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Research Article

Exact Multiplicity of Solutions for a Class of Singular Generalized One-Dimensional p-Laplacian Problem

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We describe the existence of positive solutions for a class of singular generalized one-dimensional p-Laplacian problem. By applying the related fixed point theory in cone, some new and general results on the existence of positive solutions to the singular generalized p-Laplacian problem are obtained. Note that the nonlinear term f involves the first-order derivative explicitly.

1. Introduction

Recently, increasing attention is paid to question of positive solution for singular boundary value problems [1–11]. One notices that the singular boundary value problems for ordinary differential equation describe many phenomena in applied mathematics and physical science, which can be found in the theory of nonlinear diffusion generated by nonlinear sources and in the thermal ignition of gases [12–17]. Moreover, there are excellent results of nonsingular problem, see [18–21] and the references therein.

Moreover, the nonlocal p-Laplacian problems for ordinary differential equation have been studied extensively. There are many papers dealing with the existence of positive solutions for the nonlocal p-Laplacian boundary value problem, in which the nonlinear term f is independent of the first-order derivative with different boundary conditions. For example, Ma et al. [22] established some existence results of positive solutions for the problem

$$\begin{gathered}\left(\varphi_p\left(u'\right)\right)' + q(t) f(t,u) = 0, \quad t \in (0,1), \\ u'(0) = \sum_{i=1}^{m-2} \alpha_i u'(\xi_i), \quad u(1) = \sum_{i=1}^{m-2} \beta_i u(\xi_i).\end{gathered} \tag{1}$$

The main tool is the monotone iterative technique. By applications of fixed point index theory, Cheung and Ren [23] investigated two classes of quasilinear nonlocal problems:

$$\begin{gathered}\left(\varphi_p\left(u'\right)\right)' + h(t) f(t,u) = 0, \quad 0 < t < 1, \\ u'(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \\ u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad u'(1) = 0.\end{gathered} \tag{2}$$

Sufficient conditions for the existence of twin positive solutions are established. In [16], Feng and Ge studied nonlocal problem for the one-dimensional p-Laplacian

$$\begin{gathered}\left(\varphi_p\left(u'\right)\right)' + q(t) f\left(t,u,u'\right) = 0, \quad t \in (0,1), \\ u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i).\end{gathered} \tag{3}$$

They obtained sufficient conditions for the existence of at least three solutions to the above problem by using the Avery and Peterson fixed point theorem.

However, there are not many concerning the boundary value problems, in which a generalized p-Laplacian equation with a nonlocal boundary conditions. The motivation for the present work stems from many recent investigations in [16, 23–25] and references therein. Our purpose of this paper is to establish some sufficient conditions for the existence of

triple positive solutions to a class of a singular generalized one-dimensional p-Laplacian problem

$$\left(r(t)\varphi_p\left(u'\right)\right)' + h(t) f\left(t, u(t), u'(t)\right) = 0, \quad t \in (a,b),$$
$$\alpha_1 u(a) - \beta_1 u'(a) = \sum_{i=1}^{m-2} \gamma_i u(\xi_i), \quad u'(b) = 0, \tag{$\mathscr{P}$}$$

where $\varphi_p(u)$ is p-Laplacian operator; that is, $\varphi_p(u) = |u|^{p-2}u$, $p > 1$, $(\varphi_p)^{-1}(u) = \varphi_q(u)$, $1/p + 1/q = 1$, and $\alpha_1, \beta_1, \gamma_i, \xi_i, r, h, f$ satisfy

- (A_1) $\alpha_1, \beta_1 \in [0,\infty)$, $\gamma_i \in [0,\infty)$ satisfy $\alpha_1 > \sum_{i=1}^{m-2}\gamma_i$, $a < \xi_1 < \cdots < \xi_{m-2} < b$ $(m > 3)$,
- (A_2) $r(t) \in C^1(a,b)$, $r(t)$ is positive and nondecreasing on $[a,b]$,
- (A_3) $h \in L^1((a,b),[0,\infty))$ may be singular at $t = 0$ and/or $t = 1$, and $0 < \int_\Gamma h(t)dt < \infty$, where $\Gamma \subseteq [a,b]$,
- (A_4) $f : [a,b] \times [0,\infty) \times (-\infty,\infty) \to [0,\infty)$ is an L^1-Carathédory function; that is, for each $(t,x,y) \in [a,b]\times[0,\infty)\times(-\infty,\infty)$, the mapping $t \to f(t,x,y)$ is Lebesgue measurable on $[a,b]$; for a.e. $t \in [a,b]$, the mapping $(t,x,y) \to f(t,x,y)$ is continuous on $[0,\infty)\times(-\infty,\infty)$.

Under the above assumptions, some new and general results on the existence of multiple positive solutions to singular one-dimensional p-Laplacian are obtained. Our results develop some results of Cheung and Ren [23] and include and improve the main results of Feng and Ge [16] and Zhang [25].

The rest of the paper is organized as follows. In Section 2, we provide some background material from the theory of cones in the Banach spaces, which are useful later. Some lemmas and criteria for the existence of three positive solutions for one-dimensional p-Laplacian problems are established in Section 3. Finally, we give an example to illustrate our main results.

2. Preliminaries

Let γ and θ be nonnegative continuous convex functionals on a cone K, α a nonnegative continuous concave functional on K, β a nonnegative continuous functional on K, and m_1, m_2, m_3, and m_4 positive numbers. We define the following convex sets and a closed set

$$\begin{aligned} P(\gamma, m_4) &= \{u \in K : \gamma(u) < m_4\}, \\ P(\gamma, \alpha, m_2, m_4) &= \{u \in K : m_2 \le \alpha(u), \gamma(u) \le m_4\}, \\ P(\gamma, \theta, \alpha, m_2, m_3, m_4) &= \{u \in K : m_2 \le \alpha(u), \theta(u) \le m_3, \gamma(u) \le m_4\}, \\ Q(\gamma, \beta, m_1, m_4) &= \{u \in K : m_1 \le \beta(u), \gamma(u) \le m_4\}. \end{aligned} \tag{4}$$

Now we state the fixed point theorem due to Avery and Peterson [26].

Lemma 1. *Let K be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on K, α a nonnegative continuous concave functional on K, and β a nonnegative continuous functional on K satisfying $\beta(\rho u) \le \rho\beta(u)$ for $0 \le \rho \le 1$, such that, for some positive numbers ϵ and m_4,*

$$\alpha(u) \le \beta(u), \quad \|u\| \le \epsilon\gamma(u) \quad \forall u \in \overline{P(\gamma, m_4)}. \tag{5}$$

Suppose that $T : \overline{P(\gamma, m_4)} \to \overline{P(\gamma, m_4)}$ is completely continuous and there are positive numbers m_1, m_2, and m_3 with $m_1 < m_2$ such that

- (B_1) $\{u \in P(\gamma, \theta, \alpha, m_2, m_3, m_4) : \alpha(u) > m_2\} \ne \emptyset$, $\alpha(Tu) > m_2$ *for* $u \in P(\gamma, \theta, \alpha, m_2, m_3, m_4)$,
- (B_2) $\alpha(Tu) > m_2$ *for* $u \in P(\gamma, \alpha, m_2, m_4)$ *with* $\theta(Tu) > m_3$,
- (B_3) $0 \notin Q(\gamma, \beta, m_1, m_4)$ *and* $\beta(Tu) < m_1$ *for* $u \in Q(\gamma, \beta, m_1, m_4)$ *with* $\beta(u) = m_1$.

Then T has at least three fixed points $u_1, u_2, u_3 \in \overline{P(\gamma, m_4)}$ such that

$$\begin{gathered} \gamma(u_i) \le m_4 \quad (i = 1,2,3), \\ m_2 < \alpha(u_1), \quad m_1 < \beta(u_2) \quad \text{with } \alpha(u_2) < m_2, \\ \beta(u_3) < m_1. \end{gathered} \tag{6}$$

Denote that $L^+[a,b] = \{v \in L[a,b] : v(t) \ge 0, t \in [a,b]\}$.

Lemma 2. *Assume that (A_1) and (A_2) hold. If $g \in L^+[a,b]$, then the boundary value problem*

$$\left(r(t)\varphi_p\left(u'\right)\right)' + g(t) = 0, \quad t \in (a,b), \tag{7}$$

$$\alpha_1 u(a) - \beta_1 u'(a) = \sum_{i=1}^{m-2} \gamma_i u(\xi_i), \quad u'(b) = 0, \tag{8}$$

has a unique solution

$$\begin{aligned} u(t) = {} & \int_a^t \varphi_q\left(\frac{1}{r(s)}\int_s^b g(\tau)\,d\tau\right)ds + \frac{1}{\alpha_1 - \sum_{i=1}^{m-2}\gamma_i} \\ & \times\left(\beta_1\varphi_q\left(\frac{1}{r(a)}\int_a^b g(s)\,ds\right)\right. \\ & \left. + \sum_{i=1}^{m-2}\gamma_i\int_a^{\xi_i}\varphi_q\left(\frac{1}{r(s)}\int_s^b g(\tau)\,d\tau\right)ds\right). \end{aligned} \tag{9}$$

Proof. For a.e. $t \in [a,b]$, integrating (7) from t to b, in view of (8), we have

$$u'(t) = \varphi_q\left(\frac{1}{r(t)}\int_t^b g(s)\,ds\right). \tag{10}$$

Integrating this from a to t yields

$$u(t) = u(a) + \int_a^t \varphi_q\left(\frac{1}{r(s)}\int_s^b g(\tau)\,d\tau\right)ds. \tag{11}$$

Again, it follows from (8) that

$$u(a) = \frac{1}{\alpha_1 - \sum_{i=1}^{m-2}\gamma_i} \times \left(\beta_1 \varphi_q\left(\frac{1}{r(a)}\int_a^b g(s)\,ds\right) + \sum_{i=1}^{m-2}\gamma_i \int_a^{\xi_i} \varphi_q\left(\frac{1}{r(s)}\int_s^b g(\tau)\,d\tau\right)ds\right). \tag{12}$$

The proof is complete. □

Lemma 3. *Assume that (A_1) and (A_2) hold. If $g \in L^+[a,b]$, then the unique solution of the problems* (7) *and* (8) *satisfies*

$$u(t) \geq 0, \quad \min_{t\in[\sigma,b-\sigma]} u(t) \geq \tau_1 \max_{t\in[a,b]} u(t), \tag{13}$$

where $\tau_1 = \beta_1/(\beta_1 + (b-a)(\alpha_1 - \sum_{i=1}^{m-2}\gamma_i))$, $\sigma \in (a,(a+b)/2)$.

Proof. The proof is similar to Lemma 3.6 in [25], we omit it. □

Let E be the Banach space $C^1[a,b]$ with the norm $\|u\| = \max\{\|u\|_0, \|u'\|_0\}$, where $\|u\|_0 = \max_{0\leq t\leq 1}|u(t)|$. Define a cone $K \subset E$ by

$$\begin{aligned} K = \Big\{ & u \in E : u(t) \geq 0 \text{ for } t \in [a,b], \\ & u''(t) \leq 0, u'(t) \geq 0 \text{ for } t \in (a,b), \\ & \alpha_1 u(a) - \beta_1 u'(a) = \sum_{i=1}^{m-2}\gamma_i u(\xi_i), \\ & u'(b) = 0, \min_{t\in[\sigma,b-\sigma]} u(t) \geq \tau_1 \max_{t\in[a,b]} u(t) \Big\}. \end{aligned} \tag{14}$$

Let the assumptions (A_1) and (A_2) hold. Denote that

$$\lambda = \int_a^b \left(\frac{b-s}{r(s)}\right)^{1/(p-1)} ds + \frac{1}{\alpha_1 - \sum_{i=1}^{m-2}\gamma_i} \times \left(\beta_1\left(\frac{b-a}{r(a)}\right)^{1/(p-1)} + \sum_{i=1}^{m-2}\gamma_i \int_a^{\xi_i}\left(\frac{b-s}{r(s)}\right)^{1/(p-1)} ds\right). \tag{15}$$

Evidently, $\lambda \geq 0$. We take

$$u_0(t) = \int_a^t \left(\overline{m}\frac{b-s}{r(s)}\right)^{1/(p-1)} ds + \frac{1}{\alpha_1 - \sum_{i=1}^{m-2}\gamma_i} \times \left(\beta_1\left(\overline{m}\frac{b-a}{r(a)}\right)^{1/(p-1)} + \sum_{i=1}^{m-2}\gamma_i \int_a^{\xi_i}\left(\overline{m}\frac{b-s}{r(s)}\right)^{1/(p-1)} ds\right), \tag{16}$$

where $\overline{m}$ satisfies $(\overline{m})^{1/(p-1)} := m_2/\tau_1^2\lambda$, $m_2 > 0$ is constant. It is not difficult to check that $u_0(t) \geq 0$ for a.e. $t \in [a,b]$, $u''(t) \leq 0$, $u'(t) \geq 0$ for a.e. $t \in (a,b)$, and $u_0(t)$ satisfies $\alpha_1 u_0(a) - \beta_1 u_0'(a) = \sum_{i=1}^{m-2}\gamma_i u_0(\xi_i)$, $u_0'(b) = 0$. That is to say $u_0 \in K$. Hence, $K \setminus \{0\} \neq \emptyset$.

To obtain the existence of solutions for the problem $(\mathscr{P})$, the following priori estimate is needful.

Lemma 4. *Assume that (A_1) holds. If $u \in K$, then*

$$\max_{t\in[a,b]} u(t) \leq \tau_2 \max_{t\in[a,b]} |u'(t)|, \tag{17}$$

where $\tau_2 = (\beta_1 + \alpha_1(b-a))/(\alpha_1 - \sum_{i=1}^{m-2}\gamma_i)$.

Proof. The proof is similar to Lemma 3.1 in [21], we omit it. □

Let the nonnegative continuous convex functionals γ and θ, the nonnegative continuous concave functional α, and the nonnegative continuous functional β be defined on cone K by

$$\alpha(u) = \min_{t\in[\sigma,b-\sigma]} u(t), \qquad \beta(u) = \theta(u) = \max_{t\in[a,b]} u(t), \qquad \gamma(u) = \max_{t\in[a,b]} |u'(t)|. \tag{18}$$

In the view of assumption (A_3), that is, $0 < \int_a^b h(\tau)d\tau < \infty$ and (A_3). Thus, we can define an operator T by

$$\begin{aligned} (Tu)(t) &:= \int_a^t \varphi_q\left(\frac{1}{r(s)}\int_s^b h(\tau) f(\tau, u(\tau), u'(\tau))\,d\tau\right)ds \\ &\quad + \frac{1}{\alpha_1 - \sum_{i=1}^{m-2}\gamma_i} \times \left(\beta_1 \varphi_q\left(\frac{1}{r(a)}\int_a^b h(\tau) f(\tau, u(\tau), u'(\tau))\,d\tau\right) + \sum_{i=1}^{m-2}\gamma_i \int_a^{\xi_i}\varphi_q\left(\frac{1}{r(s)}\int_s^b h(\tau) \times f(\tau, u(\tau), u'(\tau))\,d\tau\right)ds\right) \\ &\quad \text{for a.e. } t \in [a,b]. \end{aligned} \tag{19}$$

Lemma 5. *Assume that (A_1)–(A_4) hold. Then $T : K \to K$ is completely continuous.*

Proof. For any $u \in K$, by the definition of Tu, we see that $Tu \in C^1[a,b]$, $(\varphi_p((Tu)'))' \in L^1[a,b]$. From (A_1)–(A_4) and Lemma 3, we have $(Tu)(t) \geq 0$, $(\varphi_p((Tu)'))' \leq 0$ implies that $(Tu)(t)$ is concave on $[a,b]$, $-(\varphi_p((Tu)'))' = h(t)f(t,(Tu)(t), (Tu)'(t)) \geq 0$, $\alpha_1 Tu(a) - \beta_1 Tu'(a) = \sum_{i=1}^{m-2}\gamma_i Tu(\xi_i)$, and

$Tu'(b) = 0$. So we can conclude that $Tu \in K$. In what follows, we prove that T is compact. Let $R > 0$, there exists a set Ω such that $\Omega \subset \{u \in K : \|u'\|_0 \leq R\}$. For any $u \in \Omega$, the assumptions imply that

$$0 \leq \int_a^b h(t) f(t, u(t), u'(t)) dt \leq \max_{(t,u,u') \in [a,b]\times[0,\tau_2 R]\times[-R,R]} f(t,u,u') \int_a^b h(t) dt < \infty. \tag{20}$$

We take the arguments to show that the operator T is completely continuous. In view of Lemma 4 and the continuity of f in u and u', assume that $u_n, u_0 \in K$ satisfy $\|u_n - u_0\| \to 0$ $(n \to \infty)$. Moreover, $\varphi_q(v)$ is an increasing and continuously function, and we obtain that

$$\begin{aligned}
&\|Tu_n - Tu_0\|_0 \\
&= \max_{a\leq t\leq b} |Tu_n(t) - Tu_0(t)| \\
&\leq \int_a^b \left(\varphi_q\left(\frac{1}{r(s)} \int_s^b h(\tau) f(\tau, u_n(\tau), u_n'(\tau)) d\tau \right) \right. \\
&\qquad \left. - \varphi_q\left(\frac{1}{r(s)} \int_s^b h(\tau) f(\tau, u_0(\tau), u_0'(\tau)) d\tau \right) \right) ds \\
&+ \frac{\beta_1}{\alpha_1 - \sum_{i=1}^{m-2} \gamma_i} \\
&\times \left(\varphi_q\left(\frac{1}{r(a)} \int_a^b h(\tau) f(\tau, u_n(\tau), u_n'(\tau)) d\tau \right) \right. \\
&\qquad \left. - \varphi_q\left(\frac{1}{r(a)} \int_a^b h(\tau) f(\tau, u_0(\tau), u_0'(\tau)) d\tau \right) \right) \\
&+ \frac{1}{\alpha_1 - \sum_{i=1}^{m-2} \gamma_i} \sum_{i=1}^{m-2} \gamma_i \\
&\times \int_a^{\xi_i} \left(\varphi_q\left(\frac{1}{r(s)} \int_s^b h(\tau) f(\tau, u_n(\tau), u_n'(\tau)) d\tau \right) ds \right. \\
&\qquad - \varphi_q\left(\frac{1}{r(s)} \int_s^b h(\tau) \right. \\
&\qquad \left. \left. \times f(\tau, u_0(\tau), u_0'(\tau)) d\tau \right) ds \right) \longrightarrow 0 \\
&\qquad (n \longrightarrow \infty),
\end{aligned}$$

$$\begin{aligned}
&\|(Tu_n)' - (Tu_0)'\|_0 \\
&= \max_{a\leq t\leq b} |(Tu_n)'(t) - (Tu_0)'(t)| \\
&\leq \left| \varphi_q\left(\frac{1}{r(a)} \int_a^b h(\tau) f(\tau, u_n(\tau), u_n'(\tau)) d\tau \right) \right. \\
&\qquad \left. - \varphi_q\left(\frac{1}{r(a)} \int_a^b h(\tau) f(\tau, u_0(\tau), u_0'(\tau)) d\tau \right) \right| \longrightarrow 0 \\
&\qquad (n \longrightarrow \infty).
\end{aligned} \tag{21}$$

Therefore,

$$\|Tu_n - Tu_0\| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{22}$$

This means that the operator $T : K \to K$ is continuous.

Since $h(t)$ may be singular at $t = a$ and/or $t = b$, choose two sequences $\{\mu_n\}_{n=1}^{\infty}, \{\nu_n\}_{n=1}^{\infty} \subset (a,b)$ satisfying $\mu_n \leq \nu_n$ for any $n \geq 1$, such that $\mu_n \to a$ and $\alpha_n \to b$ as $n \to \infty$, respectively. Define

$$h_n(t) = \begin{cases} \inf\limits_{a\leq t\leq \mu_n} h(t), & a \leq t \leq \mu_n, \\ h(t), & \mu_n \leq t \leq \nu_n, \\ \inf\limits_{\nu_n\leq t\leq b} h(t), & \nu_n \leq t \leq b, \end{cases} \tag{23}$$

and an operator sequence T_n by

$$\begin{aligned}
&(T_n u)(t) \\
&= \int_a^t \varphi_q\left(\frac{1}{r(s)} \int_s^b h_n(\tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \\
&+ \frac{1}{\alpha_1 - \sum_{i=1}^{m-2} \gamma_i} \\
&\times \left(\beta_1 \varphi_q\left(\frac{1}{r(a)} \int_a^b h_n(\tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) \right. \\
&\qquad + \sum_{i=1}^{m-2} \gamma_i \\
&\qquad \times \int_a^{\xi_i} \varphi_q\left(\frac{1}{r(s)} \int_s^b h_n(\tau) \right. \\
&\qquad \left. \left. \times f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \right) \\
&\qquad \text{for a.e. } t \in [a,b].
\end{aligned} \tag{24}$$

Clearly, $h_n(t) : [a,b] \to [0,\infty)$ is a piecewise continuous function, and the operator $T_n : K \to K$ is well defined. Further, we can see that $T_n : K \to K$ is completely continuous.

Let $r > 0$, $\mathbb{B}_r := \{u \in K : \|u'\|_0 \le r\}$ and $M_r = \max\{f(t,u,u') : u \in K, t \in [a,b], u \in [0,\tau_2 r], u' \in [-r,r]\}$. We will prove that T_n approaches T uniformly on $\mathbb{B}_r$. From the absolute continuity of integral, we obtain

$$\lim_{n\to\infty}\int_{\Gamma(n)}\varphi_q\left(\frac{1}{r(s)}\int_s^b h(\tau)f\left(\tau,u(\tau),u'(\tau)\right)d\tau\right)ds = 0, \tag{25}$$

where $\Gamma(n) = [a,\mu_n]\cup[\nu_n,b]$. For any $u \in \mathbb{B}_r$ and a.e. $t \in [a,\mu_n]$, one has that

$$\begin{aligned}
&\|T_n u - Tu\|_0 \\
&= \max_{a\le t\le\mu_n}\left|T_n u(t) - Tu(t)\right| \\
&\le \int_a^{\mu_n}\varphi_q\left(\frac{1}{r(s)}\int_s^b h_n(\tau)f\left(\tau,u(\tau),u'(\tau)\right)d\tau\right)ds \\
&\quad - \varphi_q\left(\frac{1}{r(s)}\int_s^b h(\tau)f\left(\tau,u(\tau),u'(\tau)\right)d\tau\right)ds \\
&\quad + \frac{\beta_1}{\alpha_1-\sum_{i=1}^{m-2}\gamma_i} \\
&\quad \times\left(\varphi_q\left(\frac{1}{r(a)}\int_a^b h_n(\tau)f\left(\tau,u(\tau),u'(\tau)\right)d\tau\right)\right. \\
&\qquad \left.-\varphi_q\left(\frac{1}{r(a)}\int_a^b h(\tau)f\left(\tau,u(\tau),u'(\tau)\right)d\tau\right)\right) \\
&\quad + \frac{1}{\alpha_1-\sum_{i=1}^{m-2}\gamma_i}\sum_{i=1}^{m-2}\gamma_i \\
&\quad \times\int_a^{\xi_i}\left(\varphi_q\left(\frac{1}{r(s)}\int_s^b h_n(\tau)f\left(\tau,u(\tau),u'(\tau)\right)d\tau\right)\right. \\
&\qquad \left.-\varphi_q\left(\frac{1}{r(s)}\int_s^b h(\tau)f\left(\tau,u(\tau),u'(\tau)\right)d\tau\right)ds\right) \\
&\le \int_{\Gamma(n)}\varphi_q\left(\frac{1}{r(s)}\int_s^b h_n(\tau)f\left(\tau,u(\tau),u'(\tau)\right)d\tau\right)ds \\
&\quad - \varphi_q\left(\frac{1}{r(s)}\int_s^b h(\tau)f\left(\tau,u(\tau),u'(\tau)\right)d\tau\right)ds \\
&\quad + \frac{\beta_1}{\alpha_1-\sum_{i=1}^{m-2}\gamma_i} \\
&\quad \times\left(\varphi_q\left(\frac{1}{r(a)}\int_a^b h_n(\tau)f\left(\tau,u(\tau),u'(\tau)\right)d\tau\right)\right. \\
&\qquad \left.-\varphi_q\left(\frac{1}{r(a)}\int_a^b h(\tau)f\left(\tau,u(\tau),u'(\tau)\right)d\tau\right)\right) \\
&\quad + \frac{1}{\alpha_1-\sum_{i=1}^{m-2}\gamma_i}\sum_{i=1}^{m-2}\gamma_i \\
&\quad \times\int_a^{\xi_i}\left(\varphi_q\left(\frac{1}{r(s)}\int_s^b h_n(\tau)f\left(\tau,u(\tau),u'(\tau)\right)d\tau\right)\right. \\
&\qquad \left.-\varphi_q\left(\frac{1}{r(s)}\int_s^b h(\tau)f\left(\tau,u(\tau),u'(\tau)\right)d\tau\right)ds\right).
\end{aligned} \tag{26}$$

For $u \in \mathbb{B}_r$ and a.e. $t \in [\nu_n,b]$, we have the same result. It is easy to see that, for any $u \in \mathbb{B}_r$ and a.e. $t \in [\mu_n,\nu_n]$, there is $\|T_n u - Tu\|_0 = 0$. Similarly, we can obtain that, for any $u \in \mathbb{B}_r$ and a.e. $t \in [a,\mu_n]$, $t \in [\mu_n,\nu_n]$, $t \in [\nu_n,b]$, respectively,

$$\begin{aligned}
&\left\|(T_n u)' - (Tu)'\right\|_0 \\
&\quad = \max_t\left|(T_n u)'(t) - (Tu)'(t)\right| \\
&\quad \le \left|\varphi_q\left(\frac{1}{r(a)}\int_a^b h_n(\tau)f\left(\tau,u(\tau),u'(\tau)\right)d\tau\right)\right. \\
&\qquad \left.-\varphi_q\left(\frac{1}{r(a)}\int_a^b h(\tau)f\left(\tau,u(\tau),u'(\tau)\right)d\tau\right)\right|.
\end{aligned} \tag{27}$$

In view of the continuity of function φ_q, from the above arguments, we have

$$\|T_n u - Tu\| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{28}$$

That is to say, the sequence T_n is uniformly an approximate T on any bounded subset of K. Applying the Arzela-Ascoli lemma to the operator T, we can conclude that $T\Omega$ is relatively compact; that is, T is completely continuous. □

3. Main Results

We are ready to apply the Avery and Peterson fixed point theorem to the operator T to give sufficient conditions for the existence of at least three positive solutions to the problem $(\mathscr{P})$. For convenience, we introduce the following notation:

$$\begin{gathered}
L = \frac{1}{r(a)}\int_a^b h(\tau)\,d\tau, \qquad M = \frac{\alpha_1-\sum_{i=1}^{m-2}\gamma_i}{\beta_1}, \\
N = b - a + \frac{\beta_1+\sum_{i=1}^{m-2}\gamma_i(\xi_i-a)}{\alpha_1-\sum_{i=1}^{m-2}\gamma_i}.
\end{gathered} \tag{29}$$

Theorem 6. *Suppose (A_1)–(A_4) hold and $f(t,0,0) \not\equiv 0$ for a.e. $t \in [a,b]$. If there exist positive numbers m_1, m_2, and m_4 with $m_1 < m_2 \le \tau_1^2\lambda((b-a)/r(a))^{-1/(p-1)}m_4$ such that the following conditions are satisfied:*

(C_1) $f(t,\mu,\nu) \le (1/L)\varphi_p(m_4)$, $(t,\mu,\nu) \in [a,b]\times[0,\tau_2 m_4]\times[-m_4,m_4]$,

(C_2) $f(t,\mu,\nu) > (1/L)\,\varphi_p(m_2/\tau_1 M)$, $(t,\mu,\nu) \in [\sigma, b-\sigma] \times [m_2, m_2\tau_1^{-2}] \times [-m_4, m_4]$,

(C_3) $f(t,\mu,\nu) \le (1/L)\,\varphi_p(m_1/N)$, $(t,\mu,\nu) \in [a,b] \times [0, m_1] \times [-m_4, m_4]$,

then the problem $(\mathscr{P})$ *has at least three positive solutions* u_1, u_2, *and* u_3 *satisfying*

$$\begin{gathered}\max_{t\in[a,b]} \left|u_i'(t)\right| \le m_4 \quad \text{for } i = 1,2,3,\\ m_2 < \min_{t\in[\sigma,b-\sigma]} u_1(t), \quad m_1 < \max_{t\in[a,b]} u_2(t)\\ \text{with} \min_{t\in[\sigma,b-\sigma]} u_2(t) < m_2,\\ \max_{t\in[a,b]} u_3(t) < m_1.\end{gathered} \tag{30}$$

Proof. It is known that boundary value problem $(\mathscr{P})$ has a solution u if and only if u solves the operator equation $u = Tu$. By the definition of operator T, from Lemma 4 and the concavity of u, the functionals defined above satisfy $\sigma(b-\sigma)\theta(u) \le \alpha(u) \le \theta(u) = \beta(u)$, $\|u\| = \max\{\theta(u), \gamma(u)\} \le \max\{1, \tau_2\}\gamma(u)$ for any $u \in \overline{P(\gamma, m_4)} \subset K$. We also note that $\beta(\rho u) \le \rho\beta(u)$ for $0 \le \rho \le 1$, which suffices to show that the conditions of Lemma 1 hold with respect to T. First of all, we show that if the assumption (C_1) is satisfied, then

$$T : \overline{P(\gamma, m_4)} \longrightarrow \overline{P(\gamma, m_4)}. \tag{31}$$

In fact, for $u \in \overline{P(\gamma, m_4)}$, there is $\gamma(u) = \max_{t\in[a,b]}|u'(t)| \le m_4$. With Lemma 4, there is $\max_{t\in[a,b]}u(t) \le \tau_2\max_{t\in[a,b]}|u'(t)| \le \tau_2 m_4$, and the assumption (C_1) implies that $f(t,u(t),u'(t)) \le (1/L)\varphi_p(m_4)$ for a.e. $t \in [a,b]$. On the other hand, for $u \in K$, we obtain that $Tu \in K$, and so $(Tu)(t)$ is concave on $[a,b]$, and there is $\max_{t\in[a,b]}|(Tu)'(t)| = |(Tu)'(a)|$. By using of $r(t)$ is positive and monotone increasing on a.e. $[a,b]$, and so

$$\begin{aligned}\gamma(Tu) &= \max_{t\in[a,b]} \left|(Tu)'(t)\right|\\ &= \varphi_q\left(\frac{1}{r(a)}\int_a^b h(\tau) f\left(\tau, u(\tau), u'(\tau)\right) d\tau\right)\\ &\le \varphi_q\left(\frac{1}{L}\varphi_p(m_4)\frac{1}{r(a)}\int_a^b h(\tau)\,d\tau\right) = m_4.\end{aligned} \tag{32}$$

Therefore, (31) is satisfied.

To check if the condition (B_1) in Lemma 1 is fulfilled, we choose $u_0(t)$, where $u_0(t)$ is defined by (16). From the hypothesis conditions of the theorem, we have

$$\begin{aligned}\alpha(u_0) &= \min_{t\in[\sigma,b-\sigma]} u_0(t) = u_0(\sigma) \ge \tau_1(\overline{m})^{1/(p-1)}\lambda\\ &\ge m_2\tau_1^{-1} > m_2,\\ \theta(u_0) &= \max_{t\in[a,b]} u_0(t) = u_0(b) = (\overline{m})^{1/(p-1)}\lambda > m_2\tau_1^{-2} > m_2,\\ \gamma(u_0) &= \max_{t\in[a,b]} \left|(u_0)'(t)\right| = \left|(u_0)'(a)\right|\\ &= \left(\overline{m}\frac{b-a}{r(a)}\right)^{1/(p-1)} = \frac{m_2\tau_1^{-2}}{\lambda}\left(\frac{b-a}{r(a)}\right)^{1/(p-1)} \le m_4.\end{aligned} \tag{33}$$

Equation (33) implies that $\alpha(u_0) \ge \tau_1\theta(u_0)$. That is, $\min_{t\in[\sigma,b-\sigma]}u_0(t) \ge \tau_1\max_{t\in[a,b]}u_0(t)$. So for $u_0 \in K$, $u_0 \in P(\gamma,\theta,\alpha,m_2,m_2\tau_1^{-2},m_4)$ and $\{u_0 \in P(\gamma,\theta,\alpha,m_2,m_2\tau_1^{-2},m_4) : \alpha(u_0) > m_2\} \ne \emptyset$ hold. Hence, for $u \in P(\gamma,\theta,\alpha,m_2,m_2\tau_1^{-2},m_4)$, we see that $m_2 \le u(t) \le m_2\tau_1^{-2}$ and $|u'(t)| \le m_4$ for a.e. $t \in [\sigma, b-\sigma]$. Hence by the assumption (C_2), one has that $f(t,u(t),u'(t)) > (1/L)\varphi_p(m_2\tau_1^{-1}/M)$ for a.e. $t \in [\sigma, b-\sigma]$. By the definition of the functional α, we obtain

$$\begin{aligned}\alpha(Tu) &= \min_{t\in[\sigma,b-\sigma]}(Tu)(t) \ge \tau_1\max_{t\in[a,b]}(Tu)(t) = \tau_1(Tu)(b)\\ &= \tau_1\left(\int_a^b \varphi_q\left(\frac{1}{r(s)}\int_s^b h(\tau) f\left(\tau,u(\tau),u'(\tau)\right)d\tau\right)ds\right.\\ &\quad + \frac{\beta_1}{\alpha_1 - \sum_{i=1}^{m-2}\gamma_i}\varphi_q\\ &\quad \times\left(\frac{1}{r(a)}\int_a^b h(\tau) f\left(\tau,u(\tau),u'(\tau)\right)d\tau\right)\\ &\quad + \frac{1}{\alpha_1 - \sum_{i=1}^{m-2}\gamma_i}\sum_{i=1}^{m-2}\gamma_i\\ &\quad \times\int_a^{\xi_i}\varphi_q\left(\frac{1}{r(s)}\int_s^b h(\tau)\right.\\ &\qquad \left.\left.\times f\left(\tau,u(\tau),u'(\tau)\right)d\tau\right)ds\right)\\ &\ge \frac{\tau_1\beta_1}{\alpha_1 - \sum_{i=1}^{m-2}\gamma_i}\varphi_q\left(\frac{1}{L}\varphi_p\left(\frac{m_2\tau_1^{-1}}{M}\right)\frac{1}{r(a)}\right.\\ &\qquad \left.\times\int_a^b h(\tau)\,d\tau\right) = m_2.\end{aligned} \tag{35}$$

Therefore, we get $\alpha(Tu) > m_2$ for $u \in P(\gamma,\theta,\alpha,m_2,m_2\tau_1^{-2},m_4)$.

Next, we show that the condition (B_2) in Lemma 1 holds. In fact, if $u \in P(\gamma,\alpha,m_2,m_4)$ with $\theta(Tu) > m_2\tau_1^{-2}$, then

$$\begin{aligned}\alpha(Tu) &= \min_{t\in[\sigma,b-\sigma]}(Tu)(t) \ge \tau_1\max_{t\in[a,b]}(Tu)(t)\\ &= \tau_1\theta(Tu) > m_2\tau_1^{-1} > m_2.\end{aligned} \tag{36}$$

Finally, we assert that the condition (B_3) in Lemma 1 also holds. Since $\beta(0) = 0 < m_1$, there holds that $0 \notin Q(\gamma,\beta,$

$m_1, m_4)$. Assume that $u \in Q(\gamma, \beta, m_1, m_4)$ with $\beta(u) = m_1$. Then, by the assumption (C_3) we get

$$\begin{aligned}
\beta(Tu) &= \max_{t\in[a,b]} (Tu)(t) = (Tu)(b) \\
&= \int_a^b \varphi_q\left(\frac{1}{r(s)}\int_s^b h(\tau) f\left(\tau, u(\tau), u'(\tau)\right) d\tau\right) ds \\
&\quad + \frac{\beta_1}{\alpha_1 - \sum_{i=1}^{m-2}\gamma_i}\varphi_q \\
&\quad \times\left(\frac{1}{r(a)}\int_a^b h(\tau) f\left(\tau, u(\tau), u'(\tau)\right) d\tau\right) \\
&\quad + \frac{1}{\alpha_1 - \sum_{i=1}^{m-2}\gamma_i}\sum_{i=1}^{m-2}\gamma_i \\
&\quad \times\int_a^{\xi_i}\varphi_q\left(\frac{1}{r(s)}\int_s^b h(\tau) f\left(\tau, u(\tau), u'(\tau)\right) d\tau\right) ds \\
&\le \int_a^b \varphi_q\left(\frac{1}{r(a)}\int_a^b h(\tau) f\left(\tau, u(\tau), u'(\tau)\right) d\tau\right) ds \\
&\quad + \frac{\beta_1}{\alpha_1 - \sum_{i=1}^{m-2}\gamma_i} \\
&\quad \times\varphi_q\left(\frac{1}{r(a)}\int_a^b h(\tau) f\left(\tau, u(\tau), u'(\tau)\right) d\tau\right) \\
&\quad + \frac{1}{\alpha_1 - \sum_{i=1}^{m-2}\gamma_i}\sum_{i=1}^{m-2}\gamma_i \\
&\quad \times\int_a^{\xi_i}\varphi_q\left(\frac{1}{r(a)}\int_a^b h(\tau) f\left(\tau, u(\tau), u'(\tau)\right) d\tau\right) ds \\
&\le \int_a^b \varphi_q\left(\frac{1}{L}\varphi_p\left(\frac{m_1}{N}\right)\frac{1}{r(a)}\int_a^b h(\tau)\, d\tau\right) ds \\
&\quad + \frac{\beta_1}{\alpha_1 - \sum_{i=1}^{m-2}\gamma_i}\varphi_q\left(\frac{1}{L}\varphi_p\left(\frac{m_1}{N}\right)\frac{1}{r(a)}\int_a^b h(\tau)\, d\tau\right) \\
&\quad + \frac{1}{\alpha_1 - \sum_{i=1}^{m-2}\gamma_i}\sum_{i=1}^{m-2}\gamma_i \\
&\quad \times\int_a^{\xi_i}\varphi_q\left(\frac{1}{L}\varphi_p\left(\frac{m_1}{N}\right)\frac{1}{r(a)}\int_a^b h(\tau)\, d\tau\right) ds \\
&\le \frac{m_1}{N}\left(b - a + \frac{\beta_1 + \sum_{i=1}^{m-2}\gamma_i(\xi_i - a)}{\alpha_1 - \sum_{i=1}^{m-2}\gamma_i}\right) = m_1.
\end{aligned} \tag{37}$$

Thus, the conditions (B_1)–(B_3) in Lemma 1 hold, and so the problem $(\mathscr{P})$ has at least three positive solutions u_1, u_2, and u_3 such that (30) holds. □

Similar to the above arguments, we will discuss the problem

$$\begin{gathered}
\left(r(t)\varphi_p\left(u'\right)\right)' + h(t) f\left(t, u(t), u'(t)\right) = 0, \quad t \in (a,b), \\
u'(a) = 0, \qquad \alpha_2 u(b) + \beta_2 u'(b) = \sum_{i=1}^{m-2}\delta_i u(\xi_i).
\end{gathered} \tag{$\mathscr{P}^*$}$$

Now, we replace the assumption (A_1) with the following:

(A_1^*) $\alpha_2, \beta_2 \in [0,\infty)$, $\delta_i \in [0,\infty)$ satisfy $\alpha_2 > \sum_{i=1}^{m-2}\delta_i$, $a < \xi_1 < \cdots < \xi_{m-2} < b$ $(m > 3)$.

We only give the main results of the problem $(\mathscr{P}^*)$, and the proofs are omitted.

Lemma 7. *Assume that (A_1^*), (A_2), (A_3), and (A_4) hold. Then the problem $(\mathscr{P}^*)$ has a unique solution*

$$\begin{aligned}
u(t) &= \int_t^b \varphi_q\left(\frac{1}{r(s)}\int_a^s h(\tau) f\left(\tau, u(\tau), u'(\tau)\right) d\tau\right) ds \\
&\quad + \frac{1}{\alpha_2 - \sum_{i=1}^{m-2}\delta_i} \\
&\quad \times\left(\beta_2\varphi_q\left(\frac{1}{r(b)}\int_a^b h(\tau) f\left(\tau, u(\tau), u'(\tau)\right) d\tau\right)\right. \\
&\quad + \sum_{i=1}^{m-2}\delta_i\int_{\xi_i}^b \varphi_q \\
&\quad \left.\times\left(\frac{1}{r(s)}\int_a^s h(\tau) f\left(\tau, u(\tau), u'(\tau)\right) d\tau\right) ds\right).
\end{aligned} \tag{38}$$

The cone is defined by

$$\begin{aligned}
K^* = \Big\{ & u \in E : u(t) \ge 0 \text{ for } t \in [a,b], u''(t) \le 0, \\
& u'(t) \ge 0 \text{ for } t \in (a,b), u'(a) = 0, \\
& \alpha_2 u(b) + \beta_2 u'(b) = \sum_{i=1}^{m-2}\delta_i u(\xi_i) \\
& \text{and } \min_{t\in[\sigma,b-\sigma]} u(t) \ge \tau_1 \max_{t\in[a,b]} u(t) \Big\}.
\end{aligned} \tag{39}$$

Lemma 8. *Assume that (A_1) holds. If $u \in K^*$, then*

$$\min_{t\in[\sigma,b-\sigma]} u(t) \ge \tau_3\|u\|_0, \quad \|u\|_0 \le \tau_4\left\|u'\right\|_0, \tag{40}$$

where $\tau_3 = \beta_2/(\beta_2 + (b-a)(\alpha_2 - \sum_{i=1}^{m-2}\delta_i))$ and $\tau_4 = (\beta_2 + \alpha_2(b-a))/(\alpha_2 - \sum_{i=1}^{m-2}\delta_i)$.

In what follows, we define integral operators T and L^*, M^*, and N^*:

$$\begin{aligned}
&(Tu)(t) \\
&:= \int_t^b \varphi_q\left(\frac{1}{r(s)}\int_a^s h(\tau) f\left(\tau, u(\tau), u'(\tau)\right) d\tau\right) ds \\
&+ \frac{1}{\alpha_2 - \sum_{i=1}^{m-2}\delta_i} \\
&\times \left(\beta_2 \varphi_q\left(\frac{1}{r(b)}\int_a^b h(\tau) f\left(\tau, u(\tau), u'(\tau)\right) d\tau\right) + \sum_{i=1}^{m-2}\delta_i \right. \\
&\left. \times \int_{\xi_i}^b \varphi_q\left(\frac{1}{r(s)}\int_a^s h(\tau) f\left(\tau, u(\tau), u'(\tau)\right) d\tau\right) ds\right) \\
&\qquad \text{for a.e. } t \in [a,b],
\end{aligned}$$

$$\begin{aligned}
L^* &= \frac{1}{r(b)}\int_a^b h(\tau)\, d\tau, \\
M^* &= \frac{\alpha_2 - \sum_{i=1}^{m-2}\delta_i}{\beta_2}, \\
N^* &= b - a + \frac{\beta_2 + \sum_{i=1}^{m-2}\delta_i (b - \xi_i)}{\alpha_2 - \sum_{i=1}^{m-2}\delta_i}.
\end{aligned} \tag{41}$$

Theorem 9. *Suppose that (A_1^*), (A_2), (A_3), and (A_4) hold and $f(t,0,0) \not\equiv 0$ for a.e. $t \in [a,b]$. If there exist positive numbers λ', m_1, m_2, and m_4 with $m_1 < m_2 \le \tau_3^2\lambda'((b-a)/r(b))^{-1/(p-1)} m_4$ such that the following conditions are satisfied:*

- (D_1) $f(t,\mu,\nu) \le (1/L^*)\varphi_p(m_4/L^*)$, $(t,\mu,\nu) \in [a,b] \times [0,\tau_4 m_4] \times [-m_4, m_4]$,
- (D_2) $f(t,\mu,\nu) > (1/L^*)\varphi_p(m_2/M^*)$, $(t,\mu,\nu) \in [a,b] \times [m_2, m_2\tau_3^{-1}] \times [-m_4, m_4]$,
- (D_3) $f(t,\mu,\nu) \le (1/L^*)\varphi_p(m_1/N^*)$, $(t,\mu,\nu) \in [a,b] \times [0, m_1] \times [-m_4, m_4]$,

then the problem $(\mathscr{P}^*)$ has at least three positive solutions u_1, u_2, and u_3 satisfying

$$\begin{gathered}
\max_{t\in[a,b]} \left|u_i'(t)\right| \le m_4 \quad \text{for } i = 1,2,3, \\
m_2 < \min_{t\in[a,b]} u_1(t), \quad m_1 < \max_{t\in[a,b]} u_2(t) \\
\text{with } \min_{t\in[a,b]} u_2(t) < m_2, \\
\max_{t\in[a,b]} u_3(t) < m_1.
\end{gathered} \tag{42}$$

From Theorem 6 or Theorem 9, we can see that, when assumptions like (C_1), (C_2), and (C_3) are imposed appropriately on the nonlinear term f, we also establish the existence results of an arbitrary odd number of positive solutions of the problem $(\mathscr{P})$ or $(\mathscr{P}^*)$. Here, we only present the form of Theorem 6.

Theorem 10. *Suppose that (A_1)–(A_4) hold and $f(t,0,0) \not\equiv 0$ for a.e. $t \in [a,b]$. If there exist positive numbers λ, m_{1i}, m_{2i}, and m_{4i} $i = 1,2,\ldots$, with*

$$\begin{aligned}
m_{11} < m_{21} &\le \frac{\tau_1^2\lambda}{((b-a)/r(a))^{1/(p-1)}} m_{41} < m_{12} < m_{22} \\
&\le \frac{\tau_1^2\lambda}{((b-a)/r(a))^{1/(p-1)}} m_{42} < \cdots < m_{1n} \\
&< m_{2n} \le \frac{\tau_1^2\lambda}{((b-a)/r(a))^{1/(p-1)}} m_{4n},
\end{aligned} \tag{43}$$

such that the following conditions are satisfied:

- (C_1) $f(t,\mu,\nu) \le (1/L)\varphi_p(m_{4i})$, $(t,\mu,\nu) \in [a,b] \times [0,\tau_2 m_{4i}] \times [-m_{4i}, m_{4i}]$,
- (C_2) $f(t,\mu,\nu) > (1/L)\varphi_p(m_{2i}/\tau_1 M)$, $(t,\mu,\nu) \in [\sigma, b-\sigma] \times [m_{2i}, m_{2i}\tau_1^{-2}] \times [-m_{4i}, m_{4i}]$,
- (C_3) $f(t,\mu,\nu) \le (1/L)\varphi_p(m_1/N)$, $(t,\mu,\nu) \in [a,b] \times [0, m_{1i}] \times [-m_{4i}, m_{4i}]$,

then the problem $(\mathscr{P})$ has at least $2n-1$ positive solutions.

4. Example

Let $p = 3$, $m = 4$, and $\sigma = 1/4$. Consider the boundary value problem

$$\begin{gathered}
\left(r(t)\varphi_p\left(u'\right)\right)' + h(t) f\left(t, u(t), u'(t)\right) = 0, \quad 0 < t < 1, \\
2u(0) - u'(0) = \frac{1}{2}u\left(\frac{1}{4}\right) + \frac{1}{2}u\left(\frac{1}{2}\right), \quad u'(1) = 0,
\end{gathered} \tag{44}$$

where

$$\begin{gathered}
r(t) = t + 1, \qquad h(t) = \frac{1}{\sqrt{t(1-t)}}, \\
f(t,\mu,\nu) = \begin{cases} 10^{-4}t + 2\mu^8 + \left(\dfrac{\nu}{3\times 10^4}\right)^3 \times 10^{-5}, & \mu \le 4, \\ 10^{-4}t + 131072 + \left(\dfrac{\nu}{3\times 10^4}\right)^3 \times 10^{-5}, & \mu > 4. \end{cases}
\end{gathered} \tag{45}$$

It is easy to check that (A_1)–(A_4) hold. By some calculations, we have $\tau_1 = 1/2$, $\tau_2 = 3$, $\lambda \approx 1.615$, $L \approx 3.142$, $M = 1$, and $N = 2.375$. If we choose $m_1 = 1/2$, $m_2 = 1$, and $m_4 = 3\times 10^4$, then $f(t,\mu,\nu)$ satisfies

$$\begin{gathered}
f(t,\mu,\nu) \le \frac{1}{L}\varphi_3(m_4) \approx 2.866\times 10^8, \\
(t,\mu,\nu) \in [0,1] \times \left[0, 9\times 10^4\right] \times \left[-3\times 10^4, 3\times 10^4\right], \\
f(t,\mu,\nu) > \frac{1}{L}\varphi_3\left(\frac{m_2}{\tau_1 M}\right) \approx 1.274,
\end{gathered}$$

$$(t,\mu,\nu) \in \left[\frac{1}{4},\frac{3}{4}\right] \times [1,4] \times \left[-3\times 10^4, 3\times 10^4\right],$$
$$f(t,\mu,\nu) \le \frac{1}{L}\varphi_3\left(\frac{m_1}{N}\right) \approx 0.014,$$
$$(t,\mu,\nu) \in [0,1] \times \left[0,\frac{1}{2}\right] \times \left[-3\times 10^4, 3\times 10^4\right]. \tag{46}$$

Thus all assumptions of Theorem 6 hold, and so the problem (44) has at least three positive solutions u_1, u_2, and u_3 such that $\max_{t\in[0,1]}|u_i'(t)| \le 3\times 10^4$ for $i = 1,2,3$, $1 < \min_{(1/4)\le t\le(3/4)} u_1(t)$, $1/2 < \max_{t\in[0,1]} u_2(t)$ with $\min_{(1/4)\le t\le(3/4)} u_2(t) < 1$, and $\max_{t\in[0,1]} u_3(t) < 1/2$.

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