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Research Article

Semigroups Characterized by Their Generalized Fuzzy Ideals

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We have characterized right weakly regular semigroups by the properties of their $(\in, \in \vee q_k)$-fuzzy ideals.

1. Introduction

Usually the models of real world problems in almost all disciplines like in engineering, medical science, mathematics, physics, computer science, management sciences, operations research, and artificial intelligence are mostly full of complexities and consist of several types of uncertainties while dealing with them in several occasion. To overcome these difficulties of uncertainties, many theories had been developed such as rough sets theory, probability theory, fuzzy sets theory, theory of vague sets, theory of soft ideals, and the theory of intuitionistic fuzzy sets. Zadeh discovered the relationships of probability and fuzzy set theory in [1] which has appropriate approach to deal with uncertainties. Many authors have applied the fuzzy set theory to generalize the basic theories of Algebra. The concept of fuzzy sets in structure of groups was given by Rosenfeld [2]. The theory of fuzzy semigroups and fuzzy ideals in semigroups was introduced by Kuroki in [3, 4]. The theoretical exposition of fuzzy semigroups and their application in fuzzy coding, fuzzy finite state machines, and fuzzy languages was considered by Mordeson. The concept of belongingness of a fuzzy point to a fuzzy subset by using natural equivalence on a fuzzy subset was considered by Murali [5]. By using these ideas, Bhakat and Das [6, 7] gave the concept of (α, β)-fuzzy subgroups by using the "belongs to" relation $\in$ and "quasi-coincident with" relation q between a fuzzy point and a fuzzy subgroup and introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroups, where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$ and $\alpha \neq \in \wedge q$. In particular, $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. These fuzzy subgroups are further studied in [8, 9]. The concept of $(\in, \in \vee q_k)$-fuzzy subgroups is a viable generalization of Rosenfeld's fuzzy subgroups. Davvaz defined $(\in, \in \vee q_k)$-fuzzy subnearrings and ideals of a near ring in [10]. Jun and Song initiated the study of (α, β)-fuzzy interior ideals of a semigroup in [11] which is the generalization of fuzzy interior ideals [12]. In [13], Kazanci and Yamak studied $(\in, \in \vee q_k)$-fuzzy bi-ideals of a semigroup.

In this paper we have characterized right regular semigroups by the properties of their right ideal, bi-ideal, generalized bi-ideal, and interior ideal. Moreover we characterized right regular semigroups in terms of their $(\in, \in \vee q_k)$-fuzzy right ideal, $(\in, \in \vee q_k)$-fuzzy bi-ideal, $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal, $(\in, \in \vee q_k)$-fuzzy bi-ideal, and $(\in, \in \vee q_k)$-fuzzy interior ideals.

Throughout this paper S denotes a semigroup. A nonempty subset A of S is called a subsemigroup of S if $A^2 \subseteq A$. A nonempty subset J of S is called a left (right) ideal of S if $SJ \subseteq I$ $(JS \subseteq I)$. J is called a two-sided ideal or simply an ideal of S if it is both left and right ideal of S. A nonempty subset B of S is called a generalized bi-ideal of S if $BSB \subseteq B$. A nonempty subset B of S is called a bi-ideal of S if it is both a subsemigroup and a generalized bi-ideal of S. A subsemigroup I of S is called an interior ideal of S if $SIS \subseteq I$.

An semigroup S is called a right weakly regular if for every $a \in S$ there exist $x, y \in S$ such that $a = axay$.

Definition 1. For a fuzzy set f of a semigroup S and $t \in (0, 1]$, the crisp set $U(f; t) = \{x \in S$ such that $f(x) \geq t\}$ is called level subset of f.

Definition 2. A fuzzy subset f of a semigroup S of the form

$$f(y) = \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x \end{cases} \quad (1)$$

is said to be a fuzzy point with support x and value t and is denoted by x_t.

A fuzzy point x_t is said to *belong* to (*resp., quasi-coincident with*) a fuzzy set f, written as $x_t \in f$ (*resp.*, $x_t q f$), if $f(x) \geq t$ (*resp.*, $f(x) + t > 1$). If $x_t \in f$ or $x_t q f$, then we write $x_t \in \vee q f$. The symbol $\overline{\in \vee q}$ means $\in \vee q$ does not hold. For any two fuzzy subsets f and g of S, $f \leq g$ means that, for all $x \in S$, $f(x) \leq g(x)$.

Generalizing the concept of $x_t q f$, Jun [12, 14] defined $x_t q_k f$, where $k \in [0,1)$, as $f(x) + t + k > 1$. $x_t \in \vee q_k f$ if $x_t \in f$ or $x_t q_k f$.

2. $(\in, \in \vee q_k)$-Fuzzy Ideals in Semigroups

Definition 3. A fuzzy subset of S is called an $(\in, \in \vee q_k)$-fuzzy subsemigroup of S if for all $x, y \in S$ and $t, r \in (0,1]$ the following condition holds: $x_t \in f$ and $y_r \in f$ imply $(xy)_{\min\{t,r\}} \in \vee q_k f$.

Lemma 4 (see [15]). *Let f be a fuzzy subset of S. Then f is an $(\in, \in \vee q_k)$-fuzzy subsemigroup of S if and only if $f(xy) \geq \min\{f(x), f(y), (1-k)/2\}$.*

Definition 5. A fuzzy subset f of S is called an $(\in, \in \vee q_k)$-fuzzy left (right) ideal of S if for all $x, y \in S$ and $t, r \in (0,1]$ the following condition holds: $y_r \in f$ implies $(xy)_t \in \vee q_k f$ ($x_t \in f$ implies $(xy)_t \in \vee q_k f$).

Lemma 6 (see [15]). *Let f be a fuzzy subset of S. Then f is an $(\in, \in \vee q_k)$-fuzzy left (right) ideal of S if and only if $f(xy) \geq \min\{f(y), (1-k)/2\}$ ($f(xy) \geq \min\{f(x), (1-k)/2\}$).*

Definition 7. A fuzzy subsemigroup f of a semigroup S is called an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S if for all $x, y, z \in S$ and $t, r \in (0,1]$ the following condition holds: $x_t \in f$ and $z_t \in f$ imply $(xyz)_t \in \vee q_k f$.

Lemma 8 (see [15]). *A fuzzy subset f of S is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S if and only if it satisfies the following conditions:*

(i) *$f(xy) \geq \min\{f(x), f(y), (1-k)/2\}$ for all $x, y \in S$ and $k \in [0,1)$;*

(ii) *$f(xyz) \geq \min\{f(x), f(z), (1-k)/2\}$ for all $x, y, z \in S$ and $k \in [0,1)$.*

Definition 9. A fuzzy subset f of a semigroup S is called an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S if for all $x, y, z \in S$ and $t, r \in (0,1]$ the following condition holds: $x_t \in f$ and $z_t \in f$ imply $(xyz)_t \in \vee q_k f$.

Lemma 10 (see [15]). *A fuzzy subset f of S is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S if and only if $f(xyz) \geq \min\{f(x), f(z), (1-k)/2\}$ for all $x, y, z \in S$ and $k \in [0,1)$.*

Definition 11. A fuzzy subsemigroup f of a semigroup S is called an $(\in, \in \vee q_k)$-fuzzy interior ideal of S if for all $x, y, z \in S$ and $t, r \in (0,1]$ the following condition holds: $y_t \in f$ imply $(xyz)_t \in \vee q_k f$.

Lemma 12 (see [15]). *A fuzzy subset f of S is an $(\in, \in \vee q_k)$-fuzzy interior ideal of S if and only if it satisfies the following condition:*

(i) *$f(xy) \geq \min\{f(x), f(y), (1-k)/2\}$ for all $x, y \in S$ and $k \in [0,1)$;*

(ii) *$f(xyz) \geq \min\{f(y), (1-k)/2\}$ for all $x, y, z \in S$ and $k \in [0,1)$.*

Example 13. Let $S = \{1, 2, 3,\}$ be a semigroup with binary operation "$\cdot$," as defined in the following Cayley table:

$\cdot$	1	2	3
1	1	1	1
2	2	2	2
3	3	3	3

(2)

Clearly $(S, \cdot)$ is regular semigroup and $\{1\}$, $\{2\}$, and $\{3\}$ are left ideals of S. Let us define a fuzzy subset δ of S as

$$\delta(1) = 0.9, \qquad \delta(2) = 0.6, \qquad \delta(3) = 0.5. \quad (3)$$

Then clearly δ is an $(\in, \in \vee q)$-fuzzy ideal of S.

Lemma 14 (see [15]). *A nonempty subset R of a semigroup S is right (left) ideal if and only if $(C_R)_k$ is an $(\in, \in \vee q_k)$-fuzzy right (left) ideal of S.*

Lemma 15. *A nonempty subset I of a semigroup S is an interior ideal if and only if $(C_I)_k$ is an $(\in, \in \vee q_k)$-fuzzy interior ideal of S.*

Lemma 16. *A nonempty subset B of a semigroup S is bi-ideal if and only if $(C_B)_k$ is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S.*

Lemma 17. *Let f and g be any fuzzy subsets of semigroup S. Then following properties hold:*

(i) $(f \wedge_k g) = (f_k \wedge g_k)$,

(ii) $(f \circ_k g) = (f_k \circ g_k)$.

Proof. It is straightforward. □

Lemma 18. *Let A and B be any nonempty subsets of a semigroup S. Then the following properties hold:*

(i) $(C_A \wedge_k C_B) = (C_{A \cap B})_k$,

(ii) $(C_A \circ_k C_B) = (C_{AB})_k$.

Proof. It is straightforward. □

3. Characterizations of Regular Semigroups

Theorem 19. *For a semigroup S, the following conditions are equivalent:*

(i) *S is regular;*

(ii) *$L_1 \cap L_2 \cap B \subseteq BL_1L_2$ for left ideals L_1, L_2, and bi-ideal B of a semigroup S.*

(iii) *$L[a] \cap L[a] \cap B[a] \subseteq B[a]L[a]L[a]$, for some a in S;*

Proof. (i) $\Rightarrow$ (ii): Let S be regular semigroup, then for an element $a \in S$ there exists $x \in S$ such that $a = axa$. Let $a \in L_1 \cap L_2 \cap B$, where B is a bi-ideal and L_1 and L_2 are left ideals of S. So $a \in L_1$, $a \in L_2$, and $a \in B$.

As $a = axa = axaxa = axaxaxa \in (BSB)SL_1SL_2 \subseteq BL_1L_2$. Thus $L_1 \cap L_2 \cap B \subseteq BL_1L_2$.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i): As $a \cup Sa$ and $a \cup a^2 \cup aSa$ are left ideal and bi-ideal of S generated by a, respectively, thus by assumption we have

$$\begin{aligned}
&(a \cup Sa) \cap (a \cup Sa) \cap \left(a^2 \cup aSa\right) \\
&\quad \subseteq \left(a \cup a^2 \cup aSa\right)(a \cup Sa)(a \cup Sa) \\
&\quad \subseteq \left(a \cup a^2 \cup aSa\right) S (a \cup Sa) \\
&\quad \subseteq \left(a \cup a^2 \cup aSa\right)(a \cup Sa) \\
&\quad = a^2 \cup aSa \cup a^3 \cup a^2Sa \cup aSa^2 \cup aSaSa \\
&\quad \subseteq a^2 \cup a^3 \cup aSa.
\end{aligned} \tag{4}$$

Thus $a = a^2 = aa = a^2a = aaa$ or $a = a^3 = aaa$ or $a = axa$, for some x in S. Hence S is regular semigroup. □

Theorem 20. *For a semigroup S, the following conditions are equivalent:*

(i) *S is regular;*

(ii) *$R \cap L_1 \cap L_2 \subseteq RL_1L_2$ for every right ideal R and bi-ideal B of a semigroup S;*

(iii) *$R[a] \cap L[a] \cap L[a] \subseteq R[a]L[a]L[a]$, for some a in S.*

Proof. (i) $\Rightarrow$ (ii): Let S be regular semigroup, then for an element $a \in S$ there exists $x \in S$ such that $a = axa$. Let $a \in R \cap L_1 \cap L_2$, where R is right ideal and L_1, and L_2 are left ideals of S. So $a \in R$, $a \in L_1$and $a \in L_2$. As $a = axaxa \in (RS)L_1SL_2 \subseteq RL_1L_2$. Thus $R \cap L_1 \cap L_2 \subseteq RL_1L_2$.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i): As $a \cup aS$ is right ideal and $a \cup Sa$ is left ideal of S generated by a, respectively, thus by assumption we have

$$\begin{aligned}
&(a \cup Sa) \cap (a \cup aS) \cap (a \cup aS) \\
&\quad \subseteq (a \cup aS)(a \cup Sa)(a \cup Sa) \\
&\quad \subseteq (a \cup aS) S (a \cup Sa) \\
&\quad \subseteq (a \cup aS)(a \cup Sa) \\
&\quad = a^2 \cup aSa \cup aSa \cup aSSa \\
&\quad \subseteq a^2 \cup aSa.
\end{aligned} \tag{5}$$

Thus $a = a^2$ or $a = axa$, for some x in S. Hence S is regular semigroup. □

Theorem 21. *For a semigroup S, the following conditions are equivalent:*

(i) *S is regular;*

(ii) *$f \circ_k g \circ_k h \geq f \wedge_k g \wedge_k h$ for every $(\in, \in \vee q_k)$-fuzzy right ideal f, $(\in, \in \vee q_k)$-fuzzy left ideals g, and h of a semigroup S.*

Proof. (i) $\Rightarrow$ (ii): Let f be $(\in, \in \vee q_k)$-fuzzy right ideal, g and h any $(\in, \in \vee q_k)$-fuzzy left ideals of S. Since S is regular, therefore for each $a \in S$ there exists $x \in S$ such that

$$a = axa = axaxa. \tag{6}$$

Thus

$$\begin{aligned}
&\left(f \circ_k g \circ_k h\right)(a) \\
&\quad = \left(f \circ g \circ h\right)(a) \wedge \frac{1-k}{2} \\
&\quad = \left(_{a=pq}\left\{f(p) \wedge (g \circ h)(q)\right\}\right) \wedge \frac{1-k}{2} \\
&\quad \geq f(a) \wedge (g \circ h)((xa)(xa)) \wedge \frac{1-k}{2} \\
&\quad \geq f(a) \wedge \left(_{(xa)(xa)=bc}\left\{g(b) \wedge h(c)\right\}\right) \wedge \frac{1-k}{2} \\
&\quad \geq f(a) \wedge g(xa) \wedge h(xa) \wedge \frac{1-k}{2} \\
&\quad \geq f(a) \wedge g(a) \wedge h(a) \wedge \frac{1-k}{2} \\
&\quad \geq f(a) \wedge g(a) \wedge h(a) \wedge \frac{1-k}{2}.
\end{aligned} \tag{7}$$

(ii) $\Rightarrow$ (i): Let $R[a]$ be right ideal, and let $L_1[a]$ and $L_2[a]$ be any two left ideals of S generated by a, respectively.

Then $(C_{R[a]})_k$ is any $(\in, \in \vee q_k)$-fuzzy right ideal, and $(C_{L_1[a]})_k$ and $(C_{L_2[a]})_k$ are any $(\in, \in \vee q_k)$-fuzzy left ideals of semigroup S, respectively. Let $a \in S$ and $b \in R[a] \cap L_1[a] \cap L_2[a]$. Then $b \in R[a]$, $b \in L_1[a]$, and $b \in L_2[a]$. Now

$$\begin{aligned}\frac{1-k}{2} &\le \left(C_{R[a]\cap L_1[a]\cap L_2[a]}\right)_k(b)\\ &= \left(\left(C_{R[a]}\right)_k \wedge_k \left(C_{L_1[a]}\right)_k \wedge_k \left(C_{L_2[a]}\right)_k\right)(b)\\ &\le \left(\left(C_{R[a]}\right)_k \circ_k \left(C_{L_1[a]}\right)_k \circ_k \left(C_{L_2[a]}\right)_k\right)(b)\\ &= \left(C_{R[a]L_1[a]L_2[a]}\right)_k(b).\end{aligned} \tag{8}$$

Thus $b \in R[a]L_1[a]L_2[a]$. Therefore $R[a] \cap L_1[a] \cap L_2[a] \subseteq R[a]L_1[a]L_2[a]$.

So by Theorem 20, S is regular. □

4. Characterizations of Right Weakly Regular Semigroups in Terms of $(\in, \in \vee q_k)$-Fuzzy Ideals

Theorem 22. *For a semigroup S, the following conditions are equivalent:*

(i) *S is right weakly regular;*

(ii) *$R \cap L \cap I \subseteq RLI$ for every right ideal, left ideal, and interior ideal of S, respectively;*

(iii) *$R[a] \cap L[a] \cap I[a] \subseteq R[a]L[a]I[a]$.*

Proof. (i) $\Rightarrow$ (ii): Let S be right weakly regular semigroup, and let R, L, and I be right ideal, left ideal, and interior ideal of S, respectively. Let $a \in R \cap L \cap I$ then $a \in R$, $a \in L$, and $a \in I$. Since S is right weakly regular semigroup so for a there exist $x, y \in S$ such that

$$\begin{aligned}a &= axay = axaxayy = axaxaxayyy\\ &\in (RS)(SL)(SIS) \subseteq RLI.\end{aligned} \tag{9}$$

Therefore $a \in RLI$. So $R \cap L \cap I \subseteq RLI$.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i): As $a \cup aS$, $a \cup Sa$, and $a \cup a^2 \cup SaS$ are right ideal, left ideal, and interior ideal of S generated by an element a of S, respectively, thus by assumption, we have

$$\begin{aligned}&(a \cup aS) \cap (a \cup Sa) \cap \left(a \cup a^2 \cup SaS\right)\\ &\subseteq (a \cup aS)(a \cup Sa)\left(a \cup a^2 \cup SaS\right)\\ &= \left(a^2 \cup aSa \cup aSa \cup aSSa\right)\left(a \cup a^2 \cup SaS\right)\\ &= a^3 \cup a^4 \cup a^2SaS \cup aSa^2 \cup aSa^3 \cup aSaSaS\\ &\quad \cup aSSa^2 \cup aSSa^3 \cup aSSaSaS\\ &\subseteq a^3 \cup a^4 \cup aSa \cup aSaS.\end{aligned} \tag{10}$$

Thus $a = a^3 = aaa = aaa^3 = aaa^2a$ or $a = axa = axaxa$ or $a = auav$, for some x, u, v in S. Hence S is right weakly regular semigroup. □

Theorem 23. *For a semigroup S, the following conditions are equivalent:*

(i) *S is right weakly regular;*

(ii) *$f \wedge g \wedge h \le f \circ g \circ h$ for every fuzzy right ideal, fuzzy left ideal, and fuzzy interior ideal of S, respectively.*

Proof. (i) $\Rightarrow$ (ii): Let f, g, and h be any $(\in, \in \vee q_k)$-fuzzy right ideal, $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal, and $(\in, \in \vee q_k)$-fuzzy interior ideal of S. Since S is right weakly regular therefore for each $a \in S$ there exist $x, y \in S$ such that

$$\begin{aligned}a &= axay = (axay)(xay) = (ax)(ay)(xay)\\ &= (ax)(axayy)(xay) = (ax)(axa)(yyxay).\end{aligned} \tag{11}$$

Then

$$\begin{aligned}&(f \circ_k g \circ_k h)(a)\\ &= (f \circ g \circ h)(a) \wedge \frac{1-k}{2}\\ &= \left(\bigvee_{a=pq} \{f(p) \wedge (g \circ h)(q)\}\right) \wedge \frac{1-k}{2}\\ &\ge f(ax) \wedge (g \circ h)((axa)(yyxay)) \wedge \frac{1-k}{2}\\ &\ge f(a) \wedge \left(\bigvee_{(axa)(yyxay)=bc} \{g(b) \wedge h(c)\}\right) \wedge \frac{1-k}{2}\\ &\ge f(a) \wedge g(axa) \wedge h(yyxay) \wedge \frac{1-k}{2}\\ &\ge f(a) \wedge g(a) \wedge h(a) \wedge \frac{1-k}{2}\\ &\ge f(a) \wedge g(a) \wedge h(a) \wedge \frac{1-k}{2}.\end{aligned} \tag{12}$$

Therefore $f \wedge_k g \wedge_k h \le f \circ_k g \circ_k h$.

Now (ii) $\Rightarrow$ (i)

(ii) $\Rightarrow$ (i): Let $R[a]$, $L[a]$, and $I[a]$ be right ideal, left ideal, and interior ideal of S generated by a, respectively.

Then $(C_{R[a]})_k$, $(C_{L[a]})_k$, and $(C_{I[a]})_k$ are $(\in, \in \vee q_k)$-fuzzy right ideal, $(\in, \in \vee q_k)$-fuzzy left ideal, and $(\in, \in \vee q_k)$-fuzzy interior ideal of semigroup S. Let $a \in S$ and $b \in R[a] \cap L[a] \cap I[a]$. Then $b \in R[a]$, $b \in L[a]$, and $b \in I[a]$. Now

$$\begin{aligned}\frac{1-k}{2} &\le \left(C_{R[a]\cap L[a]\cap I[a]}\right)_k(b)\\ &= \left(\left(C_{R[a]}\right)_k \wedge_k \left(C_{L[a]}\right)_k \wedge_k \left(C_{I[a]}\right)_k\right)(b)\\ &\le \left(\left(C_{R[a]}\right)_k \circ_k \left(C_{L[a]}\right)_k \circ_k \left(C_{I[a]}\right)_k\right)(b)\\ &= \left(C_{R[a]L[a]I[a]}\right)_k(b).\end{aligned} \tag{13}$$

Thus $b \in R[a]L[a]I[a]$. Therefore $R[a] \cap L[a] \cap I[a] \subseteq R[a]L[a]I[a]$. Hence by Theorem 22, S is right weakly regular semigroup. □

Theorem 24. *For a semigroup S, the following conditions are equivalent:*

(i) *S is right weakly regular;*

(ii) *$B\cap L\cap I\subseteq BLI$ for every bi-ideal, left ideal, and interior ideal of S, respectively;*

(iii) *$B[a]\cap L[a]\cap I[a]\subseteq B[a]L[a]I[a]$.*

Proof. (i) $\Rightarrow$ (ii): Let S be right weakly regular semigroup, and B, L, and I be bi-ideal, left ideal, and interior ideal of S, respectively. Let $a\in B\cap L\cap I$ then $a\in B$, $a\in L$, and $a\in I$. Since S is right weakly regular semigroup so for a there exist $x,y\in S$ such that

$$\begin{aligned} a &= axay = axaxaxayyy \\ &= (axa)(xa)(xayyy)\in(BS)(SL)(SIS)\subseteq BLI. \end{aligned} \tag{14}$$

Therefore $a\in BLI$.So $B\cap L\cap I\subseteq BLI$.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i): As $a\cup a^2\cup aSa$, $a\cup Sa$, and $a\cup a^2\cup SaS$ are bi-ideal, left ideal, and interior ideal of S generated by an element a of S, respectively, thus by assumption we have

$$\begin{aligned} &\left(a\cup a^2\cup aSa\right)\cap(a\cup Sa)\cap\left(a\cup a^2\cup SaS\right)\\ &\quad\subseteq\left(a\cup a^2\cup aSa\right)(a\cup Sa)\left(a\cup a^2\cup SaS\right)\\ &\quad=\left(a^2\cup aSa\cup a^3\cup a^2Sa\cup aSa^2\cup aSaSa\right)\\ &\qquad\times\left(a\cup a^2\cup SaS\right)\\ &\quad=a^3\cup a^4\cup a^2SaS\cup aSa^2\cup aSa^3\cup aSaSaS\cup a^4\cup a^5\\ &\qquad\cup a^3SaS\cup a^2Sa^2\cup a^2Sa^3\cup a^2SaSaS\cup aSa^3\\ &\qquad\cup aSa^4\cup aSa^2SaSaSaSa^2\\ &\qquad\cup aSaSa^3\cup aSaSaSaS\\ &\quad\subseteq a^3\cup a^4\cup a^5\cup aSa\cup aSaS. \end{aligned} \tag{15}$$

Thus $a=a^4=aaaa$ or $a=a^3=aaa=aaa^3=aaa^2a$ or $a=axa=axaxa$ or $a=auav$, for some x,u,v in S. Hence S is right weakly regular semigroup. □

Theorem 25. *For a semigroup S, the following conditions are equivalent:*

(i) *S is right weakly regular;*

(ii) *$f\wedge g\wedge h\le f\circ g\circ h$ for every fuzzy bi-ideal, fuzzy left ideal and fuzzy interior ideal of S, respectively;*

(iii) *$f\wedge g\wedge h\le f\circ g\circ h$ for every fuzzy generalized bi-ideal, fuzzy left ideal, and fuzzy interior ideal of S, respectively.*

Proof. (i) $\Rightarrow$ (iii): Let f, g, and h be any $(\in,\in\vee q_k)$-fuzzy generalized bi-ideal, $(\in,\in\vee q_k)$-fuzzy left ideal, and $(\in,\in\vee q_k)$-fuzzy interior ideal of S. Since S is right weakly regular for each $a\in S$ there exist $x,y\in S$ such that

$$\begin{aligned} a &= axay=(axay)(xay)=(ax)(ay)(xay)\\ &=(ax)(axayy)(xay)=(axa)(xa)(yyxay). \end{aligned} \tag{16}$$

Then

$$\begin{aligned} &(f\circ_k g\circ_k h)(a)\\ &\quad=(f\circ g\circ h)(a)\wedge\frac{1-k}{2}\\ &\quad=\left(\bigvee_{a=pq}\{f(p)\wedge(g\circ h)(q)\}\right)\wedge\frac{1-k}{2}\\ &\quad\ge f(axa)\wedge(g\circ h)((xa)(yyxay))\wedge\frac{1-k}{2}\\ &\quad\ge f(a)\wedge\left(\bigvee_{(xa)(yyxay)=bc}\{g(b)\wedge h(c)\}\right)\wedge\frac{1-k}{2}\\ &\quad\ge f(a)\wedge g(xa)\wedge h(yyxay)\wedge\frac{1-k}{2}\\ &\quad\ge f(a)\wedge g(a)\wedge h(a)\wedge\frac{1-k}{2}\\ &\quad\ge f(a)\wedge g(a)\wedge h(a)\wedge\frac{1-k}{2}. \end{aligned} \tag{17}$$

Therefore $f\wedge_k g\wedge_k h\le f\circ_k g\circ_k h$.

(iii) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i): Let $B[a]$, $L[a]$, and $I[a]$ be bi-ideal, left ideal, and interior ideal of S generated by a, respectively.

Then $(C_{B[a]})_k$, $(C_{L[a]})_k$, and $(C_{I[a]})_k$ are $(\in,\in\vee q_k)$-fuzzy bi-ideal, $(\in,\in\vee q_k)$-fuzzy left ideal, and $(\in,\in\vee q_k)$-fuzzy interior ideal of semigroup S. Let $a\in S$ and $b\in B[a]\cap L[a]\cap I[a]$. Then $b\in B[a]$, $b\in L[a]$, and $b\in I[a]$. Now

$$\begin{aligned} \frac{1-k}{2} &\le\left(C_{B[a]\cap L[a]\cap I[a]}\right)_k(b)\\ &=\left(\left(C_{B[a]}\right)_k\wedge_k\left(C_{L[a]}\right)_k\wedge_k\left(C_{I[a]}\right)_k\right)(b)\\ &\le\left(\left(C_{B[a]}\right)_k\circ_k\left(C_{L[a]}\right)_k\circ_k\left(C_{I[a]}\right)_k\right)(b)\\ &=\left(C_{B[a]L[a]I[a]}\right)_k(b). \end{aligned} \tag{18}$$

Thus $b\in B[a]L[a]I[a]$. Therefore $B[a]\cap L[a]\cap I[a]\subseteq B[a]L[a]I[a]$. Hence by Theorem 24, S is right weakly regular semigroup. □

Theorem 26. *For a semigroup S, the following conditions are equivalent:*

(i) *S is right weakly regular;*

(ii) *$Q\cap L\cap I\subseteq QLI$ for every quasi-ideal Q, left ideal L, and interior ideal I of S, respectively;*

(iii) *$Q[a]\cap L[a]\cap I[a]\subseteq Q[a]L[a]I[a]$.*

Proof. (i) $\Rightarrow$ (ii): Let S be right weakly regular semigroup, and let Q, L, and I be quasi-ideal, left ideal, and interior ideal of S, respectively. Let $a \in Q \cap L \cap I$ then $a \in Q$, $a \in L$, and $a \in I$. Since S is right weakly regular semigroup so for a there exist $x, y \in S$ such that

$$\begin{aligned} a &= axay = (axay)(xay) \\ &= a(xa)(yxay) \in Q(SL)(SIS) \subseteq QLI. \end{aligned} \tag{19}$$

Therefore $a \in QLI$. So $Q \cap L \cap I \subseteq QLI$.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ *(i)*: As $a \cup (aS \cap Sa)$, $a \cup Sa$, and $a \cup a^2 \cup SaS$ are quasi-ideal, left ideal, and interior ideal of S generated by an element a of S, respectively, thus by assumption we have

$$\begin{aligned} &(a \cup (aS \cap Sa)) \cap (a \cup Sa) \cap (a \cup a^2 \cup SaS) \\ &\quad \subseteq (a \cup (aS \cap Sa))(a \cup Sa)(a \cup a^2 \cup SaS) \\ &\quad \subseteq (a \cup aS)(a \cup Sa)(a \cup a^2 \cup SaS) \\ &\quad = (a^2 \cup aSa \cup aSa \cup aSSa)(a \cup a^2 \cup SaS) \\ &\quad = a^3 \cup a^4 \cup a^2SaS \cup aSa^2 \cup aSa^3 \cup aSaSaS \\ &\qquad \cup aSSa^2 \cup aSSa^3 \cup aSSaSaS \\ &\quad \subseteq a^3 \cup a^4 \cup aSa \cup aSaS. \end{aligned} \tag{20}$$

Thus $a = a^4 = aaaa$ or $a = a^3 = aaa = aaa^3 = aaa^2a$ or $a = axa = axaxa$ or $a = auav$, for some x, u, v in S. Hence S is right weakly regular semigroup. □

Theorem 27. *For a semigroup S, the following conditions are equivalent:*

(i) *S is right weakly regular;*

(ii) *$f \wedge g \wedge h \leq f \circ g \circ h$ for every fuzzy quasi-ideal, fuzzy left ideal, and fuzzy interior ideal of S, respectively.*

Proof. (i) $\Rightarrow$ (iii): Let f, g, and h be any $(\in, \in \vee q_k)$-fuzzy quasi-ideal, $(\in, \in \vee q_k)$-fuzzy left ideal, and $(\in, \in \vee q_k)$-fuzzy interior ideal of S. Since S is right weakly regular therefore for each $a \in S$ there exist $x, y \in S$ such that

$$a = axay = (axay)(xay) = a(xa)(yxay). \tag{21}$$

Then

$$\begin{aligned} &(f \circ_k g \circ_k h)(a) \\ &\quad = (f \circ g \circ h)(a) \wedge \frac{1-k}{2} \\ &\quad = \left(\bigvee_{a=pq} \{f(p) \wedge (g \circ h)(q)\}\right) \wedge \frac{1-k}{2} \\ &\quad \geq f(a) \wedge (g \circ h)((xa)(yxay)) \wedge \frac{1-k}{2} \\ &\quad \geq f(a) \wedge \left(\bigvee_{(xa)(yxay)=bc} \{g(b) \wedge h(c)\}\right) \wedge \frac{1-k}{2} \\ &\quad \geq f(a) \wedge g(xa) \wedge h(yxay) \wedge \frac{1-k}{2} \\ &\quad \geq f(a) \wedge g(a) \wedge h(a) \wedge \frac{1-k}{2} \\ &\quad \geq f(a) \wedge g(a) \wedge h(a) \wedge \frac{1-k}{2}. \end{aligned} \tag{22}$$

Therefore $f \wedge_k g \wedge_k h \leq f \circ_k g \circ_k h$.

(iii) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i): Let $Q[a]$, $L[a]$, and $I[a]$ be quasi-ideal, left ideal and interior ideal of S generated by a, respectively.

Then $(C_{Q[a]})_k$, $(C_{L[a]})_k$, and $(C_{I[a]})_k$ are $(\in, \in \vee q_k)$-fuzzy quasi-ideal, $(\in, \in \vee q_k)$-fuzzy left ideal, and $(\in, \in \vee q_k)$-fuzzy interior ideal of semigroup S. Let $a \in S$ and let $b \in Q[a] \cap L[a] \cap I[a]$. Then $b \in Q[a]$, $b \in L[a]$, and $b \in I[a]$. Now

$$\begin{aligned} \frac{1-k}{2} &\leq (C_{Q[a] \cap L[a] \cap I[a]})_k(b) \\ &= \left((C_{Q[a]})_k \wedge_k (C_{L[a]})_k \wedge_k (C_{I[a]})_k\right)(b) \\ &\leq \left((C_{Q[a]})_k \circ_k (C_{L[a]})_k \circ_k (C_{I[a]})_k\right)(b) \\ &= (C_{Q[a]L[a]I[a]})_k(b). \end{aligned} \tag{23}$$

Thus $b \in Q[a]L[a]I[a]$. Therefore $Q[a] \cap L[a] \cap I[a] \subseteq Q[a]L[a]I[a]$. Hence by Theorem 26, S is right weakly regular semigroup. □

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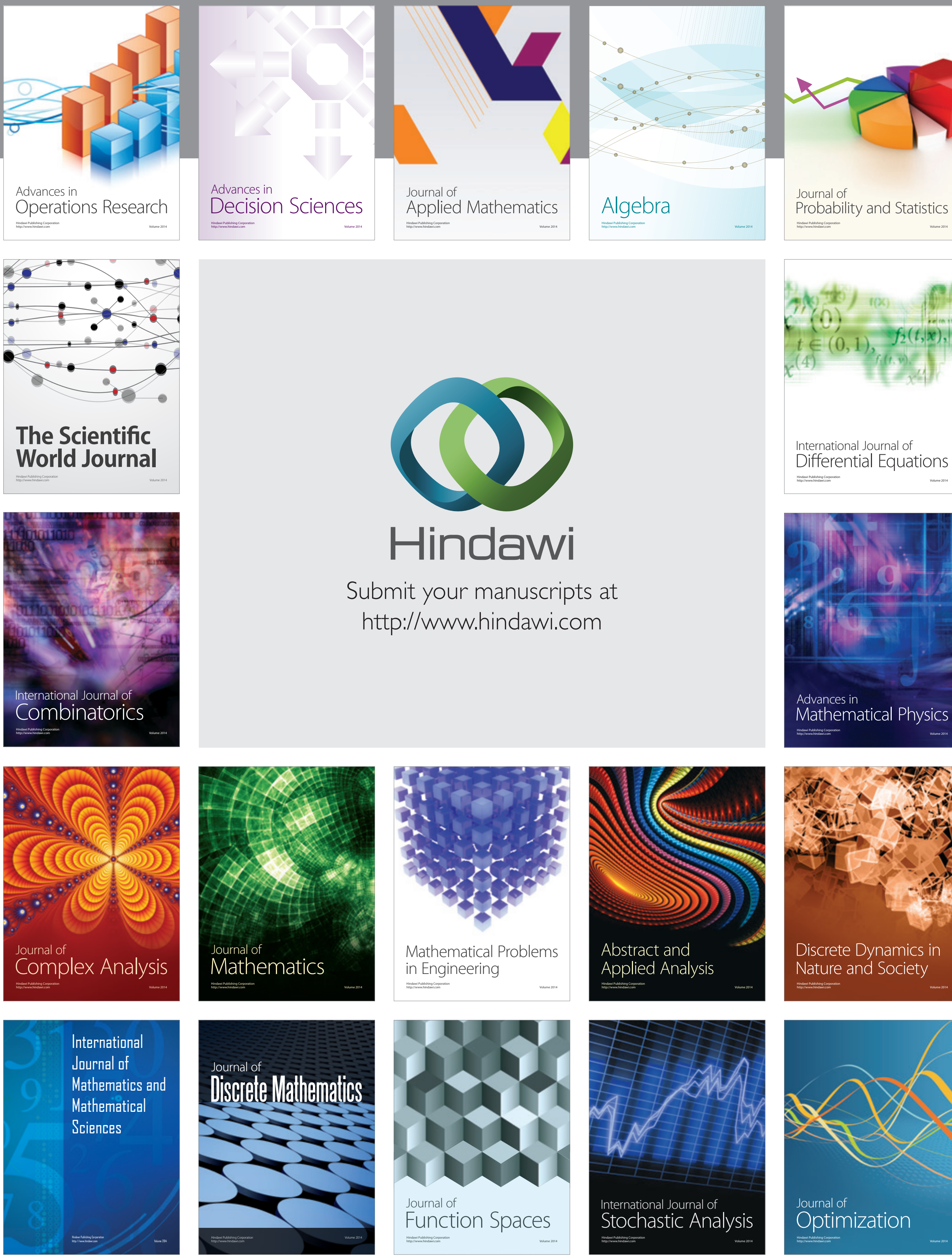
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