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Research Article

Finite p-Group with Small Abelian Subgroups

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A finite p-group G is said to have the property P, if, for any abelian subgroup M of G, there is $|MZ(G)/Z(G)| \le p$. We show that if G satisfies P, then G has the following two types: (1) G is isoclinic to some stem groups of order p^5, which form an isoclinic family. (2) G is isoclinic to a special p-group of exponent p. Elementary structures of groups with P are determined.

1. Preliminaries

Let p be a prime and G be a finite p-group. A group G is said to have the property P, if any abelian subgroup M of G satisfies $|MZ(G)/Z(G)| \le p$. It is well known that if $MZ(G)/Z(G)$ is cyclic, then M is abelian. It rises to consider the groups with property P.

Mann [1] obtained the structure of 2-groups with property P; leting any abelian subgroup M of 2-group G satisfy $|MZ(G)/Z(G)| \le p$, then G is isoclinic to a dihedral group. And for any odd prime p he showed that if any abelian subgroup M of G satisfies $|MZ(G)/Z(G)| \le p$, then $G/Z(G)$ is an elementary abelian p-group or a nonabelian group of order p^3 and exponent p.

Lemma 1 (see [1]). *Let G be a finite p-group which satisfies the property P. Then $G/Z(G)$ is an elementary abelian p-group or a nonabelian group of order p^3 and exponent p.*

This paper further discusses groups with property P. In the following all considered groups are finite p-groups among which p is an odd prime.

First we state some notions and lemmas.

Definition 2. Groups G and H are said to be isoclinic, if there exist isomorphisms $\sigma : G/Z(G) \to H/Z(H)$ and $\theta : G' \to H'$ which are compatible, in the sense that $(aZ(G))^{\sigma} = cZ(H)$ and $(bZ(G))^{\sigma} = dZ(H)$ $(c, d \in H - Z(H))$ imply that $[a,b]^{\theta} = [c,d]$.

Definition 3. A finite p-group G is called a stem group, if $Z(G) \le G'$.

Definition 4. Group G is called a special p-group, if G is an elementary abelian p-group or $G' = Z(G) = \Phi(G)$ is an elementary abelian p-group.

Lemma 5 (see [2]). *Finite p-groups are isoclinic to stem groups.*

Lemma 6 (see [3]). *Let G be a finite regular p-group, $s \ge 0$; then for any $a, b \in G$, $a^{p^s} = b^{p^s}$ if and only if $(a^{-1}b)^{p^s} = 1$.*

Lemma 7 (see [4]). *Let G be a regular p-group; then for any i, $\Omega_i(G)$ is the set of elements of order p^i.*

2. Main Results

We need the following result.

Lemma 8. *Suppose that G and H are isoclinic. If G satisfies P, then so does H.*

Proof. Assume that isomorphisms $\sigma : G/Z(G) \to H/Z(H)$ and $\theta : G' \to H'$ are compatible. And assume that M is an abelian subgroup of H. Then there exists a subgroup N of G such that $(N/Z(G))^{\sigma} = MZ(H)/Z(H)$. Suppose that $c, d \in N \setminus Z(G)$ and $(cZ(G))^{\sigma} = aZ(H)$, $(dZ(G))^{\sigma} = bZ(H)$. Then $a, b \in M \setminus Z(H)$. Then $[c,d]^{\theta} = [a,b] = 1$ since σ and θ are compatible. It follows that $|N/Z(G)| \leqslant p$ since G satisfies P. Hence $|MZ(H)/Z(H)| \leqslant p$. This completes the proof. □

Now we state our main result.

Theorem 9. *Let G be a finite p-group satisfying property P, then*

(1) *G is isoclinic to some stem groups of order p^5, which form an isoclinic family, or*

(2) *G is isoclinic to a special p-group of exponent p.*

Proof. By Lemma 1, $G/Z(G)$ is an elementary abelian p-group or a nonabelian group of order p^3 and exponent p.

(1) Suppose that $G/Z(G)$ is not abelian.

Then $G/Z(G)$ is of order p^3 and exponent p. By Lemma 5 we may suppose that $Z(G) \leqslant G'$. Noting that the nilpotent class of G is 3, thus $Z(G) \neq G'$. It follows that G' is abelian since $|G'/Z(G)| = p$. Assuming that $G/Z(G) = \langle aZ(G), bZ(G)\rangle$, then $G = \langle a, b\rangle$ since $Z(G) < G'$. Let $H = \langle a, a^b\rangle = \langle a, [a, b]\rangle$. Then H is nilpotent of class 2 and thus H is a regular p-group. Note that $a^p \in Z(G)$ since $G/Z(G)$ is of exponent p. So $(b^{-1}ab)^p = b^{-1}a^p b = a^p$, following that $[a, b]^p = (a^{-1}b^{-1}ab)^p = 1$ by Lemma 6. Hence $G' = \langle [a, b]\rangle^G = \langle [a, b], [a, b, a], [a, b, b]\rangle$ is abelian and $|G'| \leqslant p^3$.

If $|G'| = p$, then G is inner-abelian and $G/Z(G) \cong Z_p^2$, a contradiction.

If $|G'| = p^2$, there are two cases.

Case 1 ($[a, b, a] = 1$ or $[a, b, b] = 1$). If $[a, b, a] = 1$, then $\langle G', a\rangle$ is abelian and $\langle G', a\rangle/Z(G) \cong Z_p^2$, a contradiction. For $[a, b, b] = 1$, then $\langle G', b\rangle$ is abelian and $\langle G', b\rangle/Z(G) \cong Z_p^2$, a contradiction.

Case 2 ($[a, b, a] = [a, b, b]$). Then we have $[a, b, a][a, b, b^{-1}] = [a, b, ab^{-1}] = 1$. Thus $H = \langle [a, b], ab^{-1}, Z(G)\rangle$ is abelian and $H/Z(G) \cong Z_p^2$, a contradiction. Hence $|G'| = p^3$ and then $|G| = p^5$.

Now we show that all groups of order p^5 with P are isoclinic.

Suppose that $G = \langle a_1, b_1\rangle$, $H = \langle a_2, b_2\rangle$ are groups of order p^5 with P. Then $G/Z(G) = \langle a_1Z(G), b_1Z(G)\rangle \cong E_{p^3}$ and $H/Z(H) = \langle a_2Z(H), b_2Z(H)\rangle \cong E_{p^3}$.

Set $c_1 = [a_1, b_1]$, $c_2 = [a_2, b_2]$ and define $\sigma : G/Z(G) \to H/Z(H)$ such that $(a_1Z(G))^\sigma = a_2Z(H)$, $(b_1Z(G))^\sigma = b_2Z(H)$.

Then σ spans an isomorphism and $(c_1Z(G))^\sigma = [a_1Z(G), b_1Z(G)]^\sigma = [a_2Z(H), b_2Z(H)] = c_2Z(H)$. Note that $Z(G) = \langle [c_1, a_1]\rangle \times \langle [c_1, b_1]\rangle$, $Z(H) = \langle [c_2, a_2]\rangle \times \langle [c_2, b_2]\rangle$ and $G' = \langle c_1\rangle \times Z(G)$, $H' = \langle c_2\rangle \times Z(H)$.

Setting $c_1^\theta = c_2$, $[c_1, a_1]^\theta = [c_2, a_2]$, and $[c_1, b_1]^\theta = [c_2, b_2]$, then θ deduces an isomorphism from G' onto H'. Now we show that σ and θ are compatible.

For any $x, y \in G$, write $x = a_1^{i_1} b_1^{i_2} c_1^{i_3} z_1$, $y = a_1^{j_1} b_1^{j_2} c_1^{j_3} z_2$, where $z_1, z_2 \in Z(G)$. Then

$$\begin{aligned}(xZ(G))^\sigma &= \left(a_1^{i_1} b_1^{i_2} c_1^{i_3} Z(G)\right)^\sigma \\ &= \left(a_1^{i_1} Z(G)\right)^\sigma \left(b_1^{i_2} Z(G)\right)^\sigma \left(c_1^{i_3} Z(G)\right)^\sigma \\ &= a_2^{i_1} Z(H)\, b_2^{i_2} Z(H)\, c_2^{i_3} Z(H) = a_2^{i_1} b_2^{i_2} c_2^{i_3} Z(H).\end{aligned} \tag{1}$$

Similarly, $(yZ(G))^\sigma = a_2^{j_1} b_2^{j_2} c_2^{j_3} Z(H)$. Note that

$$\begin{aligned}[x, y] &= \left[a_1^{i_1} b_1^{i_2} c_1^{i_3} z_1, a_1^{j_1} b_1^{j_2} c_1^{j_3} z_2\right] \\ &= \left[a_1^{i_1} b_1^{i_2} c_1^{i_3}, a_1^{j_1} b_1^{j_2} c_1^{j_3}\right] \\ &= \left[a_1^{i_1} b_1^{i_2}, a_1^{j_1} b_1^{j_2} c_1^{j_3}\right] \left[c_1^{i_3}, a_1^{j_1} b_1^{j_2} c_1^{j_3}\right] \\ &\quad \times \left[a_1^{i_1} b_1^{i_2}, a_1^{j_1} b_1^{j_2} c_1^{j_3}, c_1^{i_3}\right] \\ &= \left[a_1^{i_1} b_1^{i_2}, a_1^{j_1} b_1^{j_2}\right] \left[a_1^{i_1}, c_1^{j_3}\right] \left[b_1^{i_2}, c_1^{j_3}\right] \left[c_1^{i_3}, a_1^{j_1}\right] \\ &\quad \times \left[c_1^{i_3}, b_1^{j_2}\right] \\ &= [c_1, a_1]^{i_3 j_1 - i_1 j_3} [c_1, b_1]^{i_3 j_2 - i_2 j_3} \left[a_1^{i_1} b_1^{i_2}, a_1^{j_1} b_1^{j_2}\right] \\ &= [c_1, a_1]^{i_3 j_1 - i_1 j_3} [c_1, b_1]^{i_3 j_2 - i_2 j_3} \\ &\quad \times \left[a_1^{i_1} b_1^{i_2}, a_1^{j_1}\right] \left[a_1^{i_1} b_1^{i_2}, b_1^{j_2}\right] \left[a_1^{i_1} b_1^{i_2}, a_1^{j_1}, b_1^{j_2}\right] \\ &= [c_1, a_1]^{i_3 j_1 - i_1 j_3} [c_1, b_1]^{i_3 j_2 - i_2 j_3} [c_1, a_1]^{\binom{i_1}{2}\binom{i_2}{1}} \\ &\quad \times [c_1, b_1]^{\binom{i_1}{1}\binom{i_2}{2}} \\ &\quad \times [c_1, a_1]^{\binom{i_2}{1}\binom{j_1}{2}} [c_1, b_1]^{\binom{j_1}{1}\binom{i_2}{2}} \\ &\quad \times [c_1, b_1]^{i_1 i_2 j_2 - i_2 j_1 j_2} c_1^{i_1 j_2 - i_2 j_1} \\ &= [c_1, a_1]^{i_3 j_1 - i_1 j_3 + \binom{i_2}{1}\binom{j_1}{2} + \binom{i_1}{2}\binom{j_2}{1}} \\ &\quad \times [c_1, b_1]^{i_3 j_2 - i_2 j_3 + i_1 i_2 j_2 - i_2 j_1 j_2 + \binom{j_1}{1}\binom{i_2}{2} + \binom{i_1}{1}\binom{j_2}{2}} c_1^{i_1 j_2 - i_2 j_1}.\end{aligned} \tag{2}$$

Set

$$\begin{gathered} f_1 = i_3 j_1 - i_1 j_3 + \binom{i_2}{1}\binom{j_1}{2} + \binom{i_1}{2}\binom{j_2}{1}, \\ f_2 = i_3 j_2 - i_2 j_3 + i_1 i_2 j_2 - i_2 j_1 j_2 + \binom{j_1}{1}\binom{i_2}{2} + \binom{i_1}{1}\binom{j_2}{2}, \\ f_3 = i_1 j_2 - i_2 j_1. \end{gathered} \tag{3}$$

Then $[x, y] = c_1^{f_3} [c_1, a_1]^{f_1} [c_1, b_1]^{f_2}$. Thus $[x, y]^\theta = c_2^{f_3} [c_2, a_2]^{f_1} [c_2, b_2]^{f_2} = [a_2^{i_1} b_2^{i_2} c_2^{i_3}, a_2^{j_1} b_2^{j_2} c_2^{j_3}]$. Hence σ and θ are compatible.

(2) Suppose that $G/Z(G)$ is an elementary abelian p-group.

Note that $G' \leq Z(G)$ since $G/Z(G)$ is abelian. By Lemma 5 we may suppose that $Z(G) = G'$. If $\exp G = p$, then G is a special p-group of exponent p. So suppose that $\exp G > p$.

Let $G/Z(G) = \langle \overline{a}_1 \rangle \times \langle \overline{a}_2 \rangle \times \cdots \times \langle \overline{a}_n \rangle$, where $\overline{a}_i = a_i Z(G)$. Then $G = \langle a_1, a_2, \ldots, a_n \rangle$ and $a_i^p \in Z(G)$. Note that $G' = \langle [a_k, a_l] \rangle^G$ and $[a_k, a_l]^p = [a_k^p, a_l] = 1$. Hence G' is an elementary abelian p-group of exponent p. Suppose that $G' = \langle c_{kl} \mid f_m(c_{kl}) = 1 \rangle$, where $c_{kl} = [a_k, a_l]$ and $f_m(c_{kl}) = 1$ indicate the laws of G'. Suppose that $K = \langle d_{kl} \mid f_m(d_{kl}) = 1 \rangle$ and $K \cong G'$. By Schreier group expansion theories, we can add elements $b_1, b_2, \ldots, b_n$ of order p such that $[b_k, b_l] = d_{kl}$ into group K, note that $H = \{b_1, b_2, \ldots, b_n \mid b_i^p = 1, [b_k, b_l] = d_{kl}\}$. Then H is a group by Schreier group expansion theories.

Define a map $\sigma : c_{kl} \mapsto d_{kl}$ for $k, l = 1, \ldots, n$. Then σ deduces an isomorphism from G' onto H'. Note that $H/H' = \langle \overline{b}_1 \rangle \times \langle \overline{b}_2 \rangle \times \cdots \times \langle \overline{b}_n \rangle$ and $G/G' = \langle \overline{a}_1 \rangle \times \langle \overline{a}_2 \rangle \times \cdots \times \langle \overline{a}_n \rangle$.

Setting $\sigma_1 : a_i G' \mapsto b_i H'$, then σ_1 spans an isomorphism from G/G' onto H/H'.

For any $xG', yG' \in G/G'$, assume that $x = a_1^{i_1} a_2^{i_2} \cdots a_n^{i_n} c_1$, $y = a_1^{j_1} a_2^{j_2} \cdots a_n^{j_n} c_2$, where $c_1, c_2 \in G'$.

Then $(xG')^{\sigma_1} = (a_1^{i_1} a_2^{i_2} \cdots a_n^{i_n} G')^{\sigma_1} = b_1^{i_1} b_2^{i_2} \cdots b_n^{i_n} H' (yG')^{\sigma_1} = (a_1^{j_1} a_2^{j_2} \cdots a_n^{j_n} G')^{\sigma_1} = b_1^{j_1} b_2^{j_2} \cdots b_n^{j_n} H'$. So $[x, y]^{\sigma_2} = [a_1^{i_1} a_2^{i_2} \cdots a_n^{i_n} c_1, a_1^{j_1} a_2^{j_2} \cdots a_n^{j_n} c_2]^{\sigma_2} = (\prod_{1 \leqslant k \leqslant l \leqslant n} [a_k, a_l]^{i_k j_l - i_l j_k})^{\sigma_2} = \prod_{1 \leqslant k \leqslant l \leqslant n} [b_k, b_l]^{i_k j_l - i_l j_k} = [b_1^{i_1} b_2^{i_2} \cdots b_n^{i_n}, b_1^{j_1} b_2^{j_2} \cdots b_n^{j_n}]$.

Thus σ_1 and σ_2 are compatible.

For any $x \in Z(H)$, write $x = b_1^{i_1} b_2^{i_2} \cdots b_n^{i_n} c$, where $c \in H'$. Then $[b_1^{i_1} b_2^{i_2} \cdots b_n^{i_n} c, b_i] = 1$.

Note that $(b_1^{i_1} b_2^{i_2} \cdots b_n^{i_n} H')^{\sigma_1^{-1}} = (\overline{b}_1^{i_1} \overline{b}_2^{i_2} \cdots \overline{b}_n^{i_n})^{\sigma_1^{-1}} = \overline{a}_1^{i_1} \overline{a}_2^{i_2} \cdots \overline{a}_n^{i_n} = a_1^{i_1} a_2^{i_2} \cdots a_n^{i_n} G'$, and similarly $(b_i H')^{\sigma_1^{-1}} = a_i G'$. Then $[b_1^{i_1} b_2^{i_2} \cdots b_n^{i_n}, b_i]^{\sigma_1^{-1}} = [a_1^{i_1} a_2^{i_2} \cdots a_n^{i_n}, a_i] = 1$. Hence $a_1^{i_1} a_2^{i_2} \cdots a_n^{i_n} \in Z(G) = G'$.

It follows that $b_1^{i_1} b_2^{i_2} \cdots b_n^{i_n} H' = (G')^{\sigma_1} = H'$ and thus $b_1^{i_1} b_2^{i_2} \cdots b_n^{i_n} \in H'$. So $Z(H) \leqslant H'$. As a result $H' = Z(H)$. Since $H = \{b_1, b_2, \ldots, b_n \mid b_i^p = 1, [b_k, b_l] = d_{kl}\}$. We know that $\mho_1(H) = \langle b_1^p, b_2^p, \ldots, b_n^p \rangle = 1$; then $\exp H = p$. Hence G is isoclinic to a special p-group of exponent p. □

Note. In the sense of isoclinism, to investigate groups with P is to consider the special p-groups of exponent p.

We need the following result.

Lemma 10. *Let G be a special p-group of exponent p. Then*

(1) *if $d(G) = n$, then $|G'| \leq p^{n(n-1)/2}$;*

(2) *for any $x, y \in G$, if $\langle xZ(G) \rangle = \langle yZ(G) \rangle$, then G satisfies P.*

Then we deduce the following.

Theorem 11. *Let G be a special p-group of exponent p and $d(G) = n$, then*

(1) *if $|G'| < p^{n-1}$, then G does not satisfy P.*

(2) *if $|G'| = p^{n(n-1)/2}$, then G satisfies P.*

Proof. Suppose that $G = \langle a_1, a_2, \ldots, a_n \rangle$. Since G' is an elementary abelian p-group, we may see G' as an additive group of the vector space on $GF(p)$.

(1) Assume that $|G'| < p^{n-1}$. Then $[a_1, a_2], [a_1, a_3], \ldots, [a_1, a_n]$ are linearly dependent. So there exist some integers which are not all $0 \bmod p$ such that $[a_1, a_2]^{k_1} [a_1, a_3]^{k_2} \cdots [a_1, a_n]^{k_{n-1}} = 1$. Thus $\langle a_1, a_2^{k_1} a_3^{k_2} \cdots a_n^{k_{n-1}} \rangle$ is abelian. However, note that $|\langle a_1, a_2^{k_1} a_3^{k_2} \cdots a_n^{k_{n-1}} \rangle Z(G)/Z(G)| \neq p$ since $a_1, a_2, \ldots, a_n$ are generators of G. So G does not have P.

(2) Assume that $|G'| = p^{n(n-1)/2}$. Then $G' = \langle [a_1, a_2], \ldots, [a_1, a_n], \ldots, [a_{n-1}, a_n] \rangle$. For any $x, y \in G$, assume that $x = a_1^{i_1} a_2^{i_2} \cdots a_n^{i_n} c_1$ and $y = a_1^{j_1} a_2^{j_2} \cdots a_n^{j_n} c_2$, where $c_1, c_2 \in G'$. Then $[x, y] = [a_1^{i_1} a_2^{i_2} \cdots a_n^{i_n} c_1, a_1^{j_1} a_2^{j_2} \cdots a_n^{j_n} c_2] = [a_1^{i_1} a_2^{i_2} \cdots a_n^{i_n}, a_1^{j_1} a_2^{j_2} \cdots a_n^{j_n}] = \prod_{1 \leqslant k \leqslant l \leqslant n} [a_k, a_l]^{i_k j_l - i_l j_k}$.

Looking at G' as an additive group of the vector space on $GF(p)$, then $[a_1, a_2], \ldots, [a_1, a_n], \ldots, [a_{n-1}, a_n]$ are linearly independent. If $[x, y] = 1$, then $i_k j_l - i_l j_k = 0 \bmod p$ for $1 \leqslant k \leqslant l \leqslant n$. Thus $i_l / j_l = i_2 / j_2 \cdots = i_n / j_n$. Setting $i_l / j_l = m$, then $x = a_1^{j_1 m} a_2^{j_2 m} \cdots a_n^{j_n m} c_1 = (a_1^{j_1} a_2^{j_2})^m c_2^{\binom{m}{2}} a_3^{j_3 m} \cdots a_n^{j_n m} c_1 = (a_1^{j_1} a_2^{j_2}, \ldots, a_n^{j_n} c_2)^m c = y^m c$, where $c \in G'$. Hence $\langle xZ(G) \rangle = \langle y^m Z(G) \rangle \leqslant \langle yZ(G) \rangle$. By Lemma 10 G satisfies the property P. □

Corollary 12. *Let G be a special p-group of exponent p. Then*

(1) *if $d(G) = 2$, then G satisfies the property P.*

(2) *if $d(G) = 3$, then G satisfies the property P if and only if $|G'| = p^3$.*

Proof. (1) If $d(G) = 2$, then G is an inner-abelian p-group. Obviously, G has the property P.

(2) Assume that $d(G) = 3$ and $G = \langle a_1, a_2, a_3 \rangle$. By Theorem 11 G satisfies P if $|G'| = p^3$, but it does not if $|G'| = p$. Assuming that $|G'| = p^2$, then $[a_1, a_2]$, $[a_1, a_3]$, $[a_2, a_3]$ are linearly dependent. So there exist integers k_1, k_2, k_3 which are not all $0 \bmod p$ such that $[a_1, a_2]^{k_1} [a_1, a_3]^{k_2} [a_2, a_3]^{k_3} = 1$. Hence $\langle a_1, a_2^{k_1} a_3^{k_2} a_2^{i k_3} a_3^{j k_3} \rangle$ is abelian.

Since a_1, a_2, and a_3 are generators of G, $|\langle a_1, a_2^{k_1} a_3^{k_2} a_2^{i k_3} a_3^{j k_3} \rangle Z(G)/Z(G)| \neq p$, then G does not satisfy P; □

Theorem 13. *Assume that $G = \langle a_1, \ldots, a_n \rangle$ $(n \geq 4)$ is a special p-group of exponent p and $|G'| = p^m$, where $n - 1 \leq m < n(n-1)/2$. Suppose that $G' = \langle [a_{k_1}, a_{l_1}], \ldots, [a_{k_{n-1}}, a_{l_{n-1}}], \ldots, [a_{k_m}, a_{l_m}] \rangle$ for $k_i, l_i \in \{1, 2, \ldots, n\}$. If G satisfies the following properties, then G has P.*

(1) *$[a_k, a_l] \neq 1$ for any $k \neq l$.*

(2) *For any distinct $l \neq m \in \{1, \ldots, n\}$ and for not all $0 \bmod p$ integers i, j, $[a_k, a_l]^i \neq [a_k, a_m]^j$.*

(3) *For $[a_{k_1}, a_{l_1}], [a_{k_2}, a_{l_2}], \ldots, [a_{k_{n-1}}, a_{l_{n-1}}]$, which satisfy $\{k_1, l_1, k_2, l_2, \ldots, k_{n-1}, l_{n-1}\} = \{1, 2, \ldots, n\}$ and $k_{i+1} \in \{k_i, l_i\}$ or $l_{i+1} \in \{k_i, l_i\}$, where $i = 1, 2, \ldots, n-2$.*

(4) *Any $[a_{k_i}, a_{l_i}]$ of $[a_{k_{m+1}}, a_{l_{m+1}}], \ldots, [a_{k_{n(n-1)/2}}, a_{l_{n(n-1)/2}}]$ can be expressed as $[a_{k_i}, a_{l_i}] = [a_{k_n}, a_{l_n}]^{t_{1i}} [a_{k_{n+1}},$*

$a_{l_{n+1}}]^{t_{2i}} \cdots [a_{k_m}, a_{l_m}]^{t_{(m-n+1)i}}$, *where* $t_{1i}, t_{2i}, \ldots, t_{(m-n+1)i}$ *are integers.*

Proof. For any $x, y \in G$, write $x = a_1^{i_1} a_2^{i_2} \cdots a_n^{i_n} c_1$ and $y = a_1^{j_1} a_2^{j_2} \cdots a_n^{j_n} c_2$, where $c_1, c_2 \in G'$. Then

$$\begin{aligned}
[x, y] &= \left[a_1^{i_1} a_2^{i_2} \cdots a_n^{i_n} c_1, a_1^{j_1} a_2^{j_2} \cdots a_n^{j_n} c_2\right] \\
&= \left[a_{k_1}, a_{l_1}\right]^{i_{k_1} j_{l_1} - i_{l_1} j_{k_1}} \left[a_{k_2}, a_{l_2}\right]^{i_{k_2} j_{l_2} - i_{l_2} j_{k_2}} \cdots \left[a_{k_{n-1}}, a_{l_{n-1}}\right]^{f_1} \\
&\quad \times \left[a_{k_n}, a_{l_n}\right]^{f_2} \cdots \left[a_{k_m}, a_{l_m}\right]^{f_3},
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
f_1 &= i_{k_{n-1}} j_{l_{n-1}} - i_{l_{n-1}} j_{k_{n-1}}, \\
f_2 &= i_{k_n} j_{l_n} - i_{l_n} j_{k_n} + \sum_{s=m+1}^{n(n-1)/2} t_{1s} \left(i_{k_s} j_{l_s} - i_{l_s} j_{k_s}\right), \\
f_3 &= i_{k_m} j_{l_m} - i_{l_m} j_{k_m} + \sum_{s=m+1}^{n(n-1)/2} t_{(n-m+1)s} \left(i_{k_s} j_{l_s} - i_{l_s} j_{k_s}\right).
\end{aligned} \tag{5}$$

Note that $[a_{k_1}, a_{l_1}], [a_{k_2}, a_{l_2}], \ldots, [a_{k_{n-1}}, a_{l_{n-1}}], [a_{k_n}, a_{l_n}], \ldots, [a_{k_m}, a_{l_m}]$ are linearly independent.

Hence if $[x, y] = 1$ then $i_{k_r} j_{l_r} - i_{l_r} j_{k_r} = i_{k_r} j_{l_r} - i_{l_r} j_{k_r} + \sum_{s=m+1}^{n(n-1)/2} t_{rs}(i_{k_s} j_{l_s} - i_{l_s} j_{k_s}) = 0 \bmod p$, where $r = 1, 2, \ldots, n, \ldots, m$. It follows that $i_1/j_1 = i_2/j_2 \cdots = i_n/j_n$ by (1) and (2). Supposing that $i_l/j_l = m$, then $x = y^m c$ and thus $\langle xZ(G)\rangle = \langle y^m Z(G)\rangle \leqslant \langle yZ(G)\rangle$. Hence $|\langle x, y, Z(G)\rangle / Z(G)| = |\langle y, Z(G)\rangle / Z(G)| = p$, following that G has P. □

Corollary 14. *Let G be a special p-group of exponent p and $d(G) = n$ $(n \geq 4)$. If $|G'| = p^{n-1}$ or p^n, then G does not satisfy the property P.*

Proof. By Theorem 13 if $|G'| = p^{n-1}$ or p^n, then G does not satisfy the conditions of Theorem 9. So G does not have P. □

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