# Local Lagrange Interpolations Using Bivariate $C^{2}$ Splines of Degree Seven on Triangulated Quadrangulations 

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#### Abstract

A local Lagrange interpolation scheme using bivariate $C^{2}$ splines of degree seven over a checkerboard triangulated quadrangulation is constructed. The method provides optimal order approximation of smooth functions.


## 1. Introduction

Suppose that $\Delta$ is a regular triangulation of a connected polygonal domain $\Omega$. For integers $0 \leq r \leq d$, we define

$$
\begin{equation*}
S_{d}^{r}(\Delta):=\left\{s \in C^{r}(\Omega):\left.s\right|_{T} \in P_{d}, \forall T \in \Delta\right\} \tag{1}
\end{equation*}
$$

where $P_{d}$ is the $((d+2)(d+1)) / 2$ dimensional space of bivariate polynomials of degree $d$. For integers $r \leq \rho \leq d$, we further define the following super spline space:

$$
\begin{equation*}
S_{d}^{r, \rho}(\Delta):=\left\{s \in C^{r}(\Omega): s \in C^{\rho}(v), \quad \forall v \in \Delta\right\} \tag{2}
\end{equation*}
$$

where, as usual, $s \in C^{\rho}(v)$ means that all polynomials on triangles sharing the vertex $v$ have common derivatives up to order $\rho$ at that vertex.

We consider the following Lagrange interpolation problem.

Problem 1. Let $\mathscr{V}:=\left\{\eta_{i}\right\}_{i=1}^{n}$ be a set of points in the plane, and let $\Delta$ be a quadrangulation with vertices at the points of $V$. Find a triangulation $\Delta$ of $\Delta$ and a set of additional points $\left\{\eta_{i}\right\}_{i=n+1}^{N}$ such that, for every choice of the data $\left\{z_{i}\right\}_{i=1}^{N}$, there is a unique spline $s \in \mathcal{S}$ satisfying

$$
\begin{equation*}
s\left(\eta_{i}\right)=z_{i}, \quad i=1, \ldots, N \tag{3}
\end{equation*}
$$

We call $\mathscr{P}:=\left\{\eta_{i}\right\}_{i=1}^{N}$ a Lagrange interpolation set for $\mathcal{S}$ and $\mathscr{P}$ and $\mathcal{S}$ a Lagrange interpolation pair. Although
constructing Lagrange interpolation pairs sounds simple at first glance, it is in fact a complex problem, especially since we want a local and stable method which has linear complexity and provides optimal order approximation. In order to construct Lagrange interpolation pairs, both $\mathcal{S}$ and $\mathscr{P}$ must be carefully chosen.

For $r=1$, the first result on local Lagrange interpolation by splines on triangulations was given by Nürnberger and Zeilfelder [1], where, by subdividing about half of the triangles with a Clough-Tocher split, a local Lagrange interpolation scheme for cubic $C^{1}$ splines on triangulations whose interior vertices have degree six was developed. Nürnberger and Zeilfelder [2] constructed a local Lagrange interpolation set for $S_{2}^{1}\left(\triangle_{\mathrm{PS} 1}\right)$, where $\triangle_{\mathrm{PS} 1}$ is the refining triangulation of Powell-Sabin type (I). Nürnberger and Zeilfelder [3] used a coloring algorithm to divide all the triangles in $\Delta$ into two kinds: white triangles and black triangles, got a new triangulation $\bar{\triangle}_{\mathrm{CT}}$ through refining all the white triangles by the Clough-Tocher refinement, and then gave a local Lagrange interpolation set for $S_{3}^{1}\left(\bar{\Delta}_{\mathrm{CT}}\right)$. As to more results, the reader is referred to several survey papers [4-6].

For $r=2$, Nürnberger et al. [7] constructed a local Lagrange interpolation set for $S_{7}^{2}\left(\bar{\Delta}_{\mathrm{CT}}\right)$. Liu and Fan [8] constructed a local Lagrange interpolation set for $S_{5}^{2,3}\left(\bar{\Delta}_{\mathrm{DCT}}\right)$, where $\bar{\Delta}_{\mathrm{DCT}}$ is the triangulation by refining some of triangles in $\triangle$ with the double Clough-Tocher splits.


Figure 1: The set $\mathscr{G}$ for a checkerboard triangulation.


Figure 2: The stable Lagrange MDS $\mathscr{M}$ for $S_{7}^{2,3}\left(Q_{\theta}\right)$ produced by Lemma 3.

For general cases $r \geq 1$, a local Lagrange interpolation of $S_{d}^{r, \rho}\left(\bar{\Delta}_{\mathrm{CT}}\right)$ has been proposed by Nürnberger et al. in [9], where $r, \rho$, and $d$ are taken as the following related values

$$
(\rho, d):= \begin{cases}(3 m+1,6 m+3), & r=2 m+1  \tag{4}\\ (3 m, 6 m+1), & r=2 m\end{cases}
$$

with $m=\lfloor r / 2\rfloor$.
Let $\diamond$ be a quadrangulation of $\Omega$ which consists of nondegenerate convex quadrilaterals. By adding one or two diagonals of each quadrilateral, some triangulated quadrangulations can be obtained. Nürnberger et al. [10] constructed a Lagrange interpolation scheme based on $C^{1}$ cubic splines on certain triangulations obtained from checkerboard quadrangulations. In [11], they also constructed a local Lagrange interpolation method based on $C^{1}$ cubic splines on certain triangulations obtained from a separable quadrangulation. Further they [12] described local Lagrange interpolation methods based on $C^{1}$ cubic splines on triangulations
obtained from arbitrary strictly convex quadrangulations by adding one or two diagonals. Their construction makes use of a fast algorithm for coloring quadrangulations, and the overall algorithm has linear complexity while providing optimal order approximation of smooth functions.

However, to the authors' knowledge, the local Lagrange interpolation schemes based on $C^{2}$ splines on any triangulated quadrangulation have not been developed. In this paper, we shall construct a Lagrange interpolation scheme on $S_{7}^{2,3}(\theta)$ over the triangulation $\forall$ obtained by adding the two diagonals of each quadrilateral from checkerboard quadrangulations.

The paper is organized as follows. In Section 2 we introduce some notation and describe the Bernstein-Bézier representation of splines. In Section 3 we introduce the checkerboard triangulations. In Section 4 several lemmas of Lagrange minimal determining sets are established. In Section 5 the main results of construction of the Lagrange interpolation pair $\mathscr{P}, S_{7}^{2,3}(\oplus)$ and error bounds for the interpolating splines are presented.


Figure 3: The point set $\mathscr{M}$ produced by Lemma 4.


Figure 4: The point set $\mathscr{M}$ produced by Lemma 5.

## 2. Preliminaries

Throughout the paper we shall make extensive use of the wellknown Bernstein-Bézier representation of splines. Let $T:=$ $\left\langle v_{1}, v_{2}, v_{3}\right\rangle$ in $\Delta$ with vertices $v_{1}, v_{2}, v_{3}$, and the corresponding polynomial piece $\left.s\right|_{T}$ is written in the form:

$$
\begin{equation*}
\left.s\right|_{T}=\sum_{i+j+k=d} c_{i j k}^{T} B_{i j k}^{d}, \tag{5}
\end{equation*}
$$

where $B_{i j k}^{d}$ are the Bernstein basis polynomials of degree $d$ associated with $T$. As usual, we identify the Bernstein-Bézier coefficients $\left\{c_{i j k}^{T}\right\}_{i+j+k=d}$ with the set of domain points $D_{T}:=$ $\left\{\xi_{i j k}^{T}:=\left(i v_{1}+j v_{2}+k v_{3}\right) / d\right\}_{i+j+k=d}$. We write $D_{d, \Delta}$ for the union of the sets of domain points associated with the triangles of $\Delta$.

Given an integer $0 \leq m \leq d$, let

$$
\begin{align*}
& R_{m}^{T}\left(v_{1}\right):=\left\{\xi_{i j k}^{T}: i=d-m\right\},  \tag{6}\\
& D_{m}^{T}\left(v_{1}\right):=\left\{\xi_{i j k}^{T}: i \geq d-m\right\} .
\end{align*}
$$

We have similar definitions at the other vertices of $T$. If $v$ is a vertex of a triangulation $\Delta$, we, respectively, define that the ring and the disk of radius $m$ around $v$ are the set

$$
\begin{equation*}
R_{m}(v):=\bigcup\left\{R_{m}^{T}(v)\right\}, \quad D_{m}(v):=\bigcup\left\{D_{m}^{T}(v)\right\} \tag{7}
\end{equation*}
$$

where the union is taken over all triangles attached to $v$.

We recall [13] that supposes that $\mathcal{S}$ is a subspace of $S_{d}^{0}(\Delta)$, and then $\mathscr{P} \subseteq D_{d, \Delta}$ is said to be a determining set for $\mathcal{S}$ that provides that, for any $s \in \mathcal{S}, \lambda_{\xi} s=0$ for all $\xi \in \mathscr{P}$ implies that $s \equiv 0$. Note that $\lambda_{\xi}$ is a linear functional. The set $\mathscr{P}$ is called a minimal determining set (MDS) for $\mathcal{S}$ if there is no smaller determining set. Further, following [7], a basis $\left\{B_{i}\right\}_{i=1}^{n}$ for a spline space $\mathcal{S}$ is called a stable local basis provided that constants $\ell, K_{1}$, and $K_{2}$ exist depending only on the smallest angle in $\triangle$ such that
(1) for each $1 \leq i \leq n$, there is a vertex $v_{i}$ of $\Delta$ for which $\operatorname{supp}\left(B_{i}\right) \subseteq \operatorname{star}^{\ell}\left(v_{i}\right)$,
(2) for all choices of the coefficient vector $c=\left(c_{1}, \ldots, c_{n}\right)$,

$$
\begin{equation*}
K_{1}\|c\|_{\infty} \leq\left\|\sum_{i=1}^{n} c_{i} B_{i}\right\|_{\infty} \leq K_{2}\|c\|_{\infty} \tag{8}
\end{equation*}
$$

Here $\operatorname{star}^{1}(v)$ is defined to be the set of all triangles surrounding vertex $v$, and $\operatorname{star}^{\ell}(v)$ is defined to be the union of the $\operatorname{star}^{1}(w)$, where $w$ are vertices of $\operatorname{star}^{\ell-1}(v)$.

It is well known that a spline $s$ in $S_{d}^{0}(\Delta)$ is uniquely determined by its Bernstein-Bézier coefficient set $\left\{c_{\xi}\right\}_{\xi} \in$ $D_{d, \Delta}$. In order to describe smoothness conditions for splines, we recall some notations introduced in [14]. Suppose that $T:=\left\langle v_{1}, v_{2}, v_{3}\right\rangle$ and $\widetilde{T}:=\left\langle v_{4}, v_{3}, v_{2}\right\rangle$ are two adjoining


Figure 5: The point set $\mathscr{M}$ produced by Lemma 6.


Figure 6: The point set $\mathscr{M}$ produced by Lemma 7.
triangles from $\Delta$, which share the oriented edge $e:=\left\langle v_{2}, v_{3}\right\rangle$, and let

$$
\begin{equation*}
\left.s\right|_{T}=\sum_{i+j+k=d} c_{i j k} B_{i j k}^{d},\left.\quad s\right|_{\widetilde{T}}=\sum_{i+j+k=d} \widetilde{c}_{i j k} \widetilde{B}_{i j k}^{d}, \tag{9}
\end{equation*}
$$

where $B_{i j k}^{d}$ and $\widetilde{B}_{i j k}^{d}$ are the Bernstein polynomials of degree $d$ on the triangles $T$ and $\widetilde{T}$, respectively. Given integers $0 \leq n \leq$ $j \leq d$, let $\tau_{j, e}^{n}$ be linear functional defined on $S_{d}^{0}(\Delta)$ by

$$
\begin{equation*}
\tau_{j, e}^{n} s:=c_{n, d-j, j-n}-\sum_{v+\mu+\kappa=n} \widetilde{c}_{v, \mu+j-n, \kappa+d-j} \widetilde{B}_{v \mu \kappa}^{n}\left(v_{1}\right) . \tag{10}
\end{equation*}
$$

These are called smoothness functionals of order $n$. According to [14], a spline $s \in S_{d}^{0}(\Delta)$ belongs to $C^{r}(T \bigcup \widetilde{T})$ for some $r>0$ if and only if

$$
\begin{equation*}
\tau_{m, e}^{n}=0, \quad n \leq m \leq d, 0 \leq n \leq r . \tag{11}
\end{equation*}
$$

## 3. Checkerboard Triangulations

Definition 1 (see [10]). Suppose that $\diamond$ is a quadrangulation consisting of quadrilaterals with largest interior angle less than $\pi$. Suppose that the quadrilaterals can be colored black and white in such a way that any two quadrilaterals sharing an edge have the opposite color. Then we call $\diamond$ a checkerboard quadrangulation. The triangulation $\Delta$ which is obtained by drawing in both diagonals of all quadrilaterals will be called a checkerboard triangulation.


Figure 7: The Lagrange MDS produced by Theorem 8.

Let $\mathscr{B}$ and $\mathscr{W}$ denote the sets of black and white quadrilaterals of $\Delta$, respectively. Following [10], throughout this paper, we also assume that all interior vertices of $\Delta$ are of degree four. This assumption can ensure that there exists $\mathscr{G} \subset$ $\mathscr{B}$ such that for every interior vertex $v$ of $\rangle$, there is a unique quadrilateral $Q \in \mathscr{B}$ sharing the vertex $v$. For $i=1,2,3,4$, let $\mathscr{W}_{i}$ be the set of white quadrilaterals which share $i$ edges with black quadrilaterals. Let $n_{B}=\# \mathscr{B}$ and $n_{i}:=\# \mathscr{W}_{i}$ for $i=1,2,3,4$, and let $n_{V}$ be the total number of vertices of $\diamond$. As shown in Figure 1, a typical checkerboard triangulation is displayed in which the quadrilaterals in the set $\mathscr{G}$ are shaded grey. It is noted that the other black quadrilaterals have not been colored.

## 4. Lagrange Minimal Determining Sets

Lemma 2 (see [10]). The set of all domain points in a triangle $T$ is a Lagrange minimal determining set for the space $P_{d}$.

Let $Q:=\left\langle v_{1}, v_{2}, v_{3}, v_{4}\right\rangle$ be a quadrilateral with vertices in counterclockwise order and $Q_{\theta}$ be the triangulated quadrangulation of $Q$ with $v_{Q}$ being the intersection point of the two diagonals of Q . As shown in Figure 2, let $T_{i}:=\left\langle v_{\mathrm{Q}}, v_{i}, v_{i+1}\right\rangle$ and $e_{i}:=\left\langle v_{i}, v_{\mathrm{Q}}\right\rangle$ for $i=1,2,3,4$, where $v_{5}=v_{1}$.

Lemma 3. The set

$$
\begin{align*}
\mathscr{M}:= & \left(\bigcup_{i=1}^{2} D_{3}^{T_{1}}\left(v_{i}\right)\right) \bigcup D_{3}^{T_{1}}\left(v_{\mathrm{Q}}\right) \bigcup D_{3}^{T_{2}}\left(v_{3}\right) \\
& \bigcup D_{3}^{T_{4}}\left(v_{4}\right) \bigcup\left\{\xi_{133}^{T_{1}}, \xi_{232}^{T_{1}}, \xi_{223}^{T_{1}}, \xi_{331}^{T_{1}}, \xi_{322}^{T_{1}}, \xi_{313}^{T_{1}}\right\}  \tag{12}\\
& \bigcup\left\{\xi_{133}^{T_{2}}, \xi_{223}^{T_{2}}, \xi_{313}^{T_{2}}, \xi_{403}^{T_{2}}\right\} \\
& \bigcup\left\{\xi_{133}^{T_{4}}, \xi_{232}^{T_{4}}, \xi_{331}^{T_{4}}, \xi_{430}^{T_{4}}\right\} \bigcup\left\{\xi_{133}^{T_{3}}\right\}
\end{align*}
$$

is a stable Lagrange MDS for $S_{7}^{2,3}\left(\mathrm{Q}_{\star}\right)$. These domain points are marked with • in Figure 2.

Proof. Equivalently, we consider the related homogenous interpolation problem. By Lemma 2, all Bernstein-Bézier coefficients of $\left.s\right|_{T_{1}}$ associated to 36 domain points in $T_{1}$ must be zero. By the $C^{2}$ smoothness conditions on the edge $\left\langle v_{\mathrm{Q}}, v_{2}\right\rangle$ and the $C^{3}$ smoothness conditions at vertex $v_{2}$, all Bernstein-Bézier coefficients of $\left.s\right|_{T_{2}}$ must be zero except for

14 coefficients $c_{i j k}^{T_{2}}(k=4,5,6,7, i+j=7-k), c_{403}^{T_{2}}, c_{313}^{T_{2}}, c_{223}^{T_{2}}$ and $c_{133}^{T_{2}}$. Let

$$
\begin{align*}
& \mathbf{C}:=\left(c_{007}^{T_{2}}, c_{106}^{T_{2}}, c_{016}^{T_{2}}, c_{205}^{T_{2}}, c_{115}^{T_{2}}, c_{025}^{T_{2}}, c_{304}^{T_{2}}, c_{214}^{T_{2}},\right. \\
&\left.c_{124}^{T_{2}}, c_{034}^{T_{2}}, c_{403}^{T_{2}}, c_{313}^{T_{2}}, c_{223}^{T_{2}}, c_{133}^{T_{2}}\right)^{T}, \tag{13}
\end{align*}
$$

then the homogenous Lagrange interpolation conditions at the associated 14 domain points lead to

$$
\begin{equation*}
\mathrm{GC}=\mathbf{0}, \tag{14}
\end{equation*}
$$

where

It is easy to see that $\operatorname{rank}(\mathbf{G})=14$. Thus all the remaining 14 Bernstein-Bézier coefficients of $\left.s\right|_{T_{2}}$ are zero. Similarly, all the Bernstein-Bézier coefficients of $\left.s\right|_{T_{4}}$ must be also zero by smoothness conditions and given Lagrange interpolation conditions. Further, using $C^{2}$ smoothness conditions along two edges $\left\langle v_{\mathrm{Q}}, v_{3}\right\rangle$ and $\left\langle v_{\mathrm{Q}}, v_{4}\right\rangle$ and $C^{3}$ smoothness conditions at vertices $v_{3}$ and $v_{4}$, all Bernstein-Bézier coefficients of $\left.s\right|_{T_{3}}$ are zero except for $c_{133}^{T_{3}}$. Using the interpolation condition at domain point $\xi_{133}^{T_{3}}: B_{133}^{7} c_{133}^{T_{3}}=0$, that is; $c_{133}^{T_{3}}=0$ since $B_{133}^{7}=14580 / 117649$. Therefore all B coefficients of $\left.s\right|_{T_{3}}$ are zero.

The construction in Lemma 3 is stable in the sense that the maximum coefficient of $s$ is bounded by $K \max _{\xi \in \mathscr{M}}|p(\xi)|$, where $K$ is a constant depending only on the smallest angle in
$\triangle$. Thus, we say that $\mathscr{M}$ is a stable Lagrange MDS for $s \in \mathcal{S}_{7}^{2,3}$ $(\theta)$. The proof is completed.

In the following lemmas, we will consider four cases depending on how many edges of $Q \in \mathscr{W}$ adjoin with the other $Q \in \mathscr{B}$.

Lemma 4. Suppose that $\oplus$ consists of two triangulated quadrangulations $Q_{1}=\left\langle v_{1}, v_{2}, v_{3}, v_{4}\right\rangle$ and $Q_{2}=\left\langle v_{3}, v_{5}, v_{6}, v_{4}\right\rangle$ sharing the edge $\left\langle v_{3}, v_{4}\right\rangle$, where $Q_{1} \in \mathscr{B}$ and $Q_{2} \in \mathscr{W}$. Let $v_{\mathrm{Q}_{1}}$ and $v_{\mathrm{Q}_{2}}$ be the points where the two diagonals of $\mathrm{Q}_{1}$ and $Q_{2}$ intersect, respectively. Let the set

$$
\begin{equation*}
\mathscr{M}:=\mathscr{M}_{\mathrm{Q}_{1}} \bigcup m_{1} \bigcup m_{2} \bigcup m_{3} \tag{16}
\end{equation*}
$$

where $\mathscr{M}_{\mathrm{Q}_{1}}$ is the set defined in Lemma 3 for the triangulated quadrangulations of $Q_{1}$ and

$$
\begin{aligned}
& m_{1}= \bigcup D_{3}^{\left\langle v_{\mathrm{Q}_{2}}, v_{4}, v_{3}\right\rangle}\left(v_{\mathrm{Q}_{2}}\right) \\
& \bigcup\left\{\xi_{331}^{\left\langle v_{\mathrm{Q}_{2}}, v_{4}, v_{3}\right\rangle}, \xi_{322}^{\left\langle v_{\mathrm{Q}_{2}}, v_{4}, v_{3}\right\rangle}, \xi_{313}^{\left\langle v_{\mathrm{Q}_{2}}, v_{4}, v_{3}\right\rangle}\right\} \\
& m_{2}= \bigcup D_{3}^{\left\langle v_{\mathrm{Q}_{2}}, v_{3}, v_{5}\right\rangle}\left(v_{5}\right) \\
& \bigcup\left\{\xi_{133}^{\left\langle v_{\left.\mathrm{Q}_{2}, v_{3}, v_{5}\right\rangle}\right\rangle}, \xi_{223}^{\left\langle v_{\left.\mathrm{Q}_{2}, v_{3}, v_{5}\right\rangle}\right.}\right. \\
&\left.\xi_{313}^{\left\langle v_{\mathrm{Q}_{2}}, v_{3}, v_{5}\right\rangle}, \xi_{403}^{\left\langle v_{\mathrm{Q}_{2}}, v_{3}, v_{5}\right\rangle}, \xi_{133}^{\left\langle v_{\mathrm{Q}_{2}}, v_{5}, v_{6}\right\rangle}\right\}
\end{aligned}
$$

$$
\begin{align*}
m_{3}= & \bigcup D_{3}^{\left\langle v_{\mathrm{Q}_{2}}, v_{6}, v_{4}\right\rangle}\left(v_{6}\right) \\
& \bigcup\left\{\xi_{133}^{\left\langle v_{\mathrm{Q}_{2}}, v_{6}, v_{4}\right\rangle}, \xi_{232}^{\left\langle v_{\mathrm{Q}_{2}}, v_{6}, v_{4}\right\rangle}, \xi_{331}^{\left\langle v_{\mathrm{Q}_{2}}, v_{6}, v_{4}\right\rangle}, \xi_{430}^{\left\langle v_{\mathrm{Q}_{2}}, v_{6}, v_{4}\right\rangle}\right\} \tag{17}
\end{align*}
$$

be a stable Lagrange MDS for $\mathcal{S}_{7}^{2,3}(\theta)$. These domain points are marked with • in Figure 3.

Proof. Using Lemma 3, we can see that all of the B coefficients of $s$ associated with domain points in $Q_{1}$ are uniquely determined by the data. Writing $\left.s\right|_{\left\langle v_{Q_{2}}, v_{4}, v_{3}\right\rangle}$ in B form and using $C^{1}, C^{2}$, and $C^{3}$ smoothness conditions, all B coefficients are determined except for the B coefficients associated with $m_{1}$. Because GC $=b$, where

$$
\begin{aligned}
& b:=\left(p\left(\xi_{700}^{\left\langle v_{\mathrm{Q}_{2}}, v_{4}, v_{3}\right\rangle}\right), p\left(\xi_{610}^{\left\langle v_{\mathrm{Q}_{2}}, v_{4}, v_{3}\right\rangle}\right), p\left(\xi_{601}^{\left\langle v_{\mathrm{Q}_{2}}, v_{4}, v_{3}\right\rangle}\right), p\left(\xi_{520}^{\left\langle v_{\mathrm{Q}_{2}}, v_{4}, v_{3}\right\rangle}\right), p\left(\xi_{511}^{\left\langle v_{\mathrm{Q}_{2}}, v_{4}, v_{3}\right\rangle}\right), p\left(\xi_{502}^{\left\langle v_{\mathrm{Q}_{2}}, v_{4}, v_{3}\right\rangle}\right),\right. \\
& p\left(\xi_{430}^{\left\langle v_{\mathrm{Q}_{2}}, v_{4}, v_{3}\right\rangle}\right), p\left(\xi_{421}^{\left\langle v_{\mathrm{Q}_{2}}, v_{4}, v_{3}\right\rangle}\right), p\left(\xi_{412}^{\left\langle v_{\mathrm{Q}_{2}}, v_{4}, v_{3}\right\rangle}\right), p\left(\xi_{403}^{\left\langle v_{\mathrm{Q}_{2}}, v_{4}, v_{3}\right\rangle}\right), p\left(\xi_{331}^{\left\langle v_{\mathrm{Q}_{2}}, v_{4}, v_{3}\right\rangle}\right), \\
& \left.p\left(\xi_{322}^{\left\langle v_{\mathrm{Q}_{2}}, v_{4}, v_{3}\right\rangle}\right), p\left(\xi_{313}^{\left\langle v_{\mathrm{Q}_{2}}, v_{4}, v_{3}\right\rangle}\right)\right)^{T},
\end{aligned}
$$

It is easy to see that $\operatorname{rank}(\mathbf{G})=13$. Thus, all of the remaining B coefficients of $m_{1}$ are uniquely determined by the data. Then, using Lemma 3 again, it is easy to see that all of the B coefficients of $s$ associated with domain points in $Q_{2}$ are uniquely and stable determined by the data. Therefore, $\mathscr{M}$ is a stable Lagrange MDS for $\delta_{7}^{2,3}(\theta)$. The proof is completed.

Lemma 5. Suppose that $\theta$ consists of three triangulated quadrangulations $Q_{1}, Q_{2}$, and $Q_{3}$ as in Figure 4, where $Q_{1}, Q_{3} \in \mathscr{B}$ and $Q_{2} \in \mathscr{W}$. Let the set

$$
\begin{equation*}
\mathscr{M}:=\mathscr{M}_{\mathrm{Q}_{1}} \bigcup \mathscr{M}_{\mathrm{Q}_{3}} \bigcup m_{1} \bigcup m_{2} \tag{19}
\end{equation*}
$$

where $\mathscr{M}_{\mathrm{Q}_{i}}$ is the set defined in Lemma 3 for the triangulated quadrangulations of $Q_{i}$ and

$$
\begin{align*}
m_{1}= & \bigcup D_{3}^{\left\langle v_{Q_{2}}, v_{3}, v_{2}\right\rangle}\left(v_{\mathrm{Q}_{2}}\right) \\
& \bigcup\left\{\xi_{331}^{\left\langle v_{\mathrm{Q}_{2}}, v_{3}, v_{2}\right\rangle}, \xi_{322}^{\left\langle v_{\mathrm{Q}_{2}}, v_{3}, v_{2}\right\rangle}, \xi_{313}^{\left\langle v_{\mathrm{Q}_{2}}, v_{3}, v_{2}\right\rangle}\right\}, \\
m_{2}= & \left\{\xi_{133}^{\left\langle v_{\mathrm{Q}_{2}}, v_{6}, v_{3}\right\rangle}, \xi_{430}^{\left\langle v_{\mathrm{Q}_{2}}, v_{6}, v_{3}\right\rangle}, \xi_{331}^{\left\langle v_{\mathrm{Q}_{2}}, v_{6}, v_{3}\right\rangle}\right.  \tag{20}\\
& \left.\xi_{133}^{\left\langle v_{\left.\mathrm{Q}_{2}, v_{2}, v_{5}\right\rangle},\right.} \xi_{403}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}, \xi_{313}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}\right\}
\end{align*}
$$

be a stable Lagrange MDS for $\mathcal{S}_{7}^{2,3}(\oplus)$. These domain points are marked with • in Figure 4.

Proof. Using Lemma 3, we can see that all of the B coefficients of $s$ associated with domain points in $Q_{1}$ and $Q_{3}$ are uniquely determined by the data. Arguing as in Lemma 4, all the B coefficients of $\left.s\right|_{\left\langle v_{Q_{2}}, v_{3}, v_{2}\right\rangle}$ associated with domain points are uniquely determined by the data too. Writing $\left.s\right|_{\left\langle v_{Q_{2}}, v_{2}, v_{5}\right\rangle}$ in B form and using $C^{1}, C^{2}$ and $C^{3}$ smoothness conditions, all B coefficients are determined except for $m:=\left\{c_{403}^{\left\langle v_{Q_{2}}, v_{2}, v_{5}\right\rangle}, c_{313}^{\left\langle v_{Q_{2}}, v_{2}, v_{5}\right\rangle}, c_{223}^{\left\langle v_{Q_{2}}, v_{2}, v_{5}\right\rangle}, c_{133}^{\left\langle v_{Q_{2}}, v_{2}, v_{5}\right\rangle}\right\}$. The Lagrange interpolation and $C^{1}$ and $C^{2}$ smoothness conditions at $e:=\left\langle v_{\mathrm{Q}_{2}}, v_{5}\right\rangle$ imply that these coefficients must satisfy the linear system $\mathbf{G C}=b$, where

$$
\begin{align*}
& \mathbf{C}:=\left(c_{403}^{\left\langle v_{Q_{2}}, v_{2}, v_{5}\right\rangle}, c_{313}^{\left\langle v_{Q_{2}}, v_{2}, v_{5}\right\rangle},\right. \\
& \left.c_{223}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}, c_{133}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}, c_{331}^{\left\langle v_{\mathrm{Q}_{2}}, v_{5}, v_{6}\right\rangle}\right)^{T}, \\
& b:=\left(p\left(\xi_{403}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}\right), p\left(\xi_{313}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}\right)\right. \text {, } \\
& \left.p\left(\xi_{133}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}\right), 0,0\right)^{T}, \\
& \mathbf{G}=\left(\begin{array}{ccccc}
\frac{1280 \times 3^{3}}{7^{6}} & 0 & 0 & 0 & 0 \\
\frac{5 \times 3^{7}}{7^{6}} & \frac{20 \times 3^{6}}{7^{6}} & \frac{30 \times 3^{5}}{7^{6}} & \frac{20 \times 3^{4}}{7^{6}} & 0 \\
\frac{5 \times 3^{3}}{7^{6}} & \frac{60 \times 3^{3}}{\gamma^{6}} & \frac{270 \times 3^{3}}{7^{6}} & \frac{540 \times 3^{3}}{7^{6}} & 0 \\
\gamma_{1} & -1 & 0 & 0 & \beta_{1} \\
\gamma_{1}^{2} & 0 & -1 & 0 & 2 \beta_{1} \gamma_{1}
\end{array}\right), \tag{21}
\end{align*}
$$

where $v_{2}=\beta_{1} v_{6}+\gamma_{1} v_{Q_{2}}$. The determinant of the matrix $\mathbf{G}$ is

$$
\begin{equation*}
-5 \times 10^{3} \beta_{1}\left(8 \gamma_{1}+9\right) \tag{22}
\end{equation*}
$$

By the geometric meaning of the $\beta_{1}$ and $\gamma_{1}$, we have $\beta_{1}<$ 0 and $\gamma_{1}>0$. Thus $\operatorname{det}(\mathbf{G}) \neq 0$. So, all B coefficients of $\left.s\right|_{\left\langle v_{\mathrm{O}_{2}}, v_{2}, v_{5}\right\rangle}$ are uniquely determined. By the same principle, all B coefficients of $\left.s\right|_{\left\langle v_{Q_{2}}, v_{6}, v_{3}\right\rangle}$ are uniquely determined. Using Lemma 3 again, it is easy to see that all B coefficients of $Q_{2}$ are uniquely and stable determined by the data. Therefore $\mathscr{M}$ is a stable Lagrange MDS for $\mathcal{S}_{7}^{2,3}(\Theta)$. The proof is completed.

Lemma 6. Suppose that $\theta$ consists of three triangulated quadrangulations $Q_{1}, Q_{2}$, and $Q_{3}$ as in Figure 5, where $Q_{1}, Q_{3} \in \mathscr{B}$ and $Q_{2} \in \mathscr{W}$. Let the set

$$
\begin{align*}
\mathscr{M}:= & \mathscr{M}_{\mathrm{Q}_{1}} \bigcup \mathscr{M}_{\mathrm{Q}_{3}} \bigcup m_{1} \bigcup m_{2} \\
& -\bigcup D_{3}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{7}\right\rangle}\left(v_{2}\right), \tag{23}
\end{align*}
$$

where $\mathscr{M}_{\mathrm{Q}_{\mathrm{i}}}$ is the set defined in Lemma 3 for the triangulated quadrangulations of $Q_{i}$ and

$$
\begin{align*}
m_{1}= & \left\{\xi_{133}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{3}\right\rangle}, \xi_{403}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}, \xi_{313}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}\right\} \\
m_{2}= & \bigcup D_{3}^{\left\langle v_{\mathrm{Q}_{2}}, v_{5}, v_{6}\right\rangle}\left(v_{6}\right)  \tag{24}\\
& \bigcup\left\{\xi_{133}^{\left\langle v_{\mathrm{Q}_{2}}, v_{5}, v_{6}\right\rangle}, \xi_{223}^{\left\langle v_{\mathrm{Q}_{2}}, v_{5}, v_{6}\right\rangle}, \xi_{313}^{\left\langle v_{\mathrm{Q}_{2}}, v_{5}, v_{6}\right\rangle}, \xi_{403}^{\left\langle v_{\mathrm{Q}_{2}}, v_{5}, v_{6}\right\rangle}\right\},
\end{align*}
$$

be a stable Lagrange MDS for $\mathcal{S}_{7}^{2,3}(\theta)$. These domain points are marked with • in Figure 5.

Proof. Using Lemma 3 and $C^{1}, C^{2}$, and $C^{3}$ smoothness conditions, we can see that all of the B coefficients of $s$ associated with domain points in $Q_{1}$ and $Q_{3}$ are uniquely determined by the data. Writing $\left.s\right|_{\left\langle v_{Q_{2}}, v_{3}, v_{6}\right\rangle}$ in B form and using $C^{1}, C^{2}$, and $C^{3}$ smoothness conditions, all $B$ coefficients are determined except for the B coefficients of $m:=\bigcup D_{3}^{\left\langle v_{Q_{2}}, v_{3}, v_{2}\right\rangle}\left(v_{\mathrm{Q}_{2}}\right) \bigcup\left\{\xi_{331}^{\left\langle v_{\mathrm{Q}_{2}}, v_{3}, v_{2}\right\rangle}, \xi_{322}^{\left\langle v_{\mathrm{Q}_{2}}, v_{3}, v_{2}\right\rangle}, \xi_{313}^{\left\langle v_{Q_{2}}, v_{3}, v_{2}\right\rangle}\right\}$ associated. The Lagrange interpolation and $C^{1}$ and $C^{2}$ smoothness conditions at $e:=\left\langle v_{\mathrm{Q}_{2}}, v_{3}\right\rangle$ and $e:=\left\langle v_{\mathrm{Q}_{2}}, v_{2}\right\rangle$ imply that these coefficients must satisfy the linear system GC $=b$, where

$$
\begin{aligned}
& \mathbf{C}:=\left(c_{700}^{\left\langle v_{Q_{2}}, v_{3}, v_{2}\right\rangle}, c_{610}^{\left\langle v_{Q_{2}}, v_{3}, v_{2}\right\rangle}, c_{601}^{\left\langle v_{Q_{2}}, v_{3}, v_{2}\right\rangle}, c_{520}^{\left\langle v_{Q_{2}}, v_{3}, v_{2}\right\rangle}, c_{511}^{\left\langle v_{Q_{2}}, v_{3}, v_{2}\right\rangle}, c_{502}^{\left\langle v_{Q_{2}}, v_{3}, v_{2}\right\rangle}, c_{432}^{\left\langle v_{Q_{2}}, v_{3}, v_{2}\right\rangle},\right. \\
& \left.c_{421}^{\left\langle v_{\mathrm{Q}_{2}}, v_{3}, v_{2}\right\rangle}, \mathcal{C}_{412}^{\left\langle v_{\mathrm{Q}_{2}}, v_{3}, v_{2}\right\rangle}, c_{403}^{\left\langle v_{\mathrm{Q}_{2}}, v_{3}, v_{2}\right\rangle}, c_{331}^{\left\langle v_{\mathrm{Q}_{2}}, v_{3}, v_{2}\right\rangle}, c_{322}^{\left\langle v_{\mathrm{Q}_{2}}, v_{3}, v_{2}\right\rangle}, c_{313}^{\left\langle v_{\mathrm{Q}_{2}}, v_{3}, v_{2}\right\rangle}, c_{313}^{\left\langle v_{\mathrm{Q}_{2}}, v_{6}, v_{3}\right\rangle}, c_{331}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}\right)^{T}, \\
& b:=\left(p\left(\xi_{700}^{\left\langle v_{\mathrm{O}_{2}}, v_{3}, v_{2}\right\rangle}\right), p\left(\xi_{610}^{\left\langle v_{\mathrm{O}_{2}}, v_{3}, v_{2}\right\rangle}\right), p\left(\xi_{601}^{\left\langle v_{\mathrm{Q}_{2}}, v_{3}, v_{2}\right\rangle}\right), p\left(\xi_{520}^{\left\langle v_{\mathrm{Q}_{2}}, v_{3}, v_{2}\right\rangle}\right), p\left(\xi_{511}^{\left\langle v_{\mathrm{Q}_{2}}, v_{3}, v_{2}\right\rangle}\right), p\left(\xi_{502}^{\left\langle v_{\mathrm{O}_{2}}, v_{3}, v_{2}\right\rangle}\right),\right. \\
& \left.p\left(\xi_{430}^{\left\langle v_{Q_{2}}, v_{3}, v_{2}\right\rangle}\right), p\left(\xi_{421}^{\left\langle v_{\mathrm{Q}_{2}}, v_{3}, v_{2}\right\rangle}\right), p\left(\xi_{412}^{\left\langle v_{\mathrm{Q}_{2}}, v_{3}, v_{2}\right\rangle}\right), p\left(\xi_{403}^{\left\langle v_{\mathrm{Q}_{2}}, v_{3}, v_{2}\right\rangle}\right), p\left(\xi_{331}^{\left\langle v_{\mathrm{Q}_{2}}, v_{3}, v_{2}\right\rangle}\right), p\left(\xi_{322}^{\left\langle v_{\mathrm{Q}_{2}}, v_{3}, v_{2}\right\rangle}\right), 0,0,0\right)^{T},
\end{aligned}
$$

where $v_{6}=\beta_{1} v_{2}+\gamma_{1} v_{\mathrm{Q}_{2}}$ and $v_{3}=\beta_{2} v_{5}+\gamma_{2} v_{\mathrm{Q}_{2}}$. The determinant of the matrix $\mathbf{G}$ is

$$
\begin{equation*}
-7.07759 \times 10^{-10} \beta_{2} \gamma_{2} \tag{26}
\end{equation*}
$$

By the geometric meaning of the $\beta_{2}$ and $\gamma_{2}$, we have $\beta_{2}<0$ and $\gamma_{2}>0$. Thus $\operatorname{det}(\mathbf{G}) \neq 0$. So, all B coefficients of $\left.s\right|_{\left\langle v_{\mathbb{K}_{2}}, v_{3}, v_{2}\right\rangle}$ are uniquely determined. Applying Lemmas 3 and 4 , it is easy to see that all B coefficients of $Q_{2}$ are uniquely and stable determined by the data. Therefore $\mathscr{M}$ is a stable Lagrange MDS for $\delta_{7}^{2,3}(\theta)$. The proof is completed.

Lemma 7. Suppose that $\oplus$ consists of three triangulated quadrangulations $Q_{1}, Q_{2}, Q_{3}$, and $Q_{4}$ as in Figure 6, where $Q_{1}, Q_{3}, Q_{4} \in \mathscr{B}$ and $Q_{2} \in \mathscr{W}$. Let the set

$$
\begin{align*}
\mathscr{M}:= & \mathscr{M}_{\mathrm{Q}_{1}} \bigcup \mathscr{M}_{\mathrm{Q}_{3}} \bigcup \mathscr{M}_{\mathrm{Q}_{4}} \bigcup m_{1}  \tag{27}\\
& -\bigcup D_{3}^{\left\langle v_{\mathrm{Q}_{4}}, v_{2}, v_{9}\right\rangle}\left(v_{2}\right) \bigcup D_{3}^{\left\langle v_{\mathrm{Q}_{4}}, v_{10}, v_{5}\right\rangle}\left(v_{5}\right)
\end{align*}
$$

where $\mathscr{M}_{\mathrm{Q}_{i}}$ is the set defined in Lemma 3 for the triangulated quadrangulations of $Q_{i}$ and

$$
\begin{align*}
& m_{1}= \bigcup D_{3}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}\left(v_{\mathrm{Q}_{2}}\right) \\
& \bigcup\left\{\xi_{322}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}, \xi_{133}^{\left\langle v_{\mathrm{Q}_{2}}, v_{6}, v_{3}\right\rangle}, \xi_{430}^{\left\langle v_{\mathrm{Q}_{2}}, v_{6}, v_{3}\right\rangle}\right.  \tag{28}\\
&\left.\xi_{313}^{\left\langle v_{\mathrm{Q}_{2}}, v_{5}, v_{6}\right\rangle}, \xi_{430}^{\left\langle v_{\mathrm{Q}_{2}}, v_{3}, v_{2}\right\rangle}, \xi_{331}^{\left\langle v_{\mathrm{Q}_{2}}, v_{3}, v_{2}\right\rangle}\right\}
\end{align*}
$$

be a stable Lagrange MDS for $\delta_{7}^{2,3}(\otimes)$. These domain points are marked with • in Figure 6.

Proof. Using Lemma 3 and $C^{1}, C^{2}$, and $C^{3}$ smoothness conditions, we can see that all of the $B$ coefficients of $s$ associated with domain points in $Q_{1}, Q_{3}$, and $Q_{4}$ are uniquely determined by the data. Writing $\left.s\right|_{\left\langle v_{0_{2}}, v_{2}, v_{5}\right\rangle}$ in B form and using $C^{1}, C^{2}$, and $C^{3}$ smoothness conditions, all B -coefficients are determined except for the B-coefficients of $m:=\bigcup D_{3}^{\left\langle v_{Q_{2}}, v_{2}, v_{5}\right\rangle}\left(v_{\mathrm{Q}_{2}}\right) \bigcup\left\{\xi_{331}^{\left\langle\nu_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}, \xi_{322}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}, \xi_{313}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}\right\}$ associated. The Lagrange interpolation and $C^{1}, C^{2}$ smoothness conditions at $e:=\left\langle v_{\mathrm{Q}_{2}}, v_{2}\right\rangle$ and $e:=\left\langle v_{\mathrm{Q}_{2}}, v_{5}\right\rangle$ imply that these coefficients must satisfy the linear system $\mathbf{G C}=b$, where

$$
\begin{aligned}
& \mathbf{C}:=\left(c_{700}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}, c_{610}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}, c_{601}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}, c_{520}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}, c_{511}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}, c_{502}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}, c_{430}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle},\right. \\
& \left.c_{421}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}, c_{412}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}, c_{403}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}, c_{331}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}, c_{322}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}, c_{313}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}, c_{313}^{\left\langle v_{\mathrm{Q}_{2}}, v_{3}, v_{2}\right\rangle}, c_{331}^{\left\langle v_{\mathrm{Q}_{2}}, v_{5}, v_{6}\right\rangle}\right)^{T}, \\
& b:=\left(p\left(\xi_{700}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}\right), p\left(\xi_{610}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}\right), p\left(\xi_{601}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}\right), p\left(\xi_{520}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}\right), p\left(\xi_{511}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}\right), p\left(\xi_{502}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}\right),\right. \\
& \left.p\left(\xi_{430}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}\right), p\left(\xi_{421}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}\right), p\left(\xi_{412}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}\right), p\left(\xi_{403}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}\right), p\left(\xi_{322}^{\left\langle v_{\mathrm{Q}_{2}}, v_{2}, v_{5}\right\rangle}\right), 0,0,0,0\right)^{T} \text {, }
\end{aligned}
$$

where $v_{2}=\beta_{1} v_{6}+\gamma_{1} v_{\mathrm{Q}_{2}}, v_{3}=\beta_{2} v_{5}+\gamma_{2} v_{\mathrm{Q}_{2}}$. The determinant of the matrix $\mathbf{G}$ is

$$
\begin{equation*}
-1.31659 \times 10^{-8} \beta_{1} \beta_{2} \gamma_{1} \gamma_{2} \tag{30}
\end{equation*}
$$

By the geometric meaning of the $\beta_{1}, \beta_{2}, \gamma_{1}$, and $\gamma_{2}$, we have $\beta_{1}<0, \beta_{2}<0, \gamma_{1}>0$, and $\gamma_{2}>0$. Thus $\operatorname{det}(\mathbf{G}) \neq 0$. So, all B coefficients of $\left.s\right|_{\left\langle v_{\mathcal{Q}_{2}}, v_{3}, v_{2}\right\rangle}$ are uniquely determined. Applying Lemmas 3 and 4 , it is easy to see that all B coefficients of $Q_{2}$ are uniquely and stable determined by the data. Therefore, $\mathscr{M}$ is a stable Lagrange MDS for $\mathcal{S}_{7}^{2,3}(\theta)$. The proof is completed.

## 5. Construction of a Lagrange Interpolation Pair

Theorem 8. Suppose that $\oplus$ is checkerboard triangulation. Then

$$
\begin{align*}
\operatorname{dim} \delta_{7}^{2,3}(\Theta)= & 10 n_{V}+25 n_{B}+22 n_{1}  \tag{31}\\
& +19 n_{2}+16 n_{3}+13 n_{4} .
\end{align*}
$$

Moreover, the following set $\mathscr{M}$ of domain points is a stable Lagrange MDS:
(1) if $Q \in \mathscr{G}$, choose points as in Lemma 3,
(2) if $Q \in \mathscr{B} \backslash \mathscr{G}$, choose points as in Lemma 3, leaving out the points in the sets $D_{3}^{T}$ whenever $v$ is a vertex of $Q$ which is interior to $\diamond$,
(3) suppose that $Q:=\left\langle v_{1}, v_{2}, v_{3}, v_{4}\right\rangle \in \mathscr{W}$, and let $e_{i}:=$ $\left\langle v_{i}, v_{i+1}\right\rangle$ for $i=1,2,3,4$, where $v_{5}=v_{1}$ and $v_{\mathrm{Q}}$ is the point where the two diagonals of $Q$ intersect:
(a) if $Q \in \mathscr{W}_{1}$, choose points as in Lemma 4,
(b) if $Q \in \mathscr{W}_{2}$, choose points as in Lemmas 5 and 6,
(c) if $Q \in \mathscr{W}_{3}$, choose points as in Lemma 7,
(d) if $Q$ sharing four edges with black quadrilaterals, say $e_{1}, e_{2}, e_{3}$, and $e_{4}$, choose the points $\xi_{223}^{T_{1}}, \xi_{403}^{T_{2}}$, $\xi_{403}^{T_{3}}$, and $D_{3}^{T_{1}}\left(v_{\mathrm{Q}}\right)$.

Proof. To establish that $\mathscr{M}$ is a Lagrange MDS, suppose $\delta_{7}^{2,3}(\theta)$ and that we are given values for $s(\xi)$ for all $\xi \in$ $\mathscr{M}$. We need to show that all of the B coefficients of $s$ are uniquely determined. By Lemma 3, all B-coefficient $s$ associated with domain points lying in quadrilaterals $Q \in \mathscr{G}$ is uniquely determined. Now consider $Q \in \mathscr{B} \backslash \mathscr{G}$. For each vertex $v \in Q$ which is an interior vertex of $\diamond$, the B coefficients corresponding to domain points in the disk $D_{3}^{T}$ are already uniquely determined by $C^{1}, C^{2}$, and $C^{3}$ continuity from the neighboring pieces. Leaving the corresponding basis functions out, we can then argue exactly as in Lemma 3 to see that all B coefficients of $s$ corresponding to the remaining domain points in $Q$ are uniquely determined.

Now suppose that $Q \in \mathscr{W}$. If $Q \in \mathscr{W}_{1}$, then, using Lemma 4, we can see that all B coefficients of $s$ corresponding to the remaining domain points in $Q$ are uniquely determined. If $Q \in \mathscr{V}_{2}$, then, using Lemmas 5 and 6 , we can see that all B-coefficients of $s$ corresponding to the remaining domain points in $Q$ are uniquely determined. If $Q \in \mathscr{V}_{3}$, then using Lemma 7, we can see that all B coefficients of $s$ corresponding to the remaining domain points in $Q$ are uniquely determined. If $Q$ shares four edges with black quadrilaterals, we can then argue exactly as in Lemma 7 to see that all B coefficients of $s$ corresponding to the remaining domain points in $Q$ are uniquely determined. Since we have shown that $\mathscr{M}$ is a MDS, it follows that $\operatorname{dim} \mathcal{S}_{7}^{2,3}(\theta)=\# \mathscr{M}$.

Finally, we note that all of the above computations are stable in the sense that the size of the computed B coefficient is bounded by a constant depending only on the smallest angle in the triangulation $\Delta$. This follows from that the computations of Lemmas 2-7 are stable, and the fact that computing coefficients from $C^{1}, C^{2}$, and $C^{3}$ smoothness conditions is automatically stable.

According to the Theorem 8, we have presented a Lagrange MDS for Figure 1 (see Figure 7).

Theorem 9. Let $\oplus$ be a checkerboard triangulation, and let $\mathscr{M}$ be the set defined in Theorem 8. Then there exists a stable local basis $\left\{B_{i}\right\}_{i=1}^{n}$ of $\mathcal{S}_{7}^{2,3}(\oplus)$, where $n:=\# \mathscr{M}$.

Proof. It is clear that if $\mathscr{M}$ is a MDS for $\mathcal{S}_{7}^{2,3}(\theta)$, then, for each $\xi \in \mathcal{M}$, there exists a unique spline $B_{\xi} \in S$, satisfying $\lambda_{\eta} B_{\xi}=\delta_{\xi, \eta}$, for all $\eta \in \mathscr{M}$. The splines $B_{\xi}$ obviously form a basis for $\mathcal{S}_{7}^{2,3}(\theta)$, which is commonly called the dual basis corresponding to $\mathscr{M}$.

The proof of Theorem 8 shows that all B coefficients of $B_{\xi}$ are uniquely and stable determined. So we should give the support of $B_{\xi}$. As in [15], suppose that $\xi$ lies in a quadrilateral $Q$. Then we claim that
(1) $\operatorname{supp}\left(B_{\xi}\right)=Q$ if $\xi=\mathscr{W}$,
(2) $\operatorname{supp}\left(B_{\xi}\right) \subset \operatorname{star}\left(Q_{\xi}\right)$ if $\xi \in \mathscr{B} \backslash \mathscr{G}$,
(3) $\operatorname{supp}\left(B_{\xi}\right) \subset \operatorname{star}^{2}\left(Q_{\xi}\right)$ otherwise.

We are now ready to discuss interpolation. Suppose that $\theta$ is a checkerboard triangulation, and that $B_{\xi}$ are the dual basis function of Theorem 9 corresponding to Lagrange MDS
for $\mathcal{S}_{7}^{2,3}(\theta)$ defined in Theorem 8. Then for every $f \in C(\Omega)$, there is a unique spline $s=\mathscr{J} f \in \mathcal{S}$ which satisfies $s(\eta)=$ $\mathscr{F} f(\eta)=f(\eta)$, for all $\eta \in \mathscr{M}$. This defines a linear projector $\mathscr{F}$ mapping $C(\Omega)$ onto $\mathcal{S}$. We now give an error bound for this interpolation method.

Theorem 10. Suppose $f$ lies in the Sobolev space $W_{\infty}^{m+1}(\Omega)$ for some $0 \leq m \leq 7$. Then

$$
\begin{equation*}
\left\|D_{x}^{\alpha} D_{y}^{\beta}(f-\mathscr{I} f)\right\|_{\Omega} \leq K|\Delta|^{m+1-\alpha-\beta}|f|_{m+1, \Omega} \tag{32}
\end{equation*}
$$

for $0 \leq \alpha+\beta \leq m$. Here $|\cdot|_{m+1, \Omega}$ is usual Sobolev seminorm, and $|\Delta|$ is the maximum of the diameters of the triangles in $\Delta$. The constant $K$ depends only on the smallest angle in $\Delta$.

Proof. It was shown in Section 10 of [16] that if a space of splines $\mathcal{S}$ of degree $d$ contains $\mathscr{P}_{d}$ and has a stable local basis, then it provides optimal order approximations of smooth functions.

The result of Theorem 10 can also be established with the weak-interpolation methods described in [17].

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