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Research Article

Properties (B) and (gB) for Bounded Linear Operators

M. H. M. Rashid

Department of Mathematics & Statistics, Faculty of Science, Mu'tah University, P.O. Box 7, Al-Karak, Jordan

Correspondence should be addressed to M. H. M. Rashid; malik_okasha@yahoo.com

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We shall consider properties which are related to Weyl type theorem for bounded linear operators $T \in \mathscr{L}(\mathscr{X})$, defined on a complex Banach space $\mathscr{X}$. These properties, that we call *property* (B), means that the set of all poles of the resolvent of T of finite rank in the usual spectrum are exactly those points λ of the spectrum for which $T-\lambda$ is an upper semi-Fredholm with index less than or equal to 0 and we call *property* (gB), means that the set of all poles of the resolvent of T in the usual spectrum are exactly those points λ of the spectrum for which $T-\lambda$ is an upper semi-B-Fredholm with index less than or equal to 0. Properties (B) and (gB) are related to a strong variants of classical Weyl's theorem, the so-called property (b) and property (gb) We shall characterize properties (B) and (gB) in several ways and we shall also describe the relationships of it with the other variants of Weyl type theorems. Our main tool is localized version of the single valued extension property. Also, we consider the properties (B) and (gB) in the frame of polaroid type operators.

1. Introduction and Preliminary

Throughout this paper, $\mathscr{X}$ denotes an infinite-dimensional complex Banach space, $\mathscr{L}(\mathscr{X})$ the algebra of all bounded linear operators on $\mathscr{X}$. For an operator $T \in \mathscr{L}(\mathscr{X})$ we shall denote by $\alpha(T)$ the dimension of the *kernel* $\ker(T)$, and by $\beta(T)$ the codimension of the *range* $\mathbb{R}(T)$. Let

$$\mathrm{SF}_+(\mathscr{X}) := \{T \in \mathscr{L}(\mathscr{X}) : \alpha(T) < \infty,\ \mathbb{R}(T) \text{ is closed}\} \tag{1}$$

be the class of all *upper semi-Fredholm* operators, and let

$$\mathrm{SF}_-(\mathscr{X}) := \{T \in \mathscr{L}(\mathscr{X}) : \beta(T) < \infty\} \tag{2}$$

be the class of all *lower semi-Fredholm* operators. The class of all *semi-Fredholm* operators is defined by $\mathrm{SF}_\pm(\mathscr{X}) := \mathrm{SF}_+(\mathscr{X}) \cup \mathrm{SF}_-(\mathscr{X})$, while the class of all *Fredholm* operators is defined by $\mathrm{SF}(\mathscr{X}) := \mathrm{SF}_+(\mathscr{X}) \cap \mathrm{SF}_-(\mathscr{X})$. If $T \in \mathrm{SF}_\pm(\mathscr{X})$, the *index* of T is defined by

$$\mathrm{ind}(T) := \alpha(T) - \beta(T). \tag{3}$$

Recall that a bounded operator T is said *bounded below* if it injective and has closed range. Evidently, if T is bounded below then $T \in \mathrm{SF}_+(\mathscr{X})$ and $\mathrm{ind}(T) \le 0$. Define

$$\begin{aligned} \mathrm{SF}_+^-(\mathscr{X}) &:= \{T \in \mathrm{SF}_+(\mathscr{X}) : \mathrm{ind}(T) \le 0\}, \\ \mathrm{SF}_-^+(\mathscr{X}) &:= \{T \in \mathrm{SF}_-(\mathscr{X}) : \mathrm{ind}(T) \ge 0\}. \end{aligned} \tag{4}$$

The set of *Weyl* operators is defined by

$$W(\mathscr{X}) := \mathrm{SF}_+^-(\mathscr{X}) \cap \mathrm{SF}_-^+(\mathscr{X}) = \{T \in \mathrm{SF}(\mathscr{X}) : \mathrm{ind}(T) = 0\}. \tag{5}$$

The classes of operators defined above generate the following spectra. Denote by

$$\sigma_a(T) := \{\lambda \in \mathbb{C} : T-\lambda \text{ is not bounded below}\} \tag{6}$$

the approximate point spectrum, and by

$$\sigma_s(T) := \{\lambda \in \mathbb{C} : T-\lambda \text{ is not surjective}\} \tag{7}$$

the surjectivity spectrum of $T \in \mathscr{L}(\mathscr{X})$. The Weyl spectrum is defined by

$$\sigma_w(T) := \{\lambda \in \mathbb{C} : T-\lambda \notin W(\mathscr{X})\}, \tag{8}$$

the Weyl essential approximate point spectrum is defined by

$$\sigma_{\mathrm{SF}_+^-}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin \mathrm{SF}_+^-(\mathcal{X})\}, \tag{9}$$

while the Weyl essential surjectivity spectrum is defined by

$$\sigma_{\mathrm{SF}_-^+}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin \mathrm{SF}_-^+(\mathcal{X})\}. \tag{10}$$

Obviously, $\sigma_w(T) = \sigma_{\mathrm{SF}_+^-}(T) \cup \sigma_{\mathrm{SF}_-^+}(T)$ and from basic Fredholm theory we have

$$\sigma_{\mathrm{SF}_+^-}(T) = \sigma_{\mathrm{SF}_-^+}(T^*), \qquad \sigma_{\mathrm{SF}_-^+}(T) = \sigma_{\mathrm{SF}_+^-}(T^*). \tag{11}$$

Note that $\sigma_{\mathrm{SF}_+^-}(T)$ is the intersection of all approximate point spectra $\sigma_a(T + K)$ of compact perturbations K of T, while $\sigma_{\mathrm{SF}_-^+}(T)$ is the intersection of all surjectivity spectra $\sigma_s(T + K)$ of compact perturbations K of T, see, for instance, [1, Theorem 3.65].

Recall that the *ascent*, $a(T)$, of an operator T is the smallest non-negative integer p such that $\ker(T^p) = \ker(T^{p+1})$. If such integer does not exist we put $a(T) = \infty$. Analogously, the *descent*, $d(T)$, of an operator T is the smallest non-negative integer q such that $\mathbb{R}(T^q) = \mathbb{R}(T^{q+1})$, and if such integer does not exist we put $d(T) = \infty$. It is well known that if $a(T)$ and $d(T)$ are both finite then $a(T) = d(T)$ [2, Proposition 1.49]. Moreover, $0 < a(T - \lambda) = d(T - \lambda) < \infty$ precisely when λ is a pole of the resolvent of T, see Dowson [2, Theorem 1.54].

The class of all *upper semi-Browder* operators is defined by

$$B_+(\mathcal{X}) := \{T \in \mathrm{SF}_+(\mathcal{X}) : a(T) < \infty\}, \tag{12}$$

while the class of all *lower semi-Browder* operators is defined by

$$B_-(\mathcal{X}) := \{T \in \mathrm{SF}_+(\mathcal{X}) : d(T) < \infty\}. \tag{13}$$

The class of all *Browder* operators is defined by

$$\begin{aligned} B(\mathcal{X}) &:= B_+(\mathcal{X}) \cap B_-(\mathcal{X}) \\ &= \{T \in \mathrm{SF}(\mathcal{X}) : a(T),\ d(T) < \infty\}. \end{aligned} \tag{14}$$

We have

$$\begin{gathered} B(\mathcal{X}) \subseteq W(\mathcal{X}), \qquad B_+(\mathcal{X}) \subseteq \mathrm{SF}_+^-(\mathcal{X}), \\ B_-(\mathcal{X}) \subseteq \mathrm{SF}_-^+(\mathcal{X}), \end{gathered} \tag{15}$$

see [1, Theorem 3.4]. The *Browder spectrum* of $T \in \mathcal{L}(\mathcal{X})$ is defined by

$$\sigma_b(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin B(\mathcal{X})\}, \tag{16}$$

the *upper Browder* spectrum is defined by

$$\sigma_{ub}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin B_+(\mathcal{X})\}, \tag{17}$$

and analogously the *lower Browder spectrum* is defined by

$$\sigma_{lb}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin B_-(\mathcal{X})\}. \tag{18}$$

Clearly, $\sigma_b(T) = \sigma_{ub}(T) \cup \sigma_{lb}(T)$ and $\sigma_w(T) \subseteq \sigma_b(T)$.

For $T \in B(X)$ and a nonnegative integer n define $T_{[n]}$ to be the restriction of T to $\mathbb{R}(T^n)$ viewed as a map from $\mathbb{R}(T^n)$ into $\mathbb{R}(T^n)$ (in particular, $T_{[0]} = T$). If for some integer n the range space $\mathbb{R}(T^n)$ is closed and $T_{[n]}$ is an *upper (a lower) semi-Fredholm operator*, then T is called an upper (a lower) semi-B-Fredholm operator. In this case the index of T is defined as the index of the semi-Fredholm operator $T_{[n]}$, see [3]. Moreover, if $T_{[n]}$ is a Fredholm operator, then T is called a *B-Fredholm operator.* A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator. An operator T is said to be a *B-Weyl operator* if it is a B-Fredholm operator of index zero. The *B-Weyl spectrum* $\sigma_{BW}(T)$ of T is defined by $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda$ is not a B-Weyl operator$\}$.

An operator $T \in \mathcal{L}(\mathcal{X})$ is called *Drazin invertible* if it has a finite ascent and descent. The *Drazin spectrum* $\sigma_D(T)$ of an operator T is defined by $\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda$ is not a Drazinin vertible$\}$. Define also the set $LD(X)$ by $LD(X) = \{T \in \mathcal{L}(\mathcal{X}) : a(T) < \infty$ and $\mathbb{R}(T^{a(T)+1})$ is closed$\}$ and $\sigma_{LD}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin LD(X)\}$. Following [4], an operator $T \in \mathcal{L}(\mathcal{X})$ is said to be *left Drazin invertible* if $T \in LD(X)$. We say that $\lambda \in \sigma_a(T)$ is a *left pole* of T if $T - \lambda \in LD(X)$, and that $\lambda \in \sigma_a(T)$ is a left pole of T of finite rank if λ is a left pole of T and $\alpha(T - \lambda) < \infty$. Let $\pi_a(T)$ denotes the set of all left poles of T and let π_a^0 denotes the set of all left poles of T of finite rank. From Theorem 2.8 of [4] it follows that if $T \in \mathcal{L}(\mathcal{X})$ is left Drazin invertible, then T is an upper semi-B-Fredholm operator of index less than or equal to 0.

Let $\pi(T)$ be the set of all poles of the resolvent of T and let $\pi^0(T)$ be the set of all poles of the resolvent of T of finite rank, that is $\pi^0(T) = \{\lambda \in \pi(T) : \alpha(T - \lambda) < \infty\}$. According to [5], a complex number λ is a pole of the resolvent of T if and only if $0 < \max\{a(T - \lambda), d(T - \lambda)\} < \infty$. Moreover, if this is true then $a(T - \lambda) = d(T - \lambda)$.

According also to [5], the space $\mathbb{R}((T - \lambda)^{a(T-\lambda)+1})$ is closed for each $\lambda \in \pi(T)$. Hence we have always $\pi(T) \subset \pi_a(T)$ and $\pi^0(T) \subset \pi_a^0(T)$. We say that *Browder's theorem* holds for $T \in \mathcal{L}(\mathcal{X})$ if $\Delta(T) = \pi^0(T)$, and that *a-Browder's theorem* holds for $T \in \mathcal{L}(\mathcal{X})$ if $\Delta_a(T) = \pi_a^0(T)$. Following [6], we say that *generalized Weyl's theorem* holds for $T \in \mathcal{L}(\mathcal{X})$ if $\Delta^g(T) = \sigma(T) \setminus \sigma_{BW}(T) = E(T)$, where $E(T) = \{\lambda \in \operatorname{iso}\sigma(T) : \alpha(T - \lambda) > 0\}$ is the set of all isolated eigenvalues of T, and that *generalized Browder's theorem* holds for $T \in \mathcal{L}(\mathcal{X})$ if $\Delta^g(T) = \pi(T)$. It is proved in Theorem 2.1 of [7] that generalized Browder's theorem is equivalent to Browder's theorem. In [4, Theorem 3.9], it is shown that an operator satisfying generalized Weyl's theorem satisfies also Weyl's theorem, but the converse does not hold in general. Nonetheless and under the assumption $E(T) = \pi(T)$, it is proved in Theorem 2.9 of [8] that generalized Weyl's theorem is equivalent to Weyl's theorem.

Let $\mathrm{SBF}_+(X)$ be the class of all upper semi-B-Fredholm operators, $\mathrm{SBF}_+^-(X) = \{T \in \mathrm{SBF}_+(X) : \operatorname{ind}(T) \le 0\}$. The *upper B-Weyl spectrum* of T is defined by $\sigma_{\mathrm{SBF}_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \mathrm{SBF}_+^-(X)\}$. We say that *generalized a-Weyl's theorem* holds for $T \in \mathcal{L}(\mathcal{X})$ if $\Delta_a^g(T) = \sigma_a(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T) = E_a(T)$, where $E_a(T) = \{\lambda \in \operatorname{iso}\sigma_a(T) : \alpha(T - \lambda) > 0\}$

is the set of all eigenvalues of T which are isolated in $\sigma_a(T)$ and that $T \in \mathscr{L}(\mathscr{X})$ obeys *generalized a-Browder's theorem* if $\Delta_a^g(T) = \pi_a(T)$. It is proved in [7, Theorem 2.2] that generalized a-Browder's theorem is equivalent to a-Browder's theorem, and it is known from [4, Theorem 3.11] that an operator satisfying generalized a-Weyl's theorem satisfies a-Weyl's theorem, but the converse does not hold in general and under the assumption $E_a(T) = \pi_a(T)$ it is proved in [8, Theorem 2.10] that generalized a-Weyl's theorem is equivalent to a-Weyl's theorem.

Following [9], we say that $T \in \mathscr{L}(\mathscr{X})$ possesses *property* (w) if $\Delta_a(T) = E^0(T)$. The property (w) has been studied in [1]. In Theorem 2.8 of [10], it is shown that property (w) implies Weyl's theorem, but the converse is not true in general. We say that $T \in \mathscr{L}(\mathscr{X})$ possesses *property* (gw) if $\Delta_a^g(T) = E(T)$. Property (gw) has been introduced and studied in [11, 12]. Property (gw) extends property (w) to the context of B-Fredholm theory, and it is proved in [11] that an operator possessing property (gw) possesses property (w) but the converse is not true in general. According to [13], an operator $T \in \mathscr{L}(\mathscr{X})$ is said to possess *property* (gb) if $\Delta_a^g(T) = \pi(T)$, and is said to possess *property* (b) if $\Delta_a(T) = \pi^0(T)$. It is shown in Theorem 2.3 of [13] that an operator possessing property (gb) possesses property (b) but the converse is not true in general, see also [14]. Following [15], we say an operator $T \in \mathscr{L}(\mathscr{X})$ is said to be satisfies *property* (R) if $\pi_a^0(T) = E^0(T)$. In Theorem 2.4 of [15], it is shown that T satisfies property (w) if and only if T satisfies a-Browder's theorem and T satisfies property (R).

The *single valued extension property* plays an important role in local spectral theory, see the recent monograph of Laursen and Neumann [16] and Aiena [1]. In this article we shall consider the following local version of this property, which has been studied in recent papers, [10, 17] and previously by Finch [18].

Let $H(\sigma(T))$ be the space of all functions that analytic in an open neighborhoods of $\sigma(T)$. Following [18] we say that $T \in \mathscr{L}(\mathscr{X})$ has the single-valued extension property (SVEP) at point $\lambda \in \mathbb{C}$ if for every open neighborhood U_λ of λ, the only analytic function $f : U_\lambda \to \mathscr{X}$ which satisfies the equation $(T - \mu) f(\mu) = 0$ is the constant function $f \equiv 0$. It is well-known that $T \in \mathscr{L}(\mathscr{X})$ has SVEP at every point of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, from the identity Theorem for analytic function it easily follows that $T \in \mathscr{L}(\mathscr{X})$ has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has SVEP at every isolated point of $\sigma(T)$. In [17, Proposition 1.8], Laursen proved that if T is of finite ascent, then T has SVEP.

Theorem 1 (see [19, Theorem 1.3]). *If $T \in SF_\pm(\mathbb{X})$ the following statements are equivalent:*

(i) *T has SVEP at λ_0;*

(ii) *$a(T - \lambda_0) < \infty$;*

(iii) *$\sigma_a(T)$ does not cluster at λ_0;*

(iv) *$H_0(T - \lambda_0)$ is finite dimensional.*

By duality we have.

Theorem 2. *If $T \in SF_\pm(\mathbb{X})$ the following statements are equivalent:*

(i) *T^* has SVEP at λ_0;*

(ii) *$d(T - \lambda_0) < \infty$;*

(iii) *$\sigma_s(T)$ does not cluster at λ_0.*

In this paper we shall consider properties which are related to Weyl type theorem for bounded linear operators $T \in \mathscr{L}(\mathscr{X})$, defined on a complex Banach space $\mathscr{X}$. These properties, that we call *property* (B), means that the set of all poles of the resolvent of T of finite rank in the usual spectrum are exactly those points λ of the spectrum for which $T - \lambda$ is an upper semi-Fredholm with index less than or equal to 0 (see Definition 3) and we call *property* (gB), means that the set of all poles of the resolvent of T in the usual spectrum are exactly those points λ of the spectrum for which $T - \lambda$ is an upper semi-B-Fredholm with index less than or equal to 0 (see Definition 3). Properties (B) and (gB) are related to a strong variants of classical Weyl's theorem, the so-called property (b) and property (gb) introduced by Berkani and Zariouh [13] and more extensively studied in recent papers [12, 14, 20, 21]. We shall characterize properties (B) and (gB) in several ways and we shall also describe the relationships of it with the other variants of Weyl type theorems. Our main tool is localized version of the single valued extension property. Also, we consider the properties (B) and (gB) in the frame of polaroid type operators.

2. Properties (B) and (gB)

Definition 3. Let $T \in \mathscr{L}(\mathscr{X})$. We say that T satisfies

(i) property (B) if $\sigma(T) \setminus \sigma_{\mathrm{SF}_+^-}(T) = \pi^0(T)$;

(ii) property (gB) if $\sigma(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T) = \pi(T)$.

Theorem 4. *Let $T \in \mathscr{L}(\mathscr{X})$. If T satisfies property (gB), then T satisfies property (B).*

Proof. Suppose that T satisfies property (gB), then $\sigma(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T) = \pi(T)$. If $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{SF}_+^-}(T)$, then $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T) = \pi(T)$. Since $T-\lambda$ is an upper semi-Fredholm, then $\alpha(T-\lambda) < \infty$. As $\lambda \in \pi(T)$, then $a(T-\lambda) = d(T-\lambda) < \infty$. So, it follows from [1, Theorem 3.4] that $\beta(T-\lambda) = \alpha(T-\lambda) < \infty$. Hence $\lambda \in \pi^0(T)$ and this implies that $\sigma(T)\setminus\sigma_{\mathrm{SF}_+^-}(T) \subseteq \pi^0(T)$. To show the other inclusion, let $\lambda \in \pi^0(T)$ be arbitrary. Then $\lambda \in \sigma(T) \setminus \sigma_b(T)$ and hence $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{SF}_+^-}(T)$. Therefore, $\sigma(T) \setminus \sigma_{\mathrm{SF}_+^-}(T) = \pi^0(T)$, that is, T satisfies property (B). □

Theorem 5. *Let $T \in \mathscr{L}(\mathscr{X})$. Then the following assertions hold.*

(i) *If T satisfies property (B), then T satisfies property (b).*

(ii) *If T satisfies property (gB), then T satisfies property (gb).*

Proof. (i) Assume that T satisfies property (B), then $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \pi^0(T)$. If $\lambda \in \Delta_a(T)$, then $\lambda \in \sigma(T) \setminus \sigma_{SF_+^-}(T) = \pi^0(T)$ and so $\Delta_a(T) \subseteq \pi^0(T)$. To show the other inclusion, let $\lambda \in \pi^0(T)$ be an arbitrary. Then $\lambda \in \pi_a^0(T) = \sigma_a(T) \setminus \sigma_{ub}(T)$ and hence $\lambda \in \Delta_a(T)$. Therefore, $\pi^0(T) = \Delta_a(T)$, that is, T satisfies property (b).

(ii) Assume that T satisfies property (gB), then $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \pi(T)$. If $\lambda \in \Delta_a^g(T)$, then $\lambda \in \sigma(T) \setminus \sigma_{SBF_+^-}(T) = \pi(T)$ and so $\Delta_a^g(T) \subseteq \pi(T)$. To show the other inclusion, let $\lambda \in \pi(T)$ be an arbitrary. Then $\lambda \in \pi_a(T) = \sigma_a(T) \setminus \sigma_{LD}(T)$ and hence $\lambda \in \Delta_a^g(T)$. Therefore, $\pi(T) = \Delta_a^g(T)$, that is, T satisfies property (gb). □

The following example shows the converse of the previous theorem does not hold in general.

Example 6. Consider the operator $T = R \oplus S$ that defined on $\mathcal{X} = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$, where R is the right unilateral shift operator and $S(x_1, x_2, \ldots) = (x_2/2, x_3/3, \ldots)$. Then $\sigma(T) = D(0,1)$, where $D(0,1)$ is the unit disc of $\mathbb{C}$. Hence, iso $\sigma(T) = \varnothing$ and so, $E^0(T) = \pi^0(T) = \pi(T) = E(T) = \varnothing$. Moreover, $\sigma_a(T) = \sigma_{SF_+^-}(T) = \sigma_{SBF_+^-}(T) = C(0,1) \cup \{0\}$, where $C(0,1)$ is the unit circle of $\mathbb{C}$. Since $\Delta_a(T) = \varnothing = \pi^0(T)$ and $\Delta_a^g(T) = \pi(T)$, then T satisfies both property (b) and property (gb). On the other hand, since $\sigma(T) \setminus \sigma_{SF_+^-}(T) \neq \pi^0(T)$ and $\sigma(T) \setminus \sigma_{SBF_+^-}(T) \neq \pi(T)$, then T does not satisfy property (B) nor the property (gB).

As a consequence of Theorem 5, we have.

Corollary 7. *Let $T \in \mathcal{L}(\mathcal{X})$. Then the following assertions hold.*

(i) *If T satisfies property (B), then a-Browder's theorem holds for T.*

(ii) *If T satisfies property (gB), then generalized a-Browder's theorem holds for T.*

Theorem 8. *Let $T \in \mathcal{L}(\mathcal{X})$. Then the following assertions hold.*

(i) *T satisfies property (B) if and only if T satisfies property (b) and $\sigma(T) = \sigma_a(T)$.*

(ii) *If T satisfies property (gB) if and only if T satisfies property (gb) and $\sigma(T) = \sigma_a(T)$.*

Proof. (i) If T satisfies property (B), then T satisfies property (b) and consequently, $\Delta_a(T) = \pi^0(T)$ and $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \pi^0(T)$. Hence $\sigma(T) = \pi^0(T) \cup \sigma_{SF_+^-}(T)$ and $\sigma_a(T) = \pi^0(T) \cup \sigma_{SF_+^-}(T)$. This implies that $\sigma(T) = \sigma_a(T)$. Conversely, assume that T satisfies property (b) and $\sigma(T) = \sigma_a(T)$. Then

$$\pi^0(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \sigma(T) \setminus \sigma_{SF_+^-}(T). \tag{19}$$

That is, T satisfies property (B).

(ii) If T satisfies property (gB), then T satisfies property (gb) and consequently, $\Delta_a^g(T) = \pi(T)$ and $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \pi(T)$. Hence $\sigma(T) = \pi(T) \cup \sigma_{SBF_+^-}(T)$ and $\sigma_a(T) = \pi(T) \cup \sigma_{SBF_+^-}(T)$. This implies that $\sigma(T) = \sigma_a(T)$. Conversely, assume that T satisfies property (gb) and $\sigma(T) = \sigma_a(T)$. Then

$$\pi(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \sigma(T) \setminus \sigma_{SBF_+^-}(T). \tag{20}$$

That is, T satisfies property (gB). □

Theorem 9. *Let $T \in \mathcal{L}(\mathcal{X})$. Then the following assertions are equivalent.*

(i) *T satisfies property (gB).*

(ii) *T satisfies property (B) and $\pi(T) = \pi_a(T)$.*

Proof. (i)⇒(ii) Assume that T satisfies property (gB), then T satisfies property (B). As T satisfies property (gB), then T satisfies property (gb), and hence it follows from [13, Corollary 2.9] that T satisfies generalized a-Browder's theorem and $\pi(T) = \pi_a(T)$.

(ii)⇒(i) Assume that T satisfies property (B) and then by Corollary 7, T satisfies a-Browder's theorem. As we know from [7, Theorem 2.2] that a-Browder's theorem is equivalent to generalized a-Browder's theorem, then T satisfies generalized a-Browder's theorem. Hence we have $\Delta_a^g(T) = \pi^a(T)$. Since T satisfies property (B), then Theorem 8 implies that T satisfies property (b) and $\sigma(T) = \sigma_a(T)$. By hypothesis $\pi(T) = \pi_a(T)$, it then follows that $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \pi(T)$ and T satisfies property (gB). □

Theorem 10. *Suppose that T^* has SVEP at every $\lambda \notin \sigma_{SBF_+^-}(T)$. Then the following assertions are equivalent:*

(i) *$E(T) = \pi(T)$;*

(ii) *$E_a(T) = \pi_a(T)$;*

(iii) *$E(T) = \pi_a(T)$.*

Consequently, property (gw), property (gb), property (gB), generalized a-Weyl's theorem and generalized Weyl's theorem are equivalent for T.

Proof. Suppose that T^* has SVEP at every $\lambda \notin \sigma_{SBF_+^-}(T)$. We prove first the equality $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$. If $\lambda \notin \sigma_{SBF_+^-}(T)$ then $T - \lambda$ is an upper semi-B-Fredholm operator and $\operatorname{ind}(T - \lambda) \leq 0$. As T^* has SVEP, then it follows from Corollary 2.8 of [22] that $T - \lambda$ is a B-Weyl operator and so $\lambda \notin \sigma_{BW}(T)$. Therefore, $\sigma_{SBF_+^-}(T) \subseteq \sigma_{BW}(T)$. Since the other inclusion is always verified, we have the equality. Now we prove that $\sigma_D(T) = \sigma_{BW}(T)$. Since $\sigma_{SBF_+^-}(T) \subseteq \sigma_{SF_+^-}(T)$ is always verified. Then T^* has SVEP at every $\lambda \notin \sigma_{SF_+^-}(T)$. This implies that T satisfies Browder's theorem. As we know from Theorem 2.1 of [7] that Browder's theorem is equivalent to generalized Browder's theorem, we have $\sigma_{BW}(T) = \sigma_D(T)$. On the other hand, as T^* has SVEP at every $\lambda \notin \sigma_{SBF_+^-}(T)$, then $\sigma(T) = \sigma_a(T)$. From this we deduce that $E(T) = E_a(T)$ and

$$\pi_a(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \sigma(T) \setminus \sigma_D(T) = \pi(T), \tag{21}$$

from which the equivalence of (i), (ii) and (iii) easily follows. To show the last statement observed that the SVEP of T^* at the points $\lambda \notin \sigma_{\mathrm{SBF}_+^-}(T)$ entails that generalized a-Browder's theorem (and hence generalized Browder's theorem) holds for T, see [20, Corollary 2.7]. Therefor,

$$\begin{aligned}\pi(T) = \Delta^g(T) = E(T) = \sigma(T) \setminus \sigma_{LD}(T) \\ = \Delta_a^g(T) = E_a(T).\end{aligned} \tag{22}$$

That is, property (gB), property (gb), property (gw), generalized a-Weyl's theorem and generalized Weyl's theorem are equivalent for T. □

Dually, we have.

Theorem 11. *Suppose that T has SVEP at every $\lambda \notin \sigma_{\mathrm{SBF}_-^+}(T)$. Then the following assertions are equivalent:*

(i) $E(T^*) = \pi(T^*)$;

(ii) $E_a(T^*) = \pi_a(T^*)$;

(iii) $E(T^*) = \pi_a(T^*)$.

Consequently, property (gB), property (gw), property (gb), generalized a-Weyl's theorem and generalized Weyl's theorem are equivalent for T^*.

Proof. Suppose that T has SVEP at every $\lambda \notin \sigma_{\mathrm{SBF}_-^+}(T)$. We prove first the equality $\sigma_{\mathrm{SBF}_-^+}(T^*) = \sigma_{BW}(T^*)$. If $\lambda \notin \sigma_{\mathrm{SBF}_-^+}(T)$ then $T-\lambda$ is a lower semi-B-Fredholm operator and ind$(T-\lambda) \geq 0$. As T has SVEP, then it follows from Theorem 2.5 of [22] that $T-\lambda$ is a B-Weyl operator and so $\lambda \notin \sigma_{BW}(T)$. As $\sigma_{BW}(T) = \sigma_{BW}(T^*)$. Then $\lambda \notin \sigma_{BW}(T^*)$. So $\sigma_{BW}(T^*) \subseteq \sigma_{\mathrm{SBF}_-^+}(T)$. As $\sigma_{\mathrm{SBF}_-^+}(T) = \sigma_{\mathrm{SBF}_+^-}(T^*)$, then $\sigma_{BW}(T^*) \subseteq \sigma_{\mathrm{SBF}_+^-}(T^*)$. Since the other inclusion is always verified, it then follows that $\sigma_{BW}(T^*) = \sigma_{\mathrm{SBF}_+^-}(T^*)$. Now we show that $\sigma_{BW}(T^*) = \sigma_D(T^*)$. Since we have always $\sigma_{\mathrm{SBF}_-^+}(T) \subseteq \sigma_{\mathrm{SF}_-^+}(T)$, then T has SVEP at every $\lambda \in \sigma_{\mathrm{SF}_-^+}(T)$. Hence T^* satisfies generalized Browder's theorem. So $\sigma_D(T^*) = \sigma_{BW}(T^*)$. Finally, we have $\sigma_{BW}(T^*) = \sigma_{\mathrm{SBF}_+^-}(T^*) = \sigma_D(T^*)$ and $\sigma(T^*) = \sigma_a(T^*)$, from which we obtain $E(T^*) = E_a(T^*)$ and $\pi(T^*) = \pi_a(T^*)$. The SVEP at every $\lambda \in \sigma_{\mathrm{SBF}_-^+}(T)$ ensure by Corollary 2.7 of [20] that generalized a-Browder's theorem (and hence generalized Browder's theorem) holds for T^*. Hence

$$\begin{aligned}\pi(T^*) = \Delta^g(T^*) = E(T^*) = \sigma(T^*) \setminus \sigma_{LD}(T^*) \\ = \Delta_a^g(T^*) = E_a(T^*).\end{aligned} \tag{23}$$

That is, property (gb), property (gB), property (gw), generalized a-Weyl's theorem and generalized Weyl's theorem are equivalent for T^*. □

Corollary 12. *Suppose that T^* has SVEP at every $\lambda \notin \sigma_{\mathrm{SF}_+^-}(T)$. Then the following assertions are equivalent:*

(i) $E^0(T) = \pi^0(T)$;

(ii) $E_a^0(T) = \pi_a^0(T)$;

(iii) $E^0(T) = \pi_a^0(T)$.

Consequently, property (w), property (b), property (B), property (R), a-Weyl's theorem and Weyl's theorem are equivalent for T.

Proof. Assume that T^* has SVEP at every $\lambda \notin \sigma_{\mathrm{SF}_+^-}(T)$. Then it follows from [1, Corollary 2.5] that $\sigma(T) = \sigma_a(T)$ and by Corollary 3.53 of [1], we then have $\sigma_w(T) = \sigma_b(T) = \sigma_{\mathrm{SF}_+^-}(T) = \sigma_{ub}(T)$. Then the SVEP of T^* at every $\lambda \notin \sigma_{\mathrm{SF}_+^-}(T)$ entails that a-Browder's theorem (and hence Browder's theorem) holds for T, see [23, Theorem 2.3]. Then it follows by Theorem 2.19 of [15] that

$$\begin{gathered}\pi^0(T) = E^0(T), \qquad \pi_a^0(T) = E_a^0(T), \\ E^0(T) = \pi_a^0(T)\end{gathered} \tag{24}$$

are equivalent. Hence

$$\begin{aligned}\pi^0(T) = \Delta(T) = E^0(T) = \sigma(T) \setminus \sigma_{ub}(T) \\ = \Delta_a(T) = \pi_a^0(T) = E_a^0(T).\end{aligned} \tag{25}$$

Therefore, property (w), property (b), property (B), property (R), a-Weyl's theorem and Weyl's theorem are equivalent for T. □

Dually, we have.

Corollary 13. *Suppose that T has SVEP at every $\lambda \notin \sigma_{\mathrm{SF}_-^+}(T)$. Then the following assertions are equivalent:*

(i) $E^0(T^*) = \pi^0(T^*)$;

(ii) $E_a^0(T^*) = \pi_a^0(T^*)$;

(iii) $E^0(T^*) = \pi_a^0(T^*)$.

Consequently, property (w), property (b), property (R), property (B), Weyl's theorem and a-Weyl's theorem are equivalent for T^*.

Proof. Assume that T has SVEP at every $\lambda \notin \sigma_{\mathrm{SF}_-^+}(T)$. Then it follows from [1, Corollary 2.5] that $\sigma(T^*) = \sigma(T) = \sigma_s(T) = \sigma_a(T^*)$ and by Corollary 3.53 of [1], we then have $\sigma_w(T^*) = \sigma_w(T) = \sigma_b(T) = \sigma_{\mathrm{SF}_-^+}(T) = \sigma_{lb}(T) = \sigma_{ub}(T^*) = \sigma_{\mathrm{SF}_+^-}(T^*)$. Then the SVEP of T at every $\lambda \notin \sigma_{\mathrm{SF}_-^+}(T)$ entails that a-Browder's theorem (and hence Browder's theorem) holds for T^*, see [23, Theorem 2.3]. Then it follows by Theorem 2.20 of [15] that

$$\begin{gathered}\pi^0(T^*) = E^0(T^*), \qquad \pi_a^0(T^*) = E_a^0(T^*), \\ E^0(T^*) = \pi_a^0(T^*)\end{gathered} \tag{26}$$

are equivalent. Hence

$$\begin{aligned}\pi^0(T^*) = \Delta(T^*) = E^0(T^*) = \Delta_a(T^*) = E_a^0(T^*) \\ = \sigma_a(T^*) \setminus \sigma_{ub}(T^*) = \pi_a^0(T^*).\end{aligned} \tag{27}$$

Therefore, property (w), property (b), property (R), property (B), Weyl's theorem and a-Weyl's theorem are equivalent for T^*. □

Definition 14 (see [21]). A bounded linear operator $T \in \mathscr{L}(\mathscr{X})$ is said to satisfy property (aw) if $\Delta(T) = E_a^0(T)$ and is said to satisfy property (gaw) if $\Delta^g(T) = E_a(T)$.

Definition 15 (see [24]). A bounded linear operator $T \in \mathscr{L}(\mathscr{X})$ is said to satisfy property (z) if $\sigma(T) \setminus \sigma_{\mathrm{SF}_+^-}(T) = E_a^0(T)$ and is said to satisfy property (gz) if $\sigma(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T) = E_a(T)$.

Theorem 16. *Let $T \in \mathscr{L}(\mathscr{X})$. Then T satisfies property (gz) if and only if T satisfies property (gB) and $\pi(T) = E_a(T)$.*

Proof. Assume that T satisfies property (gz), then $\sigma(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T) = E_a(T)$. As T satisfies property (gz), then we conclude from [24, Theorem 2.7] that T satisfies property (gaw), $\sigma_{\mathrm{SBF}_+^-}(T) = \sigma_{BW}(T)$ and from [21, Theorem 3.5] that $E_a(T) = \pi(T)$. Therefore, $\sigma(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T) = \pi(T)$. That is, T satisfies property (gB). Conversely, if T satisfies property (gB) and $\pi(T) = E_a(T)$. Then $\sigma(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T) = \pi(T) = E_a(T)$. That is, T satisfies property (gz). □

Similarly we have the following result in the case of property (B), which we give without proof.

Theorem 17. *Let $T \in \mathscr{L}(\mathscr{X})$. Then T satisfies property (z) if and only if T satisfies property (B) and $\pi^0(T) = E_a^0(T)$.*

3. Properties (B) and (gB) for Polaroid Type Operators

In this section we consider classes of operators for which the isolated points of the spectrum are poles of the resolvent.

An operator $T \in \mathscr{L}(\mathscr{X})$ is said to be polaroid if isolated point of $\sigma(T)$ is a pole of the resolvent of T. $T \in \mathscr{L}(\mathscr{X})$ is said to be a-polaroid if every isolated of $\sigma_a(T)$ is a pole of the resolvent of T.

It is easily seen that if T is a-polaroid, then T is polaroid, while in general the converse is not true. It is well known that λ is a pole of the resolvent of T if and only if λ is a pole of the resolvent of T^*. Since $\sigma(T) = \sigma(T^*)$ we then have

$$T \text{ is polaroid if and only if } T^* \text{ is polaroid.} \tag{28}$$

From the proof of Theorem 10 we know that if T^* has SVEP, then $\sigma(T) = \sigma_a(T)$. Therefore, if T^* has SVEP then

$$T \text{ is } a\text{-polaroid if and only if } T \text{ is polaroid.} \tag{29}$$

If T has SVEP, we know that $\sigma(T^*) = \sigma_a(T^*)$. Therefore, if T has SVEP, then

$$T^* \ a\text{-polaroid} \Longleftrightarrow T^* \text{ polaroid} \Longleftrightarrow T \text{ polaroid.} \tag{30}$$

Theorem 18. *Suppose that T is a-polaroid. Then the following assertions hold.*

(i) *T satisfies property (z) if and only if T satisfies property (B).*

(ii) *T satisfies property (gz) if and only if T satisfies property (gB).*

Proof. (i) Note first that if T is a-polaroid then $E_a^0(T) = \pi^0(T)$. In fact, if $\lambda \in E_a^0(T)$ then λ is isolated in $\sigma_a(T)$ and hence $a(T-\lambda) = d(T-\lambda) < \infty$. Moreover, $\alpha(T-\lambda) < \infty$, so by Theorem 3.4 of [1] it follows that $\beta(T-\lambda)$ is also finite, thus $\lambda \in \pi^0(T)$. This shows $E_a^0(T) \subseteq \pi^0(T)$. Since the other inclusion is always verified, we have $E_a^0(T) = \pi^0(T)$. Therefore, it follows from Theorem 17 that T satisfies property (z) if and only if T satisfies property (B).

(ii) Note first that if T is a-polaroid then $E_a(T) = \pi(T)$. In fact, if $\lambda \in E_a(T)$ then λ is isolated in $\sigma_a(T)$ and hence $a(T-\lambda) = d(T-\lambda) < \infty$. Therefore, $\lambda \in \pi(T)$. This shows $E_a(T) \subseteq \pi(T)$. Since we have always $\pi(T) \subseteq E_a(T)$, and so $E_a(T) = \pi(T)$. Therefore, it follows from Theorem 16 that T satisfies property (gz) if and only if T satisfies property (gB). □

Theorem 19. *Suppose that $T \in \mathscr{L}(\mathscr{X})$ is polaroid.*

(i) *If T^* has SVEP, then property (B) holds for T if and only if T satisfies property (z).*

(ii) *If T has SVEP, then property (B) holds for T^* if and only if T satisfies property (z).*

Proof. (i) Since T^* has SVEP, then it follows from [1, Corollary 2.45] that $\sigma(T) = \sigma_a(T)$, $\sigma_{ub}(T) = \sigma_{\mathrm{SF}_+^-}(T)$, see also [25, Theorem 1.5]. Since T is polaroid and T^* has SVEP, then by equivalence (29), we have T is a-polaroid and so the result follows now from Theorem 18.

(ii) The SVEP for T implies by Corollary 2.45 of [1] that $\sigma(T^*) = \sigma(T) = \sigma_a(T^*)$ and $\sigma_{lb}(T) = \sigma_{ub}(T^*) = \sigma_{\mathrm{SF}_-^+}(T) = \sigma_{\mathrm{SF}_+^-}(T^*)$. Since T^* is polaroid by equivalence (28) and T has SVEP, then T^* is a-polaroid and so the result follows now from Theorem 18. □

Similarly we have the following result in the case of property (gB), which we give without proof.

Theorem 20. *Suppose that $T \in \mathscr{L}(\mathscr{X})$ is polaroid.*

(i) *If T^* has SVEP, then property (gB) holds for T if and only if T satisfies property (gz).*

(ii) *If T has SVEP, then property (gB) holds for T^* if and only if T satisfies property (gz).*

Theorem 21. *Suppose that T is polaroid and $f \in H(\sigma(T))$.*

(i) *If T^* has SVEP, then property (B) holds for $f(T)$ if and only if property (z) holds for $f(T)$.*

(ii) *If T has SVEP, then property (B) holds for $f(T^*)$ if and only if property (z) holds for $f(T^*)$.*

Proof. (i) It follows from Lemma 3.11 of [26] that $f(T)$ is polaroid. By Theorem 2.40 of [1], we have $f(T^*)$ has SVEP, hence from equivalence (29) we conclude that $f(T)$ is a-polaroid and hence it then follows by Theorem 19 that property (B) holds for $f(T)$ if and only if property (z) holds for $f(T)$.

(ii) From the equivalence (28) we know that T^* is polaroid and hence by Lemma 3.11 of [26] that $f(T^*)$ is polaroid. By Theorem 2.40 of [1], we have $f(T)$ has SVEP, hence from equivalence (29) we conclude that $f(T^*)$ is a-polaroid and hence it then follows by Theorem 19 that property (B) holds for $f(T^*)$ if and only if property (z) holds for $f(T^*)$. □

Similarly we have the following result in the case of property (gB), which we give without proof.

Theorem 22. *Suppose that T is polaroid and $f \in H(\sigma(T))$.*

(i) *If T^* has SVEP, then property (gB) holds for $f(T)$ if and only if property (gz) holds for $f(T)$.*

(ii) *If T has SVEP, then property (gB) holds for $f(T^*)$ if and only if property (gz) holds for $f(T^*)$.*

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