sqrt{s^2-4t(r-\alpha)}| < 1$, then we may choose $x_0 = x_1 = 1$ and $x_2 = x_3 = 2(r-\alpha)/-s+\sqrt{s^2-4t(r-\alpha)}$.

We can easily verify that $u_1 - u_2 = \sqrt{(s^2-4t(r-\alpha))}/(r-\alpha)$ and $u_1 u_2 = t/(r-\alpha)$; using these results and combining the fact $x_2 = x_3 = 1/u_1$ with relation (26) we observe that

$$\begin{aligned}
x_{2n+4} &= \frac{a_{2n+1}(r-\alpha)^{n+1}}{t^{n+1}}(\alpha - r)\, x_0 + \frac{a_{2n+3}(r-\alpha)^{n+2}}{t^{n+1}} x_2 \\
&= \frac{\left(u_1^{n+1} - u_2^{n+1}\right)}{\sqrt{s^2 - 4t\,(r-\alpha)}} \times \frac{(r-\alpha)^{n+1}}{t^{n+1}} (\alpha - r) \\
&\quad + \frac{\left(u_1^{n+2} - u_2^{n+2}\right)}{\sqrt{s^2 - 4t\,(r-\alpha)}} \times \frac{(r-\alpha)^{n+2}}{t^{n+1}} \times \frac{1}{u_1} \\
&= \frac{(r-\alpha)}{\sqrt{s^2-4t\,(r-\alpha)}} \times \left(\frac{r-\alpha}{t}\right)^{n+1} \\
&\quad \times \left(u_2^{n+1} - u_1^{n+1} + \frac{u_1^{n+2} - u_2^{n+2}}{u_1} \right) \\
&= \frac{1}{(u_1 - u_2)} \times \frac{1}{u_1^{n+1} u_2^{n+1}} \times \frac{1}{u_1} \left(u_1 u_2^{n+1} - u_2^{n+1} \right) \\
&= \frac{u_2^{n+1}(u_1 - u_2)}{(u_1 - u_2)\, u_1^{n+2} u_2^{n+1}} \\
&= \frac{1}{u_1^{n+2}} \\
&= (x_2)^{n+2}, \quad \text{for } n \in \mathbb{N}.
\end{aligned} \tag{28}$$

Similarly we can show that $x_{2n+5} = (x_3)^{n+2}$, for $n \in \mathbb{N}$.

The same result may be obtained in the case $u_1 = u_2$, that is, for the case $s^2 = 4t(r-\alpha)$.

Now $x = (x_k) \in \ell_1$ since $|x_2| = |x_3| < 1$; this implies that $\alpha \in \sigma_p(D(r, 0, s, 0, t), c_0^*)$ and hence $S_1 \subseteq \sigma_p(D(r, 0, s, 0, t)^*, {c_0}^*)$.

Conversely, let $\alpha \notin S_1$; therefore $|2(r-\alpha)/(-s+\sqrt{s^2-4t(r-\alpha)})| \geq 1$; that is, $|u_1| \leq 1$. We have to show that $\alpha \notin \sigma_p(D(r, 0, s, 0, t)^*, c_0^*)$; that is, to show $(x_k) \notin \ell_1$ for this it is sufficient to show that $\sum_{n=0}^{\infty} |x_n| = \sum_{n=0}^{\infty} |x_{2n}| + \sum_{n=0}^{\infty} |x_{2n+1}|$ is divergent.

Now consider $\sum_{n=0}^{\infty} |x_{2n}|$.

Here,

$$\begin{aligned}
\frac{x_{2n+6}}{x_{2n+4}} &= \frac{1}{t} \times \frac{a_{2n+3}(r-\alpha)^{n+2}(\alpha - r)\, x_0 + a_{2n+5}(r-\alpha)^{n+3} x_2}{a_{2n+1}(r-\alpha)^{n+1}(\alpha - r)\, x_0 + a_{2n+3}(r-\alpha)^{n+2} x_2} \\
&= \left(\frac{r-\alpha}{t}\right) \times \left(\frac{a_{2n+3}}{a_{2n+1}}\right) \times \left(\frac{-x_0 + (a_{2n+5}/a_{2n+3})\, x_2}{-x_0 + (a_{2n+3}/a_{2n+1})\, x_2}\right).
\end{aligned} \tag{29}$$

Thus we have $((r-\alpha)/t) = 1/u_1 u_2$.

Now we examine three cases.

Case 1 ($|u_2| < |u_1| \leq 1$). In this case $s^2 \neq 4t(r-\alpha)$ and

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{2n+3}}{a_{2n+1}} &= \lim_{n\to\infty} \frac{u_1^{n+2} - u_2^{n+2}}{u_1^{n+1} - u_2^{n+1}} \\
&= \lim_{n\to\infty} \frac{u_1^{n+2}\left(1 - (u_2/u_1)^{n+2}\right)}{u_1^{n+1}\left(1 - (u_2/u_1)^{n+1}\right)} = u_1.
\end{aligned} \tag{30}$$

Now, if $-x_0 + u_1 x_2 = 0$, then we get $x_{2n} = (1/u_1^n) x_0$ which is not in ℓ_1 since $|u_1| \leq 1$. Otherwise

$$\lim_{n\to\infty} \left| \frac{x_{2n+6}}{x_{2n+4}} \right| = \frac{1}{|u_1|\,|u_2|} |u_1| = \frac{1}{|u_2|} > 1. \tag{31}$$

Case 2 ($|u_2| = |u_1| < 1$). In this case $s^2 = 4t(r-\alpha)$ and using the formula

$$\begin{aligned}
a_{2n+1} &= \left(\frac{2\,(n+1)}{-s}\right)\left[\frac{-s}{2\,(r-\alpha)}\right]^{n+1}, \quad \text{for } n \in Z^+, \\
a_{2n} &= 0, \quad \text{for } n \in \mathbb{N}.
\end{aligned} \tag{32}$$

We have

$$\lim_{n\to\infty} \frac{a_{2n+3}}{a_{2n+1}} = \frac{-s}{2\,(r-\alpha)} = u_2 = u_1, \tag{33}$$

and so

$$\lim_{n\to\infty} \left| \frac{x_{2n+6}}{x_{2n+4}} \right| = \frac{1}{|u_1|\,|u_2|} |u_1| = \frac{1}{|u_2|} > 1. \tag{34}$$

Case 3 ($|u_2| = |u_1| = 1$). In this case $s^2 = 4t(r-\alpha)$ and so we have $|(r-\alpha)/t| = 1/(|u_1||u_2|) = 1$ and $|u_2| = |u_1| = |-s/2(r-\alpha)| = 1$ and so $|-s/2t| = 1$. Our aim is to show $\alpha \notin \sigma_p(D(r, 0, s, 0, t)^*, c_0^*)$. On the contrary we assume that $\alpha \in \sigma_p(D(r, 0, s, 0, t)^*, c_0^*)$. This implies that $\theta \neq x \in \ell_1$.

From (26)

$$\begin{aligned}
x_{2n+4} &= \frac{a_{2n+1}(r-\alpha)^{n+1}}{t^{n+1}}(\alpha - r)\,x_0 \\
&\quad + \frac{a_{2n+3}(r-\alpha)^{n+2}}{t^{n+1}}x_2, \quad \text{for } n \in Z^+ \\
&= \left(\frac{2(n+1)}{-s}\right)\left[\frac{-s}{2(r-\alpha)}\right]^{n+1} \times \frac{1}{t^{n+1}} \\
&\quad \times \left(r-\alpha\right)^{n+1} \times \left(\alpha - r\right)x_0 \\
&\quad + \left(\frac{2(n+2)}{-s}\right)\left[\frac{-s}{2(r-\alpha)}\right]^{n+2} \times \frac{1}{t^{n+1}} \times (r-\alpha)^{n+2}x_2 \\
&= \left(\frac{-s}{2t}\right)^{n+1}\left[\frac{2(n+1)}{-s}(\alpha - r)\,x_0 + (n+2)\,x_2\right] \\
&= \left(\frac{-s}{2t}\right)^{n+1}\left[\left(\frac{s}{2t}\right)(n+1)\,x_0 + (n+2)\,x_2\right].
\end{aligned} \tag{35}$$

Now, if we consider the series $\sum_{n=0}^{\infty}|x_{2n+1}|$ in place of $\sum_{n=0}^{\infty}|x_{2n}|$, one will get results parallel to all the results obtained previous just by replacing x_1 in place of x_0 and x_3 in place of x_2.

Since $\lim_{n\to\infty}|x_n| = 0$, we must have from (35) $x_0 = x_2 = 0$ and similarly we get $x_1 = x_3 = 0$. This implies $x = \theta$, a contradiction and so we must have $\alpha \notin \sigma_p(D(r,0,s,0,t)^*, c_0^*)$.

In Cases 1 and 2, by the d'Alembert ratio test we get that $\sum_{n=0}^{\infty}|x_{2n}|$ is divergent and similarly we get that $\sum_{n=0}^{\infty}|x_{2n+1}|$ is also divergent and hence $\sum_{n=0}^{\infty}|x_n| = \sum_{n=0}^{\infty}|x_{2n}| + \sum_{n=0}^{\infty}|x_{2n+1}|$ is divergent, since the sum of two absolutely divergent series is divergent. In Case 3 $\alpha \in \sigma_p(D(r,0,s,0,t)^*, c_0^*)$ leads to contradiction. Thus in all the previous cases $\alpha \notin \sigma_p(D(r,0,s,0,t)^*, c_0^*)$ and hence $\sigma_p(D(r,0,s,0,t)^*, c_0^*) \subseteq S_1$. This completes the proof of theorem. □

Lemma 8. *T has a dense range if and only if T^* is one to one, where T^* denotes the adjoint operator of T.*

Theorem 9. $\sigma_r(D(r,0,s,0,t), c_0) = S_1$, *where S_1 is defined as in Theorem 7.*

Proof. One has $\sigma_p(D(r,0,s,0,t)^*, c_0^*) = S_1$; then $D(r,0,s,0,t)^* - \alpha I$ is not one-to-one for all $\alpha \in S_1$. Therefore by Lemma 8, $D(r,0,s,0,t) - \alpha I$ have a dense range for all $\alpha \in S_1$ and hence $\sigma_r(D(r,0,s,0,t), c_0) = S_1$. □

Theorem 10. *Consider* $\sigma_c(D(r,0,s,0,t), c_0) = S_2$, *where*

$$S_2 = \left\{\alpha \in C : \left|\frac{2(r-\alpha)}{-s + \sqrt{s^2 - 4t(r-\alpha)}}\right| = 1\right\}. \tag{36}$$

Proof. The proof immediately follows from the fact that the set of spectra is the disjoint union of the point spectrum, residual spectrum, and continuous spectrum; that is,

$$\begin{aligned}
\sigma(D(r,0,s,0,t), c_0) = \sigma_p&(D(r,0,s,0,t), c_0) \\
&\cup \sigma_r(D(r,0,s,0,t), c_0) \\
&\cup \sigma_c(D(r,0,s,0,t), c_0).
\end{aligned} \tag{37}$$

□

4. The Fine Spectrum of the More Generalized Operator $D(r,0,s,0,t)$ on the Sequence Space c

Theorem 11. *Consider* $\sigma(D(r,0,s,0,t), c) = S$, *where S is defined as in Theorem 5.*

Proof. This is obtained in a similar way used in the proof of Theorem 5. □

Theorem 12. *One has* $\sigma_p(D(r,0,s,0,t), c) = \emptyset$.

Proof. This is obtained in a similar way that is used in the proof of Theorem 6. □

If $T : c \to c$ is a bounded matrix operator with the matrix A, then $T^* : c^* \to c^*$ acting on $c \oplus \ell_1$ has a matrix representation of the form $\begin{bmatrix} \chi & 0 \\ b & A^t \end{bmatrix}$, where χ is the limit of the sequence of row sums of A minus the sum of the columns of A and b is the column vector whose kth entry is the limit of the kth column of A for each $k \in \mathbb{N}$. For $D(r,0,s,0,t) : c \to c$, the matrix $D(r,0,s,0,t)^* \in B(\ell_1)$ is of the form $D(r,0,s,0,t)^* = \begin{bmatrix} r+s+t & 0 \\ 0 & D(r,0,s,0,t)^t \end{bmatrix}$.

Theorem 13. *Consider* $\sigma_p(D(r,0,s,0,t)^*, c^*) = S_1 \cup \{r+s+t\}$.

Proof. Suppose $D(r,0,s,0,t)^* x = \alpha x$ for $x \neq \theta = (0,0,0,\ldots)$ in $c_0^* \cong \ell_1$; that is, consider the following system of linear equations:

$$\begin{gathered}
(r + s + t)\,x_0 = \alpha x_0, \\
r x_1 + s x_3 + t x_5 = \alpha x_1, \\
r x_2 + s x_4 + t x_6 = \alpha x_2, \\
r x_3 + s x_5 + t x_7 = \alpha x_3, \\
r x_4 + s x_6 + t x_8 = \alpha x_4, \\
\vdots
\end{gathered} \tag{38}$$

Then, we obtain that

$$\begin{aligned}
x_{2n+4} &= \frac{a_{2n+1}(r-\alpha)^{n+1}}{t^{n+1}}(\alpha - r)\,x_1 \\
&\quad + \frac{a_{2n+3}(r-\alpha)^{n+2}}{t^{n+1}}x_3, \quad \text{for } n \in Z^+,
\end{aligned}$$

$$x_{2n+5} = \frac{a_{2n+1}(r-\alpha)^{n+1}}{t^{n+1}}(\alpha - r)\,x_2 + \frac{a_{2n+3}(r-\alpha)^{n+2}}{t^{n+1}}x_4, \quad \text{for } n \in Z^+. \tag{39}$$

If $x_0 \neq 0$, then $(r+s+t) = \alpha$. So, $\alpha = (r+s+t)$ is an eigenvalue with the corresponding eigenvector $x = (x_0, 0, 0, \ldots)$.

If $\alpha \neq (r+s+t)$, then $x_0 = 0$ and so using arguments similar to those in the proof of Theorem 7 one can see by (39) that $x \notin \ell_1$. This completes the proof. □

Theorem 14. *Consider* $\sigma_r(D(r,0,s,0,t),c) = \sigma_p(D(r,0,s,0,t)^*,c^*)$.

Theorem 15. *Consider* $\sigma_c(D(r,0,s,0,t),c) = S_2 \setminus \{r+s+t\}$, *where* S_2 *is defined as in Theorem 10.*

5. Particular Case

The spectrum of the operator Δ_2^2 over the sequence space c_0 and c may be derived as the set $\{\alpha \in C : |1-\sqrt{\alpha}| \leq 1\}$; one can directly produce the same result from the present paper since $\Delta_2^2 = D(1,0,-2,0,1)$.

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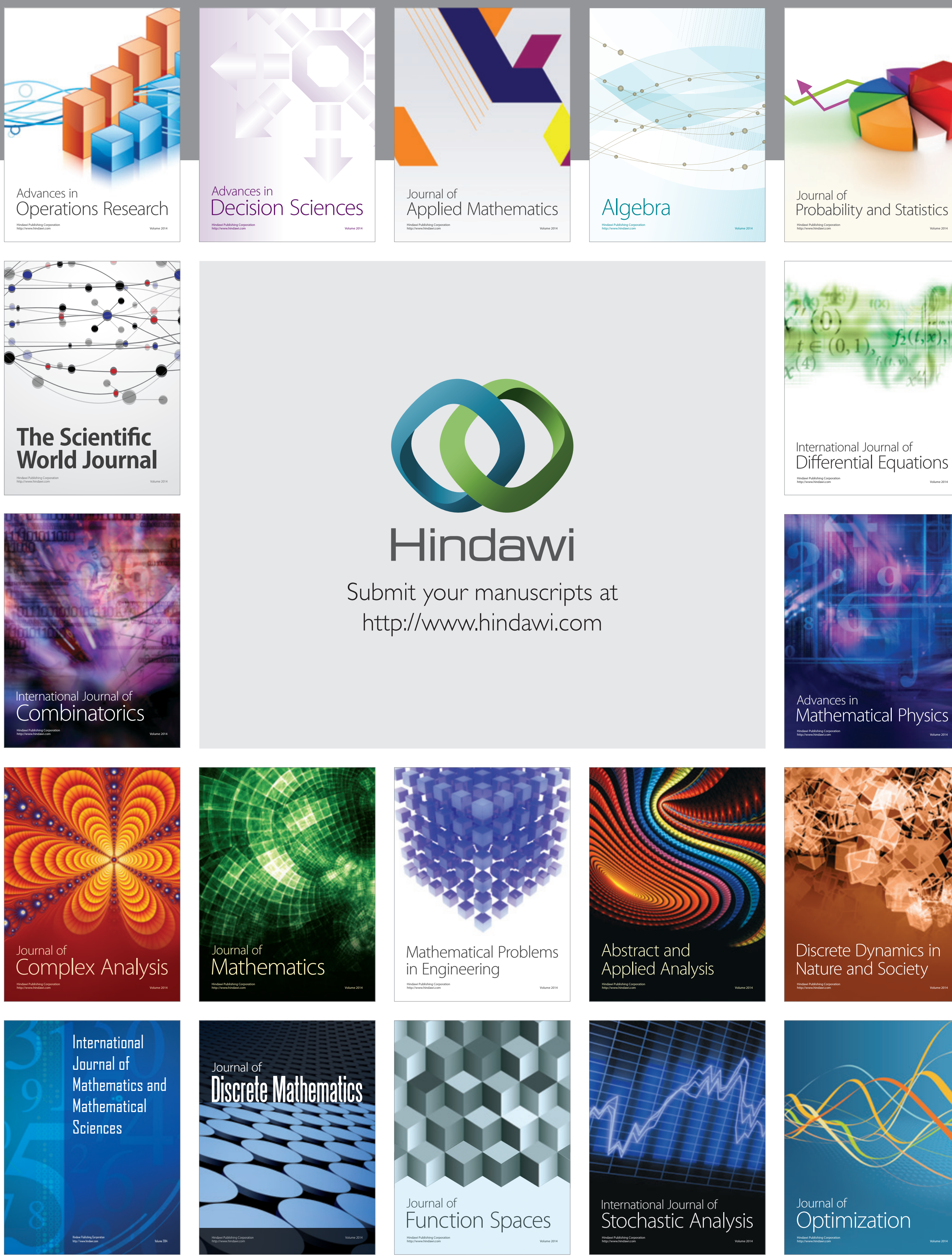
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