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Research Article

Variational Iteration Method for Nonlinear Singular Two-Point Boundary Value Problems Arising in Human Physiology

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The variational iteration method is applied to solve a class of nonlinear singular boundary value problems that arise in physiology. The process of the method, which produces solutions in terms of convergent series, is explained. The Lagrange multipliers needed to construct the correctional functional are found in terms of the exponential integral and Whittaker functions. The method easily overcomes the obstacle of singularities. Examples will be presented to test the method and compare it to other existing methods in order to confirm fast convergence and significant accuracy.

1. Introduction

In this paper, He's variational iteration method (VIM) [1, 2] is applied to obtain an approximate solution for the following nonlinear singular two-point boundary value problem (BVP):

$$\left(p(x)\,y'\right)' = p(x)\,f(x,y), \quad 0 \le x \le 1, \tag{1}$$

with the following two sets of boundary conditions:

$$y'(0) = 0, \qquad \alpha y(1) + \beta y'(1) = \gamma, \tag{2}$$

$$y(0) = A, \qquad \alpha y(1) + \beta y'(1) = \gamma, \tag{3}$$

where

$$p(x) = x^b h(x), \quad x \in [0,1]. \tag{4}$$

Here $\alpha > 0$, $\beta \ge 0$, and A and γ are finite constants. It is assumed that $p(x)$ is nonnegative, continuously differentiable on $[0,1]$, and $1/h(x)$ is analytic in the disk $|z| < r$ for some $r > 1$. It is also assumed that $f(x,y)$ and $\partial f/\partial y$ are both continuous on $[0,1] \times \mathbb{R}$ and further $\partial f/\partial y \ge 0$.

Assuming that xp'/p is analytic in $|z| < r$ for some $r > 1$, the existence-uniqueness has been established for problem (1) under the following restrictions (see [3] and the references therein):

(1) BC (2) holds for $\alpha = 1$, $\beta = 0$ and such that $b \ge 0$, and

(2) BC (3) holds for $\alpha = 1$, $\beta = 0$, and such that $0 \le b \le 1$.

Equation (1) with BC (3) arises in the study of tumor growth problems [4, 5] where $b = 0, 1, 2$, $h(x) = 1$, and $f(x,y)$ is of the form

$$f(x,y) = \frac{\theta y}{y+\kappa}, \quad \theta, \kappa > 0. \tag{5}$$

This problem also arises in the study of a steady-state oxygen diffusion in a cell with Michaleis-Menten uptake kinetics when $b = 2$ and $h(x) = 1$ [6]. Another application of this problem appeared in the study of heat sources distribution in the human head [7] given that $b = 2$, $h(x) = 1$, and

$$f(x,y) = -\delta e^{-\theta y}, \quad \theta, \delta > 0. \tag{6}$$

Problems (1)-(2) and (1)–(3) have been treated by several authors using different numerical schemes such as cubic B-splines, finite difference, and pointwise solution bounds [3, 8, 9]. The VIM has been used by Wazwaz [10] to solve the nonlinear singular Emden-Fowler boundary value problem, which is a special case of (1)-(2), where $p(x) = x^b$. Also, Ravi

Kanth and Aruna [11] used the variational iteration method to solve (1)-(2) and (1)–(3), but that was also done under the assumption that $h(x) = 1$.

The objective of this paper is to apply the VIM to obtain an approximate solution for the proposed nonlinear singular boundary value problems with more relaxed $p(x)$. The VIM has been intensively used to solve linear and nonlinear, ordinary and partial, delay and fractional order differential equations [12–18]. In a recent review article, He [19] shows how the VIM can be employed as an effective method in searching for wave solutions including solitons and compacton solutions without the need for linearization or weak nonlinearity assumptions. For nonlinear differential equations, the VIM gives an approximate solution in a series form that converges to the exact solution, if the latter exists. In case of linear differential equations, exact solution by the VIM can be readily obtained by a one iteration step because the exact Lagrange multiplier needed to construct the correctional functional can be identified.

2. Variational Iteration Method

Consider the general nonlinear differential equation given by

$$Ly(t) + Ny(t) = g(t), \tag{7}$$

where L and N are linear and nonlinear operators, respectively, and $g(t)$ is a given analytical function. As per the variational iteration method, the correction functional is constructed as follows:

$$\begin{aligned} y_{n+1}(x) &= y_n(x) \\ &\quad + \int_0^x \lambda(\xi)\left(Ly_n(\xi) + N\widetilde{y}_n(\xi) - g(\xi)\right)d\xi, \end{aligned} \tag{8}$$

where λ is a general Lagrange multiplier, which can be optimally identified by the variational theory, and $\widetilde{y}_n$ is a restricted variation and hence $\delta\widetilde{y}_n = 0$. A proper choice of the initial approximation y_0 is crucial to obtain a successive approximations y_{n+1} that converge as rapidly as possible to the exact solution.

For simplicity, (1) will be expressed in the form

$$y'' + \left(\frac{b}{x} + \frac{h'(x)}{h(x)}\right)y' = f(x, y), \tag{9}$$

and its correction functional is given by

$$\begin{aligned} y_{n+1}(x) &= y_n(x) + \int_0^x \lambda(\xi) \\ &\quad \times \left(\left(y_n(\xi)\right)_{\xi\xi} + \left(\frac{b}{\xi} + \frac{h'(\xi)}{h(\xi)}\right)\left(y_n(\xi)\right)_{\xi}\right. \\ &\quad \left. - \widetilde{f}(\xi, y_n) - g(\xi)\right)d\xi. \end{aligned} \tag{10}$$

Taking the variation with respect to y_n gives

$$\begin{aligned} \delta y_{n+1}(x) &= \delta y_n(x) + \delta\int_0^x \lambda(\xi) \\ &\quad \times \left(\left(y_n(\xi)\right)_{\xi\xi} + \left(\frac{b}{\xi} + \frac{h'(\xi)}{h(\xi)}\right)\left(y_n(\xi)\right)_{\xi}\right)d\xi. \end{aligned} \tag{11}$$

Using integration by parts in (11) gives

$$\begin{aligned} &\delta y_{n+1}(x) \\ &\quad = \delta y_n(x)\left(1 - \lambda'(x) + \left(\frac{b}{x} + \frac{h'(\xi)}{h(\xi)}\right)\lambda(x)\right) \\ &\qquad + \delta\left(y_n\right)_{\xi}(x)\lambda(x) \\ &\qquad + \int_0^x \delta y_n(\xi)\left(\lambda''(\xi) - \left(\frac{b}{\xi} + \frac{h'(\xi)}{h(\xi)}\right)\lambda'(\xi)\right. \\ &\qquad\qquad \left. + \frac{b}{\xi^2}\lambda(\xi)\right)d\xi, \end{aligned} \tag{12}$$

which in turn leads to the stationary relations

$$\begin{gathered} 1 - \lambda'(x) + \left(\frac{b}{x} + \frac{h'(x)}{h(x)}\right)\lambda(x) = 0, \\ \lambda(x) = 0, \\ \lambda''(x) - \left(\frac{b}{x} + \frac{h'(x)}{h(x)}\right)\lambda'(x) + \frac{b}{x^2}\lambda(x) = 0. \end{gathered} \tag{13}$$

Depending on b, the Lagrange multiplier can be determined by solving system (13). In the next section, we will use $h(x) = e^x$, and hence the choice of b will lead to one of the following two Lagrange multipliers.

If $b \geq 1$, the solution of (13) is given in terms of the exponential integral $E_n(x)$ [20]. For example, if $b = 1$, then

$$\lambda(\xi) = \left(E_1(\xi) - E_1(x)\right)\xi e^{\xi}, \tag{14}$$

where $E_1(x)$ is the generalized complex exponential integral given by

$$E_1(x) = \int_x^{\infty} \frac{e^{-t}}{t}dt. \tag{15}$$

If $b < 1$, the solution of (13) is given in terms of Whittaker function $M_{p,q}(x)$ [20]. For example, if $b = 1/4$, then

$$\begin{aligned} \lambda(\xi) &= \frac{4e^{-x}}{21x^{5/8}\xi^{3/8}}\left(-\alpha(x,\xi)M_{3/8,7/8}(x) - \beta(x,\xi)\right. \\ &\qquad \left. + \beta(\xi,x) + \alpha(\xi,x)M_{3/8,7/8}(\xi)\right), \end{aligned} \tag{16}$$

where

$$\alpha(m,n) = 4mn^{5/8}e^{1/2m+n},$$
$$\beta(m,n) = 7m^{11/8}n^{5/8}e^{n}, \tag{17}$$

and $M_{p,q}(x)$ is given by

$$M_{p,q}(x) = e^{-x/2}x^{q+1/2}\sum_{k=0}^{\infty}\frac{(1/2+q-p)_k}{k!\,(2q+1)_k}x^k, \tag{18}$$

in which $(a)_n$ is known as the Pochhammer symbol [20] and is defined by

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1),$$
$$(a)_0 = 1. \tag{19}$$

Now that λ is constructed, an appropriate first guess $y_0(x)$ will lead to the formation of the recurrence sequences $\{y_n(x)\}$. The solution for (1)-(2) and (1)–(3) is, then, obtained from the relation

$$y(x) = \lim_{n\to\infty} y_n(x), \tag{20}$$

provided that the limit exists. The proof that the sequences $\{y_n(x)\}$ are convergent has been well established in the literature (see, e.g., [11, 21]).

3. Numerical Examples

Example 1. Consider the nonlinear singular BVP

$$(p(x)y')' = p(x)f(x,y), \quad x\in[0,1],$$
$$y'(0) = 0, \qquad y(1)+5y'(1) = -5-\ln 5, \tag{21}$$

where

$$p(x) = x^b g(x), \quad g(x) = e^x,$$
$$f(x,y) = \frac{5x^3\left(5x^5e^y - x - b - 4\right)}{4+x^5}. \tag{22}$$

The analytic solution of (21) is $y = -\ln(x^5+4)$ for all $b\in R$. This problem is an application of oxygen diffusion corresponding to (1), (2), and (5) with $p(x) = x^2$, $\theta = 0.76129$, $\kappa = 0.03119$, $\alpha = 5$, $\beta = 1$, and $\gamma = 5$. In Tables 1 and 3, the second and third iteration solutions are compared to the exact solution when $b = 0.25$ and $b = 8$, respectively. In Tables 2 and 4, the absolute maximum errors obtained by the proposed VIM is compared with that of other methods.

Example 2. Consider the nonlinear singular BVP

$$(p(x)y')' = p(x)f(x,y), \quad x\in[0,1],$$
$$y(0) = -\ln 4, \qquad y(1)+5y'(1) = -5-\ln 5, \tag{23}$$

where $p(x)$, $g(x)$, and $f(x,y)$ are as in (22).

Table 1: Numerical results for Example 1 ($b = 0.25$).

x	y_2 solution	y_3 solution	Exact solution
0.1	−1.386471621	−1.386300039	−1.386296861
0.2	−1.386549118	−1.386377536	−1.386374358
0.3	−1.387076437	−1.386904854	−1.386901677
0.4	−1.389025850	−1.388854268	−1.388851090
0.5	−1.394251264	−1.394079679	−1.394076502
0.6	−1.405722595	−1.405550997	−1.405547818
0.7	−1.427627934	−1.427456278	−1.427453099
0.8	−1.465206626	−1.465034787	−1.465031602
0.9	−1.524162176	−1.523989964	−1.523986772
1.0	−1.609612516	−1.609441112	−1.609437912

Table 2: Maximum error for Example 1 ($b = 0.25$).

Present method	Method in [9]	Method in [3]
1.74×10^{-4} ($n = 2$)	7.79×10^{-4} ($n = 16$)	1.17×10^{-3} ($n = 16$)
3.20×10^{-6} ($n = 3$)	1.98×10^{-4} ($n = 32$)	3.04×10^{-4} ($n = 32$)

Table 3: Numerical results for Example 1 ($b = 8$).

x	y_2 solution	y_3 solution	Exact solution
0.1	−1.386409538	−1.386300604	−1.386296861
0.2	−1.386487035	−1.386378101	−1.386374358
0.3	−1.387014353	−1.386905420	−1.386901677
0.4	−1.388963767	−1.388854833	−1.388851090
0.5	−1.394189179	−1.394080244	−1.394076502
0.6	−1.405660501	−1.405551561	−1.405547818
0.7	−1.427565803	−1.427456843	−1.427453099
0.8	−1.465144370	−1.465035348	−1.465031602
0.9	−1.524099638	−1.523990525	−1.523986772
1.0	−1.609550088	−1.609441663	−1.609437912

This problem is an application of the nonlinear heat conduction model of the human head corresponding to (1), (3), and (6) with $p(x) = x^2$, $\delta = \theta = 1$, and $\gamma = 0$. In Tables 5 and 7, the second and third iteration solutions are compared to the exact solution when $b = 0.25$ and $b = 0.75$, respectively, and in Tables 6 and 8, the absolute maximum errors obtained by the proposed VIM is compared with that of other methods.

4. Conclusion

In this paper, the simplicity and reliability of He's variational iteration method have been tested against a general form of nonlinear two point singular value problems that arise in physiology. The method is shown to be highly accurate and easily implemented even with the presence of singularities. For the two examples shown in Section 3, the second and third iteration solutions obtained by the VIM were substantially more accurate than the 16 and the 32 time-step approximate solutions obtained by cubic B-spline and the finite difference methods.

Table 4: Maximum error for Example 1 ($b = 8$).

Present method	Method in [9]	Method in [3]
1.12×10^{-4} ($n = 2$)	2.52×10^{-3} ($n = 16$)	4.11×10^{-3} ($n = 16$)
3.75×10^{-6} ($n = 3$)	6.30×10^{-4} ($n = 32$)	9.76×10^{-4} ($n = 32$)

Table 5: Numerical results for Example 2 ($b = 0.25$).

x	y_2 solution	y_3 solution	Exact solution
0.1	−1.386296861	−1.386296861	−1.386296861
0.2	−1.386374358	−1.386374358	−1.386374358
0.3	−1.386901676	−1.386901676	−1.386901677
0.4	−1.388851090	−1.388851090	−1.388851090
0.5	−1.394076501	−1.394076501	−1.394076502
0.6	−1.405547818	−1.405547818	−1.405547818
0.7	−1.427453099	−1.427453099	−1.427453099
0.8	−1.465031592	−1.465031601	−1.465031602
0.9	−1.523986605	−1.523986761	−1.523986772
1.0	−1.609436979	−1.609437790	−1.609437912

Table 6: Maximum error for Example 2 ($b = 0.25$).

Present method	Method in [9]	Method in [3]
9.33×10^{-7} ($n = 2$)	5.60×10^{-4} ($n = 16$)	7.85×10^{-4} ($n = 16$)
1.22×10^{-7} ($n = 3$)	1.41×10^{-4} ($n = 32$)	1.94×10^{-4} ($n = 32$)

Table 7: Numerical results for Example 2 ($b = 0.75$).

x	y_2 solution	y_3 solution	Exact solution
0.1	−1.386296861	−1.386296861	−1.386296861
0.2	−1.386374358	−1.386374358	−1.386374358
0.3	−1.386901677	−1.386901677	−1.386901677
0.4	−1.388851090	−1.388851090	−1.388851090
0.5	−1.394076501	−1.394076502	−1.394076502
0.6	−1.405547818	−1.405547818	−1.405547818
0.7	−1.427453098	−1.427453099	−1.427453099
0.8	−1.465031590	−1.465031600	−1.465031602
0.9	−1.523986772	−1.523986737	−1.523986772
1.0	−1.609436938	−1.609437612	−1.609437912

Table 8: Maximum error for Example 2 ($b = 0.75$).

Present method	Method in [9]	Method in [3]
9.74×10^{-7} ($n = 2$)	1.04×10^{-3} ($n = 16$)	7.94×10^{-4} ($n = 16$)
3.00×10^{-7} ($n = 3$)	2.61×10^{-4} ($n = 32$)	2.00×10^{-4} ($n = 32$)

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