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Research Article

Mapping Properties of Some Classes of Analytic Functions under Certain Integral Operators

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We consider certain subclasses of analytic functions with bounded radius and bounded boundary rotation and study the mapping properties of these classes under certain integral operators.

1. Introduction

Let $\mathcal{A}$ be the class of all functions of the following form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (1)$$

which are analytic in the open unit disc

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}. \quad (2)$$

A function $f \in \mathcal{A}$ is said to be spiral-like if there exists a real number λ ($|\lambda| < \pi/2$) such that

$$\operatorname{Re}\left\{e^{i\lambda} \frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \mathbb{U}). \quad (3)$$

The class of all spiral-like functions was introduced by Spacek [1] in 1933 and we denote it by $\mathcal{S}_\lambda^*$. Later in 1969, Robertson [2] considered the class $\mathcal{C}_\lambda$ of analytic functions in $\mathbb{U}$ for which $zf'(z) \in \mathcal{S}_\lambda^*$.

Let $\mathcal{P}_k^\lambda(\rho)$ be the class of functions $p(z)$ analytic in $\mathbb{U}$ with $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} e^{i\lambda} p(z) - \rho \cos\lambda}{1-\rho} \right| d\theta \le k\pi\cos\lambda, \quad z = re^{i\theta}, \quad (4)$$

where $k \ge 2$, $0 \le \rho < 1$, λ is real with $|\lambda| < \pi/2$.

For $\lambda = 0$, this class was introduced in [3] and for $\rho = 0$, see [4]. For $k = 2$, $\lambda = 0$ and $\rho = 0$, the class $\mathcal{P}_k^\lambda(\rho)$ reduces to the class $\mathcal{P}$ of functions $p(z)$ analytic in $\mathbb{U}$ with $p(0) = 1$ and whose real part is positive.

The following definition of fractional derivative by Owa [5] (also by Srivastava and Owa [6]) will be required in our investigation.

The fractional derivative of order γ is defined, for a function f, by

$$D_z^\gamma f(z) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^\gamma} d\xi \quad (0 \le \gamma < 1), \quad (5)$$

where the function f is analytic in a simply connected region of the complex z-plane containing the origin, and the multiplicity of $(z-\xi)^{-\gamma}$ is removed by requiring $\log(z-\xi)$ to be real when $z - \xi > 0$.

It readily follows from (5) that

$$D_z^\gamma z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\gamma)} z^{k-\gamma} \quad (0 \le \gamma < 1, k \in \mathbb{N} = \{1, 2, \ldots\}). \quad (6)$$

Using $D_z^\gamma f$, Owa and Srivastava [7] introduced the operator $\Omega^\gamma : \mathcal{A} \to \mathcal{A}$, which is known as an extension of fractional derivative and fractional integral, as follows:

$$\begin{aligned}\Omega^\gamma f(z) &= \Gamma(2-\gamma) z^\gamma D_z^\gamma f(z) \\ &= z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\gamma)}{\Gamma(k+1-\gamma)} a_k z^k.\end{aligned} \quad (7)$$

Note that

$$\Omega^0 f(z) = f(z). \tag{8}$$

In [8], Al-Oboudi and Al-Amoudi defined the linear multiplier fractional differential operator (namely, generalized Al-Oboudi differential operator) $D_\delta^{n,\gamma}$ as follows:

$$\begin{aligned} D^0 f(z) &= f(z), \\ D_\delta^{1,\gamma} f(z) &= (1-\delta)\,\Omega^\gamma f(z) + \delta z(\Omega^\gamma f(z))' \\ &= D_\delta^\gamma (f(z)), \quad \delta \geq 0,\ 0 \leq \gamma < 1, \\ D_\delta^{2,\gamma} f(z) &= D_\delta^\gamma \left(D_\delta^{1,\gamma} f(z)\right), \\ &\vdots \\ D_\delta^{n,\gamma} f(z) &= D_\delta^\gamma \left(D_\delta^{n-1,\gamma} f(z)\right), \quad n \in \mathbb{N}. \end{aligned} \tag{9}$$

If f is given by (1), then by (7) and (9), we see that

$$D_\delta^{n,\gamma} f(z) = z + \sum_{k=2}^{\infty} \Psi_{k,n}(\gamma,\delta)\, a_k z^k, \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \tag{10}$$

where

$$\Psi_{k,n}(\gamma,\delta) = \left[\frac{\Gamma(k+1)\Gamma(2-\gamma)}{\Gamma(k+1-\gamma)}(1+(k-1)\delta)\right]^n. \tag{11}$$

Remark 1. (i) When $\gamma = 0$, we get Al-Oboudi differential operator [9].

(ii) When $\gamma = 0$ and $\delta = 1$, we get Sălăgean differential operator [10].

(iii) When $n = 1$ and $\delta = 0$, we get Owa-Srivastava fractional differential operator [7].

Definition 2. A function $f \in \mathscr{A}$ is said to belong to the class $\mathscr{R}_k^\lambda(\rho, b; n, \gamma, \delta)$ if and only if

$$1 + \frac{1}{b}\left(\frac{z\left(D_\delta^{n,\gamma} f(z)\right)'}{D_\delta^{n,\gamma} f(z)} - 1\right) \in \mathscr{P}_k^\lambda(\rho), \tag{12}$$

where $k \geq 2, 0 \leq \rho < 1, \lambda$ is real with $|\lambda| < \pi/2, b \in \mathbb{C} - \{0\}$ and D is the generalized Al-Oboudi differential operator.

Definition 3. A function $f \in \mathscr{A}$ is said to belong to the class $\mathscr{V}_k^\lambda(\rho, b; n, \gamma, \delta)$ if and only if

$$1 + \frac{1}{b}\frac{z\left(D_\delta^{n,\gamma} f(z)\right)''}{\left(D_\delta^{n,\gamma} f(z)\right)'} \in \mathscr{P}_k^\lambda(\rho), \tag{13}$$

where $k \geq 2, 0 \leq \rho < 1, \lambda$ is real with $|\lambda| < \pi/2, b \in \mathbb{C} - \{0\}$ and D is the generalized Al-Oboudi differential operator.

Remark 4. (i) Letting $\gamma = 0$ and $b = 1$ in Definition 2, we have the class $\mathscr{V}_k^\lambda(\rho, \delta, n)$ introduced by Dileep and Latha [11].

(ii) For $n = 0$ and $b = 1$, we obtain the classes $\mathscr{R}_k^\lambda(\rho)$ and $\mathscr{V}_k^\lambda(\rho)$, respectively, introduced and studied by Noor et al. [12] and Moulis [13].

(iii) For $\lambda = 0$ and $n = 0$, we have the classes $\mathscr{R}_k(\rho, b)$ and $\mathscr{V}_k(\rho, b)$, respectively, introduced and studied by Noor et al. [14].

(iv) For $k = 2$, $\lambda = 0$ and $n = 0$, we have the classes $\mathscr{S}_\rho^*(b)$ and $\mathscr{C}_\rho(b)$, respectively, introduced by Frasin [15].

Definition 5. Let $n \in \mathbb{N}_0$, $l = (l_1, \ldots, l_m) \in \mathbb{N}_0^m$, and $k_j > 0$ $(1 \leq j \leq m)$. One defines the integral operator $I_{n,m,l,k} : \mathscr{A}^m \to \mathscr{A}$ as

$$I_{n,m,l,k}(f_1, \ldots, f_m) = F, \tag{14}$$

$$\begin{aligned} D_\delta^{n,\gamma} F(z) = \int_0^z &\left(\frac{D_\delta^{l_1,\gamma} f_1(t)}{t}\right)^{k_1} \\ &\cdots \left(\frac{D_\delta^{l_m,\gamma} f_m(t)}{t}\right)^{k_m} dt \quad (z \in \mathbb{U}), \end{aligned} \tag{15}$$

where $f_1, \ldots, f_m \in \mathscr{A}$ and D is the generalized Al-Oboudi differential operator.

Remark 6. The integral operator $D_\delta^{n,\gamma} F$ generalizes many operators which were introduced and studied recently.

(i) For $\gamma = 0$, we have the integral operator

$$D_\delta^n F(z) = \int_0^z \left(\frac{D_\delta^{l_1} f_1(t)}{t}\right)^{k_1} \cdots \left(\frac{D_\delta^{l_m} f_m(t)}{t}\right)^{k_m} dt \tag{16}$$

introduced by Bulut [16]. Here D is the Al-Oboudi differential operator.

(ii) For $n = 0$, $\gamma = 0$ and $l_1 = \cdots = l_m = l \in \mathbb{N}_0$, we have the integral operator

$$F(z) = \int_0^z \left(\frac{D_\delta^l f_1(t)}{t}\right)^{k_1} \cdots \left(\frac{D_\delta^l f_m(t)}{t}\right)^{k_m} dt \tag{17}$$

introduced by Bulut [17]. Here D is the Al-Oboudi differential operator.

(iii) For $\gamma = 0$ and $\lambda = 1$, we have the integral operator

$$D^n F(z) = \int_0^z \left(\frac{D^{l_1} f_1(t)}{t}\right)^{k_1} \cdots \left(\frac{D^{l_m} f_m(t)}{t}\right)^{k_m} dt \tag{18}$$

introduced by Breaz et al. [18]. Here D is the Sălăgean differential operator.

(iv) For $n = 0$ and $D_\delta^{0,\gamma} f_j = D_0^{1,0} f_j = f_j \in \mathscr{A}$ $(1 \leq j \leq m)$, we have the integral operator

$$F_m(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{k_1} \cdots \left(\frac{f_m(t)}{t}\right)^{k_m} dt \tag{19}$$

introduced by D. Breaz and N. Breaz [19].

(v) For $n = 0$, $m = 1$, $k_1 = k \in [0,1]$, $k_2 = \cdots = k_m = 0$ and $D_\delta^{0,\gamma} f_1 = D_0^{1,0} f_1 = f \in \mathcal{S}^*$ (consists of functions that are analytic, univalent and starlike), we have the integral operator

$$F_k(z) = \int_0^z \left(\frac{f(t)}{t}\right)^k dt \tag{20}$$

studied by Miller et al. [20].

(vi) For $n = 0$, $m = 1$, $k_1 = 1$, $k_2 = \cdots = k_m = 0$ and $D_\delta^{0,\gamma} f_1 = D_0^{1,0} f_1 = f \in \mathcal{A}$, we have the integral operator of Alexander

$$F(z) = \int_0^z \frac{f(t)}{t} dt \tag{21}$$

introduced by Alexander [21].

Definition 7. Let $n \in \mathbb{N}_0$, $l = (l_1, \dots, l_m) \in \mathbb{N}_0^m$, and $k_j > 0$ $(1 \le j \le m)$. One defines the integral operator $J_{n,m,l,k} : \mathcal{A}^m \to \mathcal{A}$ as

$$J_{n,m,l,k}(f_1, \dots, f_m) = G, \tag{22}$$

$$\begin{aligned} D_\delta^{n,\gamma} G(z) = \int_0^z &\left[\left(D_\delta^{l_1,\gamma} f_1(t)\right)'\right]^{k_1} \\ &\cdots \left[\left(D_\delta^{l_m,\gamma} f_1(t)\right)'\right]^{k_m} dt \quad (z \in \mathbb{U}), \end{aligned} \tag{23}$$

where $f_1, \dots, f_m \in \mathcal{A}$ and D is the generalized Al-Oboudi differential operator.

Remark 8. The integral operator $D_\delta^{n,\gamma} G$ generalizes many operators which were introduced and studied recently.

(i) For $n = 0$ and $D_\delta^{0,\gamma} f_j = D_0^{1,0} f_j = f_j \in \mathcal{A}$ $(1 \le j \le m)$, we have the integral operator

$$G(z) = \int_0^z \left(f_1'(t)\right)^{k_1} \cdots \left(f_m'(t)\right)^{k_m} dt \tag{24}$$

introduced by Breaz et al. [22].

(ii) For $n = 0$, $m = 1$, $k_1 = k \in \mathbb{C}$, $k_2 = \cdots = k_m = 0$ and $D_\delta^{0,\gamma} f_1 = D_0^{1,0} f_1 = f \in \mathcal{A}$, we have the integral operator

$$G_k(z) = \int_0^z \left(f'(t)\right)^k dt \tag{25}$$

introduced by Pfaltzgraff [23] (see also Pascu and Pescar [24]).

In this paper, we investigate some propeties of the above integral operators $D_\delta^{n,\gamma} F$ and $D_\delta^{n,\gamma} G$ for the classes

$$\mathcal{R}_k^\lambda(\rho, b; n, \gamma, \delta), \qquad \mathcal{V}_k^\lambda(\rho, b; n, \gamma, \delta). \tag{26}$$

2. Main Results

Theorem 9. *Let $f_j \in \mathcal{R}_k^\lambda(\rho_j, b; n, \gamma, \delta)$ for $1 \le j \le m$ with $0 \le \rho_j < 1, b \in \mathbb{C} - \{0\}$. Also let λ be real with $|\lambda| < \pi/2, k_j > 0$ $(1 \le j \le m)$. If*

$$0 \le 1 + \sum_{j=1}^m k_j\left(\rho_j - 1\right) < 1, \tag{27}$$

then the integral operator F defined by (15) is in the class $\mathcal{V}_k^\lambda(\eta, b; n, \gamma, \delta)$ with

$$\eta = 1 + \sum_{j=1}^m k_j\left(\rho_j - 1\right). \tag{28}$$

Proof. Since $f_j \in \mathcal{A}$ $(1 \le j \le m)$, by (10), we have

$$\begin{gathered} \frac{D_\delta^{l_j,\gamma} f_j(z)}{z} = 1 + \sum_{k=2}^\infty \Psi_{k,l_j}(\gamma, \delta)\, a_{k,j} z^{k-1}, \\ \frac{D_\delta^{l_j,\gamma} f_j(z)}{z} \ne 0 \end{gathered} \tag{29}$$

for all $z \in \mathbb{U}$. By (15), we get

$$\left(D_\delta^{n,\gamma} F(z)\right)' = \left(\frac{D_\delta^{l_1,\gamma} f_1(z)}{z}\right)^{k_1} \cdots \left(\frac{D_\delta^{l_m,\gamma} f_m(z)}{z}\right)^{k_m}. \tag{30}$$

This equality implies that

$$\ln\left(D_\delta^{n,\gamma} F(z)\right)' = k_1 \ln \frac{D_\delta^{l_1,\gamma} f_1(z)}{z} + \cdots + k_m \ln \frac{D_\delta^{l_m,\gamma} f_m(z)}{z} \tag{31}$$

or equivalently

$$\begin{aligned} \ln\left(D_\delta^{n,\gamma} F(z)\right)' = {}& k_1\left[\ln D_\delta^{l_1,\gamma} f_1(z) - \ln z\right] \\ &+ \cdots + k_m\left[\ln D_\delta^{l_m,\gamma} f_m(z) - \ln z\right]. \end{aligned} \tag{32}$$

By differentiating the above equality, we get

$$\frac{\left(D_\delta^{n,\gamma} F(z)\right)''}{\left(D_\delta^{n,\gamma} F(z)\right)'} = \sum_{j=1}^m k_j \left[\frac{\left(D_\delta^{l_j,\gamma} f_j(z)\right)'}{D_\delta^{l_j,\gamma} f_j(z)} - \frac{1}{z}\right]. \tag{33}$$

Hence, we obtain from this equality that

$$\frac{z\left(D_\delta^{n,\gamma} F(z)\right)''}{\left(D_\delta^{n,\gamma} F(z)\right)'} = \sum_{j=1}^m k_j \left(\frac{z\left(D_\delta^{l_j,\gamma} f_j(z)\right)'}{D_\delta^{l_j,\gamma} f_j(z)} - 1\right). \tag{34}$$

Then by multiplying the above relation with $1/b$, we have

$$\begin{aligned}\frac{1}{b}\frac{z\left(D_\delta^{n,\gamma}F(z)\right)''}{\left(D_\delta^{n,\gamma}F(z)\right)'} &= \sum_{j=1}^m k_j\frac{1}{b}\left(\frac{z\left(D_\delta^{l_j,\gamma}f_j(z)\right)'}{D_\delta^{l_j,\gamma}f_j(z)}-1\right)\\ &= \sum_{j=1}^m k_j\left[1+\frac{1}{b}\left(\frac{z\left(D_\delta^{l_j,\gamma}f_j(z)\right)'}{D_\delta^{l_j,\gamma}f_j(z)}-1\right)\right]-\sum_{j=1}^m k_j\end{aligned} \tag{35}$$

or equivalently

$$\begin{aligned}e^{i\lambda}\left(1+\frac{1}{b}\frac{z\left(D_\delta^{n,\gamma}F(z)\right)''}{\left(D_\delta^{n,\gamma}F(z)\right)'}\right) &= \left(1-\sum_{j=1}^m k_j\right)e^{i\lambda}\\ &\quad+\sum_{j=1}^m k_je^{i\lambda}\left[1+\frac{1}{b}\left(\frac{z\left(D_\delta^{l_j,\gamma}f_j(z)\right)'}{D_\delta^{l_j,\gamma}f_j(z)}-1\right)\right].\end{aligned} \tag{36}$$

Subtracting and adding $\cos\lambda\sum_{j=1}^m k_j\rho_j$ on the left hand side and then taking real part, we have

$$\begin{aligned}&\operatorname{Re}\left\{e^{i\lambda}\left(1+\frac{1}{b}\frac{z\left(D_\delta^{n,\gamma}F(z)\right)''}{\left(D_\delta^{n,\gamma}F(z)\right)'}\right)-\eta\cos\lambda\right\}\\ &\quad=\sum_{j=1}^m k_j\operatorname{Re}\left\{e^{i\lambda}\left[1+\frac{1}{b}\left(\frac{z\left(D_\delta^{l_j,\gamma}f_j(z)\right)'}{D_\delta^{l_j,\gamma}f_j(z)}-1\right)\right]\right.\\ &\qquad\left.-\rho_j\cos\lambda\right\},\end{aligned} \tag{37}$$

where η is given by (28). Integrating (37) and then using (28), we have

$$\begin{aligned}&\int_0^{2\pi}\left|\operatorname{Re}\left\{e^{i\lambda}\left(1+\frac{1}{b}\frac{z\left(D_\delta^{n,\gamma}F(z)\right)''}{\left(D_\delta^{n,\gamma}F(z)\right)'}\right)-\eta\cos\lambda\right\}\right|d\theta\\ &\quad\le\sum_{j=1}^m k_j\int_0^{2\pi}\left|\operatorname{Re}\left\{e^{i\lambda}\left[1+\frac{1}{b}\left(\frac{z\left(D_\delta^{l_j,\gamma}f_j(z)\right)'}{D_\delta^{l_j,\gamma}f_j(z)}-1\right)\right]\right.\right.\\ &\qquad\left.\left.-\rho_j\cos\lambda\right\}\right|d\theta.\end{aligned} \tag{38}$$

Since $f_j\in\mathscr{R}_k^\lambda(\rho_j,b;n,\gamma,\delta)$ $(1\le j\le m)$, we get

$$\begin{aligned}&\int_0^{2\pi}\left|\operatorname{Re}\left\{e^{i\lambda}\left[1+\frac{1}{b}\left(\frac{z\left(D_\delta^{l_j,\gamma}f_j(z)\right)'}{D_\delta^{l_j,\gamma}f_j(z)}-1\right)\right]\right.\right.\\ &\qquad\left.\left.-\rho_j\cos\lambda\right\}\right|d\theta\le\left(1-\rho_j\right)k\pi\cos\lambda\end{aligned} \tag{39}$$

for $1\le j\le m$. Using (39) in (38), we obtain

$$\begin{aligned}&\int_0^{2\pi}\left|\operatorname{Re}\left\{e^{i\lambda}\left(1+\frac{1}{b}\frac{z\left(D_\delta^{n,\gamma}F(z)\right)''}{\left(D_\delta^{n,\gamma}F(z)\right)'}\right)-\eta\cos\lambda\right\}\right|d\theta\\ &\quad\le k\pi\cos\lambda\sum_{j=1}^m k_j\left(1-\rho_j\right).\end{aligned} \tag{40}$$

Hence, we obtain $F\in\mathscr{V}_k^\lambda(\eta,b;n,\gamma,\delta)$ with η is given by (28). □

By setting $n=0,\gamma=0,b=1,l_1=\cdots=l_m=l$ in Theorem 9, we obtain the following.

Corollary 10 (see [11, Theorem 1]). *Let $f_j\in\mathscr{V}_k^\lambda(\rho_j,\delta,n)$ for $1\le j\le m$ with $0\le\rho_j<1$. Also let λ be real with $|\lambda|<\pi/2,k_j>0$ $(1\le j\le m)$. If*

$$0\le 1+\sum_{j=1}^m k_j\left(\rho_j-1\right)<1, \tag{41}$$

then the integral operator $F(z)$ defined by (17) *is in the class $\mathscr{V}_k^\lambda(\eta)$ with*

$$\eta=1+\sum_{j=1}^m k_j\left(\rho_j-1\right). \tag{42}$$

Remark 11. Letting $\rho_1 = \rho_2 = \cdots = \rho_m = \rho$ in Corollary 10, then we have [12, Theorem 3.1].

By setting $n = 0$, $\lambda = 0$ and $D_\delta^{0,\gamma} = D_0^{1,0}$ in Theorem 9, we obtain the following.

Corollary 12. *Let $f_j \in \mathscr{R}_k(\rho_j, b)$ for $1 \le j \le m$ with $0 \le \rho_j < 1, b \in \mathbb{C} - \{0\}$. Also let $k_j > 0$ $(1 \le j \le m)$. If*

$$0 \le 1 + \sum_{j=1}^{m} k_j \left(\rho_j - 1\right) < 1, \tag{43}$$

then the integral operator $F_m(z)$ defined by (19) is in the class $\mathscr{V}_k(\eta, b)$ with

$$\eta = 1 + \sum_{j=1}^{m} k_j \left(\rho_j - 1\right). \tag{44}$$

Remark 13. In Corollary 12, letting

(i) $\rho_1 = \rho_2 = \cdots = \rho_m = \rho$, we have [14, Theorem 2.1],

(ii) $k = 2$, we have [25, Theorem 1].

Theorem 14. *Let $f_j \in \mathscr{V}_k^\lambda(\rho_j, b; n, \gamma, \delta)$ for $1 \le j \le m$ with $0 \le \rho_j < 1, b \in \mathbb{C} - \{0\}$. Also let λ be real with $|\lambda| < \pi/2, k_j > 0$ $(1 \le j \le m)$. If*

$$0 \le 1 + \sum_{j=1}^{m} k_j \left(\rho_j - 1\right) < 1, \tag{45}$$

then the integral operator G defined by (23) is in the class $\mathscr{V}_k^\lambda(\mu, b; n, \gamma, \delta)$ with

$$\mu = 1 + \sum_{j=1}^{m} k_j \left(\rho_j - 1\right). \tag{46}$$

Proof. By (23), we get

$$\begin{aligned} &\left(D_\delta^{n,\gamma} G(z)\right)' \\ &\quad = \left[\left(D_\delta^{l_1,\gamma} f_1(z)\right)'\right]^{k_1} \cdots \left[\left(D_\delta^{l_m,\gamma} f_1(z)\right)'\right]^{k_m}. \end{aligned} \tag{47}$$

This equality implies that

$$\begin{aligned} &\left(D_\delta^{n,\gamma} G(z)\right)'' \\ &\quad = \sum_{j=1}^{m} k_j \left[\left(D_\delta^{l_j,\gamma} f_j(z)\right)'\right]^{k_j} \frac{\left(D_\delta^{l_j,\gamma} f_j(z)\right)''}{\left(D_\delta^{l_j,\gamma} f_j(z)\right)'} \\ &\qquad \times \prod_{\substack{r=1 \\ (r \ne j)}}^{m} \left[\left(D_\delta^{l_r,\gamma} f_r(z)\right)'\right]^{k_r}. \end{aligned} \tag{48}$$

Thus by using (47) and (48), we obtain

$$\frac{z\left(D_\delta^{n,\gamma} G(z)\right)''}{\left(D_\delta^{n,\gamma} G(z)\right)'} = \sum_{j=1}^{m} k_j \frac{z\left(D_\delta^{l_j,\gamma} f_j(z)\right)''}{\left(D_\delta^{l_j,\gamma} f_j(z)\right)'}. \tag{49}$$

Then by multiplying the above relation with $1/b$, we have

$$\begin{aligned} &\frac{1}{b} \frac{z\left(D_\delta^{n,\gamma} G(z)\right)''}{\left(D_\delta^{n,\gamma} G(z)\right)'} \\ &\quad = \sum_{j=1}^{m} k_j \frac{1}{b} \frac{z\left(D_\delta^{l_j,\gamma} f_j(z)\right)''}{\left(D_\delta^{l_j,\gamma} f_j(z)\right)'} \\ &\quad = \sum_{j=1}^{m} k_j \left(1 + \frac{1}{b} \frac{z\left(D_\delta^{l_j,\gamma} f_j(z)\right)''}{\left(D_\delta^{l_j,\gamma} f_j(z)\right)'}\right) - \sum_{j=1}^{n} k_j \end{aligned} \tag{50}$$

or equivalently

$$\begin{aligned} &e^{i\lambda} \left(1 + \frac{1}{b} \frac{z\left(D_\delta^{n,\gamma} G(z)\right)''}{\left(D_\delta^{n,\gamma} G(z)\right)'}\right) \\ &\quad = \left(1 - \sum_{j=1}^{m} k_j\right) e^{i\lambda} \\ &\qquad + \sum_{j=1}^{m} k_j e^{i\lambda} \left(1 + \frac{1}{b} \frac{z\left(D_\delta^{l_j,\gamma} f_j(z)\right)''}{\left(D_\delta^{l_j,\gamma} f_j(z)\right)'}\right). \end{aligned} \tag{51}$$

Subtracting and adding $\cos\lambda \sum_{j=1}^{m} k_j \rho_j$ on the left hand side and then taking real part, we have

$$\begin{aligned} &\operatorname{Re}\left\{e^{i\lambda} \left(1 + \frac{1}{b} \frac{z\left(D_\delta^{n,\gamma} G(z)\right)''}{\left(D_\delta^{n,\gamma} G(z)\right)'}\right) - \mu \cos\lambda\right\} \\ &\quad = \sum_{j=1}^{m} k_j \operatorname{Re}\left\{e^{i\lambda} \left(1 + \frac{1}{b} \frac{z\left(D_\delta^{l_j,\gamma} f_j(z)\right)''}{\left(D_\delta^{l_j,\gamma} f_j(z)\right)'}\right) - \rho_j \cos\lambda\right\}, \end{aligned} \tag{52}$$

where μ is given by (46). Integrating (52) and then using (46), we have

$$\int_0^{2\pi}\left|\operatorname{Re}\left\{e^{i\lambda}\left(1+\frac{1}{b}\frac{z\left(D_\delta^{n,\gamma}G(z)\right)''}{\left(D_\delta^{n,\gamma}G(z)\right)'}\right)-\mu\cos\lambda\right\}\right|d\theta$$
$$\leq\sum_{j=1}^{m}k_j\int_0^{2\pi}\left|\operatorname{Re}\left\{e^{i\lambda}\left(1+\frac{1}{b}\frac{z\left(D_\delta^{l_j,\gamma}f_j(z)\right)''}{\left(D_\delta^{l_j,\gamma}f_j(z)\right)'}\right)\right.\right.$$
$$\left.\left.-\rho_j\cos\lambda\right\}\right|d\theta. \tag{53}$$

Since $f_j\in\mathcal{V}_k^{\lambda}(\rho_j,b;n,\gamma,\delta)$ $(1\leq j\leq m)$, we get

$$\int_0^{2\pi}\left|\operatorname{Re}\left\{e^{i\lambda}\left(1+\frac{1}{b}\frac{z\left(D_\delta^{l_j,\gamma}f_j(z)\right)''}{\left(D_\delta^{l_j,\gamma}f_j(z)\right)'}\right)-\rho_j\cos\lambda\right\}\right|d\theta$$
$$\leq\left(1-\rho_j\right)k\pi\cos\lambda \tag{54}$$

for $1\leq j\leq m$. Using (54) in (53), we obtain

$$\int_0^{2\pi}\left|\operatorname{Re}\left\{e^{i\lambda}\left(1+\frac{1}{b}\frac{z\left(D_\delta^{n,\gamma}G(z)\right)''}{\left(D_\delta^{n,\gamma}G(z)\right)'}\right)-\mu\cos\lambda\right\}\right|d\theta$$
$$\leq k\pi\cos\lambda\sum_{j=1}^{m}k_j\left(1-\rho_j\right). \tag{55}$$

Hence, we obtain $G\in\mathcal{V}_k^{\lambda}(\mu,b;n,\gamma,\delta)$ with μ given by (46). □

By setting $n=0$, $\lambda=0$ and $D_\delta^{0,\gamma}=D_0^{1,0}$ in Theorem 14, we obtain the following.

Corollary 15. *Let $f_j\in\mathcal{V}_k(\rho_j,b)$ for $1\leq j\leq m$ with $0\leq\rho_j<1, b\in\mathbb{C}-\{0\}$. Also let $k_j>0$ $(1\leq j\leq m)$. If*

$$0\leq 1+\sum_{j=1}^{m}k_j\left(\rho_j-1\right)<1, \tag{56}$$

then the integral operator $G(z)$ defined by (24) is in the class $\mathcal{V}_k(\mu,b)$ with

$$\mu=1+\sum_{j=1}^{m}k_j\left(\rho_j-1\right). \tag{57}$$

Remark 16. In Corollary 15, letting

(i) $\rho_1=\rho_2=\cdots=\rho_m=\rho$, we have [14, Theorem 2.5],

(ii) $k=2$, we have [25, Theorem 3].

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