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Research Article

A Note on Common Fixed Point Results in Uniformly Convex Hyperbolic Spaces

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It is shown that the notion of mappings satisfying condition (K) introduced by Akkasriworn et al. (2012) is weaker than the notion of asymptotically quasi-nonexpansive mappings in the sense of Qihou (2001) and is weaker than the notion of pointwise asymptotically nonexpansive mappings in the sense of Kirk and Xu (2008). We also obtain a common fixed point for a commuting pair of a mapping satisfying condition (K) and a multivalued mapping satisfying condition (C_λ) for some $\lambda \in (0,1)$. Our results properly contain the results of Abkar and Eslamian (2012), Akkasriworn et al. (2012), and many others.

1. Introduction

Let (X, d) be a metric space. A mapping $t : X \to X$ is said to be *nonexpansive* if

$$d(t(x), t(y)) \le d(x, y) \quad \forall x, y \in X. \tag{1}$$

A point $x \in X$ is called a *fixed point* of t if $x = t(x)$. We shall denote by $\mathrm{Fix}(t)$ the set of fixed points of t. The mapping t is said to be *quasi-nonexpansive* if $\mathrm{Fix}(t) \neq \emptyset$ and

$$d(t(x), y) \le d(x, y) \quad \forall x \in X, \qquad y \in \mathrm{Fix}(t). \tag{2}$$

A single-valued mapping $t : X \to X$ and a multivalued mapping $T : X \to 2^X$ are said to be *commute* if

$$t(T(x)) \subseteq T(t(x)) \quad \forall x \in X. \tag{3}$$

The first result concerning to the existence of common fixed points for a commuting pair of a single-valued quasi-nonexpansive mapping and a multivalued nonexpansive mapping was established in Hilbert spaces by Itoh and Takahashi [1]. Since then the common fixed point theory for commuting pairs of single-valued and multivalued mappings has been rapidly developed and many of papers have appeared (see, e.g., [2–11] and the references therein).

In 2008, Suzuki [12] introduced a condition on mappings, which is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness and called it condition (C). Later on, García-Falset et al. [13] introduced two generalizations of condition (C), namely, conditions (E) and (C_λ), and studied the existence of fixed points for mappings satisfying such conditions. These conditions were extended to the multivalued case by Abkar and Eslamian [11] and Espínola et al. [14]. However, these conditions still lie between nonexpansiveness and quasi-nonexpansiveness in both single-valued and multivalued cases. On the other hand, Qihou [15] introduced the notion of asymptotically quasi-nonexpansive mappings and Kirk and Xu [16] introduced the notion of pointwise asymptotically nonexpansive mappings. Both of them generalize the notion of asymptotically nonexpansive mappings in the sense of Goebel and Kirk [17].

Recently, Abkar and Eslamian [18] studied the existence of common fixed points for three different classes of generalized nonexpansive mappings including a quasi-nonexpansive single-valued mapping, a pointwise asymptotically nonexpansive single-valued mapping, and a multivalued mapping satisfying conditions (E) and (C_λ) for some $\lambda \in (0,1)$. Very recently, Akkasriworn et al. [19] introduced a condition on mappings, namely, condition (K), which is weaker than both

quasi-nonexpansiveness and asymptotically nonexpansiveness and proved the existence of common fixed points for a commuting pair of a single-valued mapping satisfying condition (K) and a multivalued mapping satisfying conditions (E) and (C_λ) for some $\lambda \in (0,1)$.

In this note, motivated by the above results, we prove that the condition (K) is even weaker than asymptotically quasi-nonexpansiveness and is weaker than pointwise asymptotically nonexpansiveness in the setting of uniformly convex hyperbolic spaces. Moreover, we also obtain a common fixed point theorem with some weaker assumptions.

2. Preliminaries

Definition 1 (see [20]). A *hyperbolic space* is a triple (X,d,W) where (X,d) is a metric space and $W : X \times X \times [0,1] \to X$ is such that for all $x,y,z,w \in X$ and $\alpha,\beta \in [0,1]$, we have

(W1) $d(z,W(x,y,\alpha)) \le (1-\alpha)d(z,x) + \alpha d(z,y)$;

(W2) $d(W(x,y,\alpha),W(x,y,\beta)) = |\alpha-\beta|d(x,y)$;

(W3) $W(x,y,\alpha) = W(y,x,1-\alpha)$;

(W4) $d(W(x,z,\alpha),W(y,w,\alpha)) \le (1-\alpha)d(x,y) + \alpha d(z,w)$.

If $x,y \in X$, and $\alpha \in [0,1]$, then we use the notation $(1-\alpha)x \oplus \alpha y$ for $W(x,y,\alpha)$. It is easy to see that for any $x,y \in X$, and $\alpha \in [0,1]$, we have

$$\begin{aligned} d\left(x,(1-\alpha)x \oplus \alpha y\right) &= \alpha d\left(x,y\right), \\ d\left(y,(1-\alpha)x \oplus \alpha y\right) &= (1-\alpha)\, d\left(x,y\right). \end{aligned} \tag{4}$$

We shall denote by $[x,y]$ the set $\{(1-\alpha)x \oplus \alpha y : \alpha \in [0,1]\}$. A nonempty subset C of X is said to be *convex* if $[x,y] \subseteq C$ for all $x,y \in C$.

Definition 2 (see [20]). The hyperbolic space (X,d,W) is called *uniformly convex* if for any $r > 0$, and $\varepsilon \in (0,2]$ there exists a $\delta \in (0,1]$ such that for all $a,x,y \in X$ with $d(x,a) \le r$, $d(y,a) \le r$, and $d(x,y) \ge r\varepsilon$, it is the case that

$$d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) \le (1-\delta)\, r. \tag{5}$$

A function $\eta : (0,\infty) \times (0,2] \to (0,1]$ providing such a $\delta := \eta(r,\varepsilon)$ for given $r > 0$ and $\varepsilon \in (0,2]$ is called *a modulus of uniform convexity.*

Obviously, uniformly convex Banach spaces are uniformly convex hyperbolic spaces. CAT(0) spaces are also uniformly convex hyperbolic spaces, see [20, Proposition 8]. From now on, X stands for a complete uniformly convex hyperbolic space having a modulus of uniform convexity η such that for a fixed $\varepsilon \in (0,2]$, $\eta(\cdot,\varepsilon)$ is a constant function on $(0,\infty)$.

The following lemma can be found in [21].

Lemma 3. *Let C be a nonempty closed convex subset of X and $x \in X$. Then there exists a unique point $x_0 \in C$ such that*

$$d\left(x,x_0\right) = d\left(x,C\right) := \inf\left\{d\left(x,y\right) : y \in C\right\}. \tag{6}$$

The following lemma, which is proved by Khamsi and Khan [22], is also needed.

Lemma 4. *Fix $a \in X$. For each $r > 0$ and for each $\varepsilon \in (0,2]$, set*

$$\Psi\left(r,\varepsilon\right) = \inf\left\{\frac{1}{2}d^2\left(a,x\right) + \frac{1}{2}d^2\left(a,y\right) - d^2\left(a,\frac{1}{2}x \oplus \frac{1}{2}y\right)\right\}, \tag{7}$$

where the infimum is taken over all $x,y \in X$ such that $d(x,a) \le r$, $d(y,a) \le r$ and $d(x,y) \ge r\varepsilon$. Then $\Psi(r,\varepsilon) > 0$ for any $r > 0$ and $\varepsilon \in (0,2]$. Moreover, for each fixed $r > 0$, we have

(i) $\Psi(r,0) = 0$;

(ii) $\Psi(r,\varepsilon)$ *is a nondecreasing function of* ε;

(iii) *If* $\lim_{n\to\infty}\Psi(r,t_n) = 0$, *then* $\lim_{n\to\infty}t_n = 0$.

We shall denote by 2^C the family of nonempty subsets of C, by $CB(C)$ the family of nonempty closed and bounded subsets of C, by $K(C)$ the family of nonempty compact subsets of C, and by $KC(C)$ the family of nonempty compact convex subsets of C. Let H be the Hausdorff distance on $CB(C)$, that is,

$$\begin{aligned} &H\left(A,B\right) \\ &\quad := \max\left\{\sup_{a\in A} d\left(a,B\right), \sup_{b\in B} d\left(b,A\right)\right\}, \quad A,B \in CB\left(C\right). \end{aligned} \tag{8}$$

Definition 5. A multivalued mapping $T : C \to 2^C$ is said to satisfy

(i) *condition* (E) if there exists $\mu \ge 1$ such that for each $x,y \in C$,

$$d\left(x,T\left(y\right)\right) \le \mu\, d\left(x,T\left(x\right)\right) + d\left(x,y\right); \tag{9}$$

(ii) *condition* (C_λ) if there exists $\lambda \in (0,1)$ such that for each $x,y \in C$,

$$\begin{aligned} &\lambda\, d\left(x,T\left(x\right)\right) \le d\left(x,y\right) \text{ implies} \\ &H\left(T\left(x\right),T\left(y\right)\right) \le d\left(x,y\right). \end{aligned} \tag{10}$$

We say that $I-T$ is *strongly demiclosed* if for every sequence $\{x_n\}$ in C which converges to $x \in C$ and such that $\lim_{n\to\infty}d(x_n,T(x_n)) = 0$, we have $x \in T(x)$.

We note that for every continuous mapping $T : C \to 2^C$, $I-T$ is strongly demiclosed but the converse is not true (see [13, Example 5]). Notice also that if T satisfies condition (E), then $I-T$ is strongly demiclosed (see [23, Proposition 2.10]).

Definition 6. A single-valued mapping $t : C \to C$ is said to

(i) satisfy *condition* (K) if $\mathrm{Fix}(t)$ is nonempty closed and convex, and for each $x \in \mathrm{Fix}(t)$ and any closed convex subset A with $t(A) \subseteq A$, the nearest point of x in A must be contained in $\mathrm{Fix}(t)$;

(ii) be *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ of positive numbers with $\lim_{n\to\infty} k_n = 1$ and such that

$$d(t^n(x), t^n(y)) \le k_n d(x, y) \quad \forall x, y \in C,\ n \in \mathbb{N}; \tag{11}$$

(iii) be *pointwise asymptotically nonexpansive* if there exists a sequence of mappings $\alpha_n : C \to [0, \infty)$ with $\lim_{n\to\infty} \alpha_n(x) = 1$ and such that

$$d(t^n(x), t^n(y)) \le \alpha_n(x) d(x, y) \quad \forall x, y \in C,\ n \in \mathbb{N}; \tag{12}$$

(iv) be *asymptotically quasi-nonexpansive* if $\mathrm{Fix}(t)$ is nonempty and there exists a sequence $\{k_n\}$ of positive numbers with $\lim_{n\to\infty} k_n = 1$ and such that

$$d(t^n(x), p) \le k_n d(x, p) \quad \forall x \in C,\ p \in \mathrm{Fix}(t),\ n \in \mathbb{N}. \tag{13}$$

3. Main Results

We begin this section by proving that every quasi-nonexpansive mapping satisfies condition (K).

Proposition 7. *Let C be a nonempty convex subset of X. If $t : C \to C$ is a quasi-nonexpansive mapping, then t satisfies condition (K).*

Proof. By [24, Theorem 4.2], $\mathrm{Fix}(t)$ is closed and convex. Let $x \in \mathrm{Fix}(t)$ and A be a closed convex subset of C with $t(A) \subseteq A$. Let $y \in A$ be such that $d(x, y) = d(x, A)$. Since t is quasi-nonexpansive, $d(x, t(y)) \le d(x, y)$. By the uniqueness of y, we have $t(y) = y$. Therefore t satisfies condition (K). □

The following two propositions show that the notion of mappings satisfying condition (K) is weaker than the notion of pointwise asymptotically nonexpansive mappings and weaker than the notion of asymptotically quasi-nonexpansive continuous mappings. For a mapping that satisfies condition (K) but is neither pointwise asymptotically nonexpansive nor asymptotically quasi-nonexpansive, see [19].

Proposition 8. *Let C be a nonempty bounded closed convex subset of X. If $t : C \to C$ is a pointwise asymptotically nonexpansive mapping, then t satisfies condition (K).*

Proof. By [9, Theorem 3.11], $\mathrm{Fix}(t)$ is nonempty closed and convex. Since C is bounded, there exists $r > 0$ such that $d(a, b) \le r$ for all $a, b \in C$. We now let $x \in \mathrm{Fix}(t)$ and A be a closed convex subset of C with $t(A) \subseteq A$. Let $y \in A$ be such that $d(x, y) = d(x, A)$. Since X is uniformly convex, then by Lemma 4 for each integers $l, m \ge 1$, we have

$$\begin{aligned} d^2\left(x, \frac{1}{2}t^l(y) \oplus \frac{1}{2}t^m(y)\right) &\le \frac{1}{2}d^2\left(x, t^l(y)\right) + \frac{1}{2}d^2\left(x, t^m(y)\right) \\ &\quad - \Psi\left(r, \frac{1}{r}d\left(t^l(y), t^m(y)\right)\right) \\ &\le \frac{1}{2}\alpha_l^2(x) d^2(x, y) + \frac{1}{2}\alpha_m^2(x) d^2(x, y) \\ &\quad - \Psi\left(r, \frac{1}{r}d\left(t^l(y), t^m(y)\right)\right). \end{aligned} \tag{14}$$

Since $d(x, y) = d(x, A)$ and A is convex, we have

$$d^2(x, y) \le d^2\left(x, \frac{1}{2}t^l(y) \oplus \frac{1}{2}t^m(y)\right). \tag{15}$$

This, together with (14), we get

$$\begin{aligned} &\Psi\left(r, \frac{1}{r}d\left(t^l(y), t^m(y)\right)\right) \\ &\quad \le \left[\frac{1}{2}\alpha_l^2(x) + \frac{1}{2}\alpha_m^2(x) - 1\right] d^2(x, y). \end{aligned} \tag{16}$$

Consequently, $\lim_{l,m\to\infty}\Psi(r, (1/r)d(t^l(y), t^m(y))) = 0$. By Lemma 4, $\lim_{l,m\to\infty} d(t^l(y), t^m(y)) = 0$. Hence $\{t^n(y)\}$ is a Cauchy sequence. Let $\lim_{n\to\infty} t^n(y) = z \in A$. Now, letting $l, m \to \infty$ in (14), we get that

$$d(x, z) \le d(x, y) = d(x, A). \tag{17}$$

Since t is continuous,

$$d(x, t(z)) = \lim_{n\to\infty} d\left(x, t^{n+1}(y)\right) = d(x, z). \tag{18}$$

By (17), (18), and the uniqueness of y, we get $y = z = t(z) \in \mathrm{Fix}(t)$. □

Proposition 9. *Let C be a nonempty bounded closed convex subset of X. If $t : C \to C$ is continuous and asymptotically quasi-nonexpansive, then t satisfies condition (K).*

Proof. Since t is continuous, $\mathrm{Fix}(t)$ is closed. Next, we show that $\mathrm{Fix}(t)$ is convex. Let u, v be two different points in $\mathrm{Fix}(t)$ and let $w = (1/2)u \oplus (1/2)v$. It is enough to show that $w \in \mathrm{Fix}(t)$. Since t is asymptotically quasi-nonexpansive,

$$\begin{aligned} d(t^n(w), u) &\le k_n d(w, u) = \frac{k_n}{2} d(u, v), \\ d(t^n(w), v) &\le k_n d(w, v) = \frac{k_n}{2} d(u, v). \end{aligned} \tag{19}$$

Let $r = d(u, v)/2$. Then, for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that if $n \ge n_0$ then

$$t^n(w) \in \overline{B}(u, r + \varepsilon) \cap \overline{B}(v, r + \varepsilon) := D_\varepsilon. \tag{20}$$

We will show that the diameters of the sets D_ε tend to 0 as ε tends to 0 and so $\lim_{n\to\infty} t^n(w) = w$, which proves that $w \in \mathrm{Fix}(t)$. Let $r_\varepsilon = d(u, D_\varepsilon)$, $d_\varepsilon = \mathrm{diam}(D_\varepsilon)$ and $d = \lim_{\varepsilon\to 0} d_\varepsilon = \inf_{\varepsilon>0} d_\varepsilon$. Then $r_\varepsilon \to r$ as $\varepsilon \to 0$. Assume that $d > 0$ and let $\xi \in (0, d/2)$. Thus, for each $\varepsilon \in (0, \xi)$ there exist $x_\varepsilon, y_\varepsilon \in D_\varepsilon$ such that $d(x_\varepsilon, y_\varepsilon) \ge d_\varepsilon - \varepsilon \ge d - \varepsilon \ge d/2$. Since $d(x_\varepsilon, u) \le r + \varepsilon$,

$d(y_\varepsilon, u) \le r+\varepsilon$ and $d(x_\varepsilon, y_\varepsilon) \ge d/2 \ge (r+\varepsilon)(d/2(r+1))$, we get that for each $\varepsilon \in (0, \xi)$,

$$\begin{aligned} r_\varepsilon &\le d\left(\frac{1}{2}x_\varepsilon \oplus \frac{1}{2}y_\varepsilon, u\right) \\ &\le \left(1-\eta\left(r+\varepsilon, \frac{d}{2(r+1)}\right)\right)(r+\varepsilon) \\ &= \left(1-\eta\left(r+1, \frac{d}{2(r+1)}\right)\right)(r+\varepsilon). \end{aligned} \tag{21}$$

By letting $\varepsilon \to 0$, we get that $r \le (1-\eta(r+1, d/2(r+1)))\, r < r$, which is a contradiction. Hence $\mathrm{Fix}(t)$ is convex. The proof of the remaining part closely follows the proof of Proposition 8, upon replacing $\alpha_m(x)$ with k_m. □

Remark 10. Continuity seems essential to the proof of Proposition 9. We do not have an example to show that it is necessary.

The following result is a consequence of [23, Theorem 3.2].

Theorem 11. *Let C be a nonempty bounded closed convex subset of X. Suppose that $T : C \to K(C)$ satisfies condition (C_λ) and $I-T$ is strongly demiclosed. Then T has a fixed point.*

Now, we are ready to prove our main theorem.

Theorem 12. *Let C be a nonempty bounded closed convex subset of X and $t : C \to C$ be a mapping satisfying condition (K). Suppose that $T : C \to KC(C)$ satisfies condition (C_λ) and $I-T$ is strongly demiclosed. If t and T commute, then there exists $z \in C$ such that $z = t(z) \in T(z)$.*

Proof. This proof is patterned after the proof of [25, Theorem 3.1]. Commutative of t and T implies that $t(T(x)) \subseteq T(x)$ for all $x \in \mathrm{Fix}(t)$. Then we have $\mathrm{Fix}(t) \cap T(x) \ne \emptyset$ for all $x \in \mathrm{Fix}(t)$ since t satisfies condition (K). Therefore, the mapping $F(\cdot) := T(\cdot) \cap \mathrm{Fix}(t) : \mathrm{Fix}(t) \to KC(\mathrm{Fix}(t))$ is well defined. Since $I-T$ is strongly demiclosed, then $I-F$ is strongly demiclosed. Next, we show that F satisfies condition (C_λ). Let $x, y \in \mathrm{Fix}(t)$ be such that

$$\lambda d(x, F(x)) \le d(x, y). \tag{22}$$

This implies that $\lambda d(x, T(x)) \le d(x, y)$ and hence $H(T(x), T(y)) \le d(x, y)$ since T satisfies condition (C_λ). We claim that $d(u, F(v)) = d(u, T(v))$ for all $u, v \in \mathrm{Fix}(t)$. Let a be the point in $T(v)$ such that $d(u, a) = d(u, T(v))$. Again by the condition (K), we have $a \in \mathrm{Fix}(t)$. This shows that $d(u, F(v)) = d(u, T(v))$. Now, for each $x, y \in \mathrm{Fix}(t)$ satisfying (22), we have

$$\begin{aligned} &H(F(x), F(y)) \\ &\quad = \max\left\{\sup_{u\in F(x)} d(u, F(y)), \sup_{v\in F(y)} d(v, F(x))\right\} \\ &\quad = \max\left\{\sup_{u\in F(x)} d(u, T(y)), \sup_{v\in F(y)} d(v, T(x))\right\} \\ &\quad \le \max\left\{\sup_{u\in T(x)} d(u, T(y)), \sup_{v\in T(y)} d(v, T(x))\right\} \\ &\quad = H(T(x), T(y)) \\ &\quad \le d(x, y). \end{aligned} \tag{23}$$

By Theorem 11, there exists $z \in \mathrm{Fix}(t)$ such that $z \in F(z)$. As a result, we have $t(z) = z \in T(z)$. □

As consequences of Proposition 7, Proposition 8, and Theorem 12, we obtain the following.

Corollary 13 (see [18, Theorem 3.2]). *Let C be a nonempty bounded closed convex subset of a complete CAT(0) space X. Let $f : C \to C$ be a pointwise asymptotically nonexpansive mapping, and $g : C \to C$ a quasi-nonexpansive mapping, and let $T : C \to KC(C)$ be a multivalued mapping satisfying conditions (E) and (C_λ) for some $\lambda \in (0, 1)$. If f, g, and T are pairwise commuting, then there exists a point $z \in C$ such that $z = f(z) = g(z) \in T(z)$.*

Corollary 14 (see [19, Theorem 3.3]). *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X. Let $t : C \to C$ be a mapping satisfying condition (K), and let $T : C \to KC(C)$ be a multivalued mapping satisfying conditions (E) and (C_λ) for some $\lambda \in (0, 1)$. If t and T commute, then there exists $z \in C$ such that $z = t(z) \in T(z)$.*

Finally, we show that the strongly demiclosedness of $I-T$ in Theorem 12 cannot be removed.

Example 15. Put $X = \mathbb{R}$ and $C = [-1/4, 1]$. Let t be the identity mapping on C and let T be the mapping on C defined by

$$T(x) = \begin{cases} 1 & x = 0, \\ -\left(\frac{1}{3}\right)x & x \in \left[\frac{-1}{4}, 0\right) \cup \left(0, \frac{3}{4}\right), \\ 1-x & x \in \left[\frac{3}{4}, 1\right]. \end{cases} \tag{24}$$

It is easy to see that t and T commute. In [13], the authors prove that either

$$\begin{aligned} &|T(x) - T(y)| \le |x-y| \quad \text{or} \\ &\left(\frac{3}{4}\right)\min\{|x-T(x)|, |y-T(y)|\} \ge |x-y|. \end{aligned} \tag{25}$$

We now let $\varepsilon \in (0, 1/4)$, then either

$$\begin{aligned} &|T(x) - T(y)| \le |x-y| \quad \text{or} \\ &\left(\frac{3}{4}+\varepsilon\right)\min\{|x-T(x)|, |y-T(y)|\} > |x-y|. \end{aligned} \tag{26}$$

This implies that T satisfies condition $(C_{(3/4)+\varepsilon})$ for all $\varepsilon \in (0, 1/4)$. Let $\{x_n\} = \{1/n\}_{n=1}^{\infty}$, then $\{x_n\}$ is an approximate fixed point sequence for T which converges to 0. But 0 is not a fixed point of T. This shows that $I - T$ is not strongly demiclosed. Obviously, T does not have a fixed point.

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