Research Article

On Lacunary Ideal Convergence in Random n -Normed Space

U. Yamancı and M. Gürdal

Department of Mathematics, Süleyman Demirel University, East Campus, 32260 Isparta, Turkey

Correspondence should be addressed to M. Gürdal; gurdalmehmet@sdu.edu.tr

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We introduce the notions of lacunary $\mathcal{I}$ -convergence and lacunary $\mathcal{I}$ -Cauchy in the topology induced by random n -normed spaces and prove some important results.

1. Introduction

Menger [1] generalized the metric axioms by associating a distribution function with each pair of points of a set. This system, called a probabilistic metric space, originally a statistical metric space, has been developed extensively by Schweizer and Sklar [2, 3]. The idea of Menger was to use distribution function instead of nonnegative real numbers as values of the metric, which was further developed by several other authors. In this theory, the notion of distance has a probabilistic nature. Namely, the distance between two points x and y is represented by a distribution function F_{xy} , and for $t > 0$, the value $F_{xy}(t)$ is interpreted as the probability that the distance from x to y is less than t . Using this concept, Šerstnev [4] introduced the concept of probabilistic normed spaces. It provides an important area into which many deterministic results of linear normed spaces can be generalized. The studies of continuity properties, linear operators, statistical convergence, and ideal convergence in probabilistic normed spaces have gained many attractions, and such studies have diverse applications into various fields [5–13].

In [14], Gähler introduced an attractive theory of 2-normed and n -normed spaces in the 1960s. Since then, many researchers have studied these subjects and obtained various results [15–19].

Since the introduction of the notion of statistical convergence for sequences of real numbers by Steinhaus [20] and Fast [21] independently, the theory has been investigated and developed by several authors. There has been an effort

to introduce several generalizations and variants of statistical convergence in different spaces. One very important generalization of this notion was introduced by Kostyrko et al. [22] by using an ideal $\mathcal{I}$ of subsets of the set of natural numbers, which they called $\mathcal{I}$ -convergence.

Another important variant of statistical convergence is the notion of lacunary statistical convergence introduced by Fridy and Orhan [23]. Recently, Mohiuddine and Aiyub [24] studied lacunary statistical convergence by introducing the concept θ -statistical convergence in random 2-normed space. Their work can be considered as a particular generalization of the statistical convergence. In [25], Mursaleen and Mohiuddine extended the idea of lacunary statistical convergence with respect to the intuitionistic fuzzy normed space, and Debnath [26] investigated lacunary ideal convergence in intuitionistic fuzzy normed linear spaces. Also, lacunary statistically convergent double sequences in probabilistic normed space were studied by Mohiuddine and Savaş in [27].

The notion of lacunary ideal convergence has not been studied previously in the setting of n -normed linear spaces. Motivated by this fact, in this paper, as a variant of $\mathcal{I}$ -convergence, the notion of lacunary ideal convergence is introduced in a random n -normed space, and some important results are established. Finally, the notions of lacunary $\mathcal{I}^F$ -Cauchy and lacunary $\mathcal{I}^{F^*}$ -Cauchy sequences are introduced and studied.

Throughout the paper, $\mathbb{N}$ will denote the set of all natural numbers. First, we recall some of the basic concepts which will be used in this paper.

Definition 1 (see [15]). Let $n \in \mathbb{N}$, and let X be a real vector space of dimension $d \geq n$, where $n \leq d < \infty$. A real-valued function $\|\cdot, \dots, \cdot\|$ on $X \times X \times \dots \times X = X^n$, satisfying the following properties:

- (i) $\|x_1, x_2, \dots, x_n\| = 0$ if and only if $x_1, x_2, \dots, x_n$ are linearly dependent,
- (ii) $\|x_1, x_2, \dots, x_n\|$ is invariant under any permutation of $x_1, x_2, \dots, x_n$,
- (iii) $\|x_1, x_2, \dots, \alpha x_n\| = |\alpha| \|x_1, x_2, \dots, x_n\|$, where $\alpha \in \mathbb{R}$ (set of real numbers),
- (iv) $\|x_1, x_2, \dots, x_{n-1}, y + z\| \leq \|x_1, x_2, \dots, x_{n-1}, y\| + \|x_1, x_2, \dots, x_{n-1}, z\|$,

is called an n -norm on X , and the pair $(X, \|\cdot, \dots, \cdot\|)$ is called an n -normed space.

Given an n -normed space $(X, \|\cdot, \dots, \cdot\|)$, one can derive a topology for it via the following definition of the limit of a sequence: a sequence $x(k)$ in X is said to be convergent to x in X if $\lim_{k \rightarrow \infty} \|x_1, \dots, x_{n-1}, x(k) - x\| = 0$ for every $x_1, \dots, x_{n-1} \in X$.

All the concepts listed later are studied in depth in the fundamental book by Schweizer and Sklar [3].

Definition 2. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, and $S = [0, 1]$ the closed unit interval. A mapping $f : \mathbb{R} \rightarrow S$ is called a distribution function if it is nondecreasing and left continuous with $\inf_{t \in \mathbb{R}} f(t) = 0$ and $\sup_{t \in \mathbb{R}} f(t) = 1$.

We denote the set of all distribution functions by D^+ such that $f(0) = 0$. If $a \in \mathbb{R}_+$, then $H_a \in D^+$, where

$$H_a(t) = \begin{cases} 1 & \text{if } t > a, \\ 0 & \text{if } t \leq a. \end{cases} \quad (1)$$

It is obvious that $H_0 \geq f$ for all $f \in D^+$.

Definition 3. A triangular norm (t -norm) is a continuous mapping $* : S \times S \rightarrow S$ such that $(S, *)$ is an abelian monoid with unit one and $c * d \leq a * b$ if $c \leq a$ and $d \leq b$ for all $a, b, c, d \in S$. A triangle function τ is a binary operation on D^+ which is commutative and associative, and $\tau(f, H_0) = f$ for every $f \in D^+$.

Definition 4 (see [28]). Let X be a linear space of dimension greater than one, τ a triangle function, and F a mapping from $\underbrace{X \times X \times \dots \times X}_n$ into D^+ . Then, F is called a probabilistic n -norm and (X, F, τ) a probabilistic n -normed space if the following conditions are satisfied:

- (i) $F(x_1, x_2, \dots, x_n; t) = H_0(t)$ if and only if $x_1, x_2, \dots, x_n \in X$ are linearly dependent, where $F(x_1, x_2, \dots, x_n; t)$ denotes the value of $F(x_1, x_2, \dots, x_n)$ at $t \in \mathbb{R}$;
- (ii) $F(x_1, x_2, \dots, x_n; t) \neq H_0(t)$ if and only if $x_1, x_2, \dots, x_n \in X$ are linearly independent;

- (iii) $F(x_1, x_2, \dots, x_n; t)$ is invariant under any permutation $x_1, x_2, \dots, x_n \in X$;
- (iv) $F(x_1, x_2, \dots, \alpha x_n; t) = F(x_1, x_2, \dots, x_n; t/|\alpha|)$ for every $t > 0, \alpha \neq 0$ and $x_1, x_2, \dots, x_n \in X$;
- (v) $F(x_1, x_2, \dots, x_n + x'_n; t) \geq \tau(F(x_1, x_2, \dots, x_n; t), F(x_1, x_2, \dots, x'_n; t))$ whenever $x_1, x_2, \dots, x_n, x'_n \in X$, and $t > 0$.

If (v) is replaced by

- (vi) $F(x_1, x_2, \dots, x_n + x'_n; t_1 + t_2) \geq F(x_1, x_2, \dots, x_n; t_1) * F(x_1, x_2, \dots, x'_n; t_2)$ for all $x_1, x_2, \dots, x_n, x'_n \in X$ and $t_1, t_2 \in \mathbb{R}_+$,

then $(X, F, *)$ is called a random n -normed space (for short, RnN space).

Remark 5. Note that every n -norm space $(X, \|\cdot, \dots, \cdot\|)$ can be made a random n -normed space in a natural way, by setting

- (i) $F(x_1, x_2, \dots, x_n; t) = H_0(t - \|x_1, x_2, \dots, x_n\|)$, for every $x_1, x_2, \dots, x_n \in X, t > 0$ and $a * b = \min\{a, b\}, a, b \in S$;
- (ii) $F(x_1, x_2, \dots, x_n; t) = t/(t + \|x_1, x_2, \dots, x_n\|)$ for every $x_1, x_2, \dots, x_n \in X, t > 0$ and $a * b = ab$ for $a, b \in S$.

Let $(X, F, *)$ be a RnN space. Since $*$ is a continuous t -norm, the system of (ε, λ) -neighborhoods of ϕ (the null vector in X)

$$\{\mathcal{N}_\phi(\varepsilon, \lambda) : \varepsilon > 0, \lambda \in (0, 1)\}, \quad (2)$$

where

$$\mathcal{N}_\phi(\varepsilon, \lambda) = \{x_1, x_2, \dots, x_{n-1} \in X : F_{x_1, x_2, \dots, x_{n-1}}(\varepsilon) > 1 - \lambda\}, \quad (3)$$

determines a first countable Hausdorff topology on X , called the F -topology. Thus, the F -topology can be completely specified by means of F -convergence of sequences. It is clear that $x_1, x_2, \dots, x_{n-1} - y \in \mathcal{N}_\phi$ means that $y \in \mathcal{N}_{x_1, x_2, \dots, x_{n-1}}$ and vice versa.

A sequence $x = (x_k)$ in X is said to have F -convergence to $L \in X$ if for every $\varepsilon > 0, \lambda \in (0, 1)$ and for each nonzero $x_1, \dots, x_{n-1} \in X$ there exists a positive integer N such that

$$x_1, \dots, x_{n-1}, x_k - L \in \mathcal{N}_\phi(\varepsilon, \lambda) \quad \text{for each } n \geq N, \quad (4)$$

or equivalently,

$$x_1, \dots, x_{n-1}, x_k \in \mathcal{N}_L(\varepsilon, \lambda) \quad \text{for each } n \geq N. \quad (5)$$

In this case, we write $F\text{-}\lim x_1, \dots, x_{n-1}, x_k = L$.

Lemma 6. Let $(X, \|\cdot, \dots, \cdot\|)$ be a real n -normed space, and let $(X, F, *)$ be RnN space induced by the random norm $F(x_1, x_2, \dots, x_n; t) = t/(t + \|x_1, x_2, \dots, x_n\|)$, where $x_1, x_2, \dots, x_n \in X$ and $t > 0$. Then, for every sequence $x = (x_k)$ and nonzero $x_1, x_2, \dots, x_{n-1}$ in X ,

$$\begin{aligned} \lim \|x_1, x_2, \dots, x_{n-1}, x - L\| &= 0 \\ \implies F\text{-}\lim x_1, x_2, \dots, x_{n-1}, (x - L) &= 0. \end{aligned} \quad (6)$$

Proof. Suppose that $\lim \|x_1, x_2, \dots, x_{n-1}, x - L\| = 0$. Then, for every $t > 0$ and for every $x_1, x_2, \dots, x_{n-1} \in X$, there exists a positive integer $N = N(t)$ such that

$$\|x_1, x_2, \dots, x_{n-1}, x_k - L\| < t \quad \text{for each } k > N. \quad (7)$$

We observe that for any given $\varepsilon > 0$,

$$\frac{\varepsilon + \|x_1, x_2, \dots, x_{n-1}, x_k - L\|}{\varepsilon} < \frac{\varepsilon + t}{\varepsilon} \quad (8)$$

which is equivalent to

$$\frac{\varepsilon}{\varepsilon + \|x_1, x_2, \dots, x_{n-1}, x_k - L\|} > \frac{\varepsilon}{\varepsilon + t} = 1 - \frac{t}{\varepsilon + t}. \quad (9)$$

Therefore, by letting $\lambda = t/(\varepsilon + t) \in (0, 1)$, we have

$$F_{x_1, x_2, \dots, x_{n-1}, x_k - L}(\varepsilon) > 1 - \lambda \quad \text{for each } k > N. \quad (10)$$

This implies that $x_1, x_2, \dots, x_{n-1}, x_k \in \mathcal{N}_L(\varepsilon, \lambda)$ for each $k > N$ as desired. $\square$

Definition 7 (see [21, 29]). A subset E of $\mathbb{N}$ is said to have density $\delta(E)$ if $\delta(E) = \lim_{n \rightarrow \infty} n^{-1} \sum_{k=1}^n \chi_E(k)$ exists. A number sequence $(x_n)_{n \in \mathbb{N}}$ is said to be statistically convergent to L if for every $\varepsilon > 0$, $\delta(\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}) = 0$. If $(x_n)_{n \in \mathbb{N}}$ is statistically convergent to L , we write $\text{st-lim } x_n = L$, which is necessarily unique.

Definition 8 (see [22]). A family $\mathcal{F} \subset 2^Y$ of subsets of a nonempty set Y is said to be an ideal in Y if (i) $\emptyset \in \mathcal{F}$; (ii) $A, B \in \mathcal{F}$ imply $A \cup B \in \mathcal{F}$; (iii) $A \in \mathcal{F}$, $B \subset A$ imply $B \in \mathcal{F}$. A nontrivial ideal $\mathcal{F}$ in Y is called an admissible ideal if it is different from $P(\mathbb{N})$ and it contains all singletons, that is, $\{x\} \in \mathcal{F}$ for each $x \in Y$.

Let $I \subset P(Y)$ be a nontrivial ideal. A class $\mathcal{F}(I) = \{M \subset Y : \exists A \in I : M = Y \setminus A\}$, called the filter associated with the ideal I , is a filter on Y .

Definition 9 (see [22]). Let $\mathcal{F} \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. Then, a sequence $(x_n)_{n \in \mathbb{N}}$ in X is said to be $\mathcal{F}$ -convergent to $L \in X$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - L\| \geq \varepsilon\}$ belongs to $\mathcal{F}$.

Definition 10. By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r := k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. Throughout this paper, the intervals determined by θ will be denoted by $I_r := (k_{r-1}, k_r]$. Let $K \subseteq \mathbb{N}$. The number

$$\delta_\theta(K) = \lim_r \frac{1}{h_r} |\{k \in I_r : k \in K\}| \quad (11)$$

is said to be the θ -density of K , provided the limit exists (see [23]).

Definition 11 (see [30]). Let $\mathcal{F} \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. A sequence $x = (x_k)$ of numbers is said to be lacunary

$\mathcal{F}$ -convergent to a number L if, for every $\varepsilon > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| \geq \varepsilon \right\} \in \mathcal{F}. \quad (12)$$

In this case, we write $\mathcal{F}\text{-lim } x = L$.

2. Lacunary $\mathcal{F}$ and $\mathcal{F}^*$ -Convergence for Sequences in RnN Spaces

In this section, we study the concepts lacunary $\mathcal{F}$ and $\mathcal{F}^*$ -convergence of a sequence in $(X, F, *)$ and prove some important results.

Definition 12. Let $(X, F, *)$ be RnN space, and let $\mathcal{F}$ be a proper ideal in $\mathbb{N}$. A sequence $x = (x_k)$ in X is said to be $\mathcal{F}$ -convergent to $L \in X$ ($\mathcal{F}$ -convergent to $L \in X$ with respect to the random n -norm F -topology) if for each $\varepsilon > 0$, $\lambda \in (0, 1)$ and each nonzero $x_1, x_2, \dots, x_{n-1} \in X$,

$$\{k \in \mathbb{N} : x_1, x_2, \dots, x_{n-1}, x_k \notin \mathcal{N}_L(\varepsilon, \lambda)\} \in \mathcal{F}. \quad (13)$$

In this case, the vector L is called the $\mathcal{F}$ -limit of the sequence $x = (x_k)$, and we write $\mathcal{F}\text{-lim } x_1, x_2, \dots, x_{n-1}, x = L$.

Definition 13. Let $(X, F, *)$ be RnN space, and let $\mathcal{F}$ be a proper ideal in $\mathbb{N}$. A sequence $x = (x_k)$ in X is said to be lacunary $\mathcal{F}$ -convergent to $L \in X$ with respect to the random n -norm F -topology if for each $\varepsilon > 0$, $\lambda \in (0, 1)$ and each nonzero $x_1, x_2, \dots, x_{n-1} \in X$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} (x_1, x_2, \dots, x_{n-1}, x_k \notin \mathcal{N}_L(\varepsilon, \lambda)) \right\} \in \mathcal{F}. \quad (14)$$

In this case, the vector L is called the $\mathcal{F}$ -limit of the sequence $x = (x_k)$, and we write $\mathcal{F}\text{-lim } x_1, x_2, \dots, x_{n-1}, x = L$.

Lemma 14. Let $(X, F, *)$ be RnN space. If a sequence $x = (x_k)$ is $\mathcal{F}$ -convergent with respect to the random n -norm F , then $\mathcal{F}$ -limit is unique.

Proof. Let us assume that

$$\begin{aligned} \mathcal{F}\text{-lim } x_1, x_2, \dots, x_{n-1}, x &= L_1, \\ \mathcal{F}\text{-lim } x_1, x_2, \dots, x_{n-1}, x &= L_2, \end{aligned} \quad (15)$$

where $L_1 \neq L_2$. Since $L_1 \neq L_2$, select $\varepsilon > 0$, $\lambda \in (0, 1)$ and each nonzero $x_1, x_2, \dots, x_{n-1} \in X$ such that $\mathcal{N}_{L_1}(\varepsilon, \lambda)$ and

$\mathcal{N}_{L_2}(\varepsilon, \lambda)$ are disjoint neighborhoods of L_1 and L_2 . Since L_1 and L_2 both are $\mathcal{F}_\theta^F$ -limit of the sequence (x_k) , we have

$$\begin{aligned} A &= \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} (x_1, x_2, \dots, x_{n-1}, x_k \notin \mathcal{N}_{L_1}(\varepsilon, \lambda)) \right\}, \\ B &= \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} (x_1, x_2, \dots, x_{n-1}, x_k \notin \mathcal{N}_{L_2}(\varepsilon, \lambda)) \right\}, \end{aligned} \quad (16)$$

both belonging to $\mathcal{F}$. This implies that the sets

$$A^c = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} (x_1, x_2, \dots, x_{n-1}, x_k \in \mathcal{N}_{L_1}(\varepsilon, \lambda)) \right\} \quad (17)$$

and $B^c = \{r \in \mathbb{N} : (1/h_r) \sum_{k \in I_r} (x_1, x_2, \dots, x_{n-1}, x_k \in \mathcal{N}_{L_2}(\varepsilon, \lambda))\}$ belong to $\mathcal{F}(\mathcal{F})$. In this way, we obtain a contradiction to the fact that the neighborhoods $\mathcal{N}_{L_1}(\varepsilon, \lambda)$ and $\mathcal{N}_{L_2}(\varepsilon, \lambda)$ of L_1 and L_2 are disjoint. Hence, we have $L_1 = L_2$. This completes the proof. $\square$

Lemma 15. Let $(X, F, *)$ be RnN space. Then one has

- (i) F^θ -lim $x_1, x_2, \dots, x_{n-1}, x_k = L$, then $\mathcal{F}_\theta^F$ -lim $x_1, x_2, \dots, x_{n-1}, x_k = L$;
- (ii) if $\mathcal{F}_\theta^F$ -lim $x_1, x_2, \dots, x_{n-1}, x_k = L_1$ and $\mathcal{F}_\theta^F$ -lim $x_1, x_2, \dots, x_{n-1}, y_k = L_2$, then $\mathcal{F}_\theta^F$ -lim $x_1, x_2, \dots, x_{n-1}, (x_k + y_k) = L_1 + L_2$;
- (iii) if $\mathcal{F}_\theta^F$ -lim $x_1, x_2, \dots, x_{n-1}, x_k = L$ and $\alpha \in \mathbb{R}$, then $\mathcal{F}_\theta^F$ -lim $x_1, x_2, \dots, x_{n-1}, \alpha x_k = \alpha L$;
- (iv) if $\mathcal{F}_\theta^F$ -lim $x_1, x_2, \dots, x_{n-1}, x_k = L_1$, and $\mathcal{F}_\theta^F$ -lim $x_1, x_2, \dots, x_{n-1}, x_k = L_2$, then $\mathcal{F}_\theta^F$ -lim $x_1, x_2, \dots, x_{n-1}, (x_k - y_k) = L_1 - L_2$.

Proof. (i) Suppose that F^θ -lim $x_1, x_2, \dots, x_{n-1}, x_k = L$. Let $\varepsilon > 0$, $\lambda \in (0, 1)$ and nonzero $x_1, x_2, \dots, x_{n-1} \in X$. Then, there exists positive integer N such that $(1/h_r) \sum_{k \in I_r} (x_1, x_2, \dots, x_{n-1}, x_k \in \mathcal{N}_L(\varepsilon, \lambda))$ for each $k > N$. Since the set

$$\begin{aligned} A &= \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} (x_1, x_2, \dots, x_{n-1}, x_k \notin \mathcal{N}_{L_1}(\varepsilon, \lambda)) \right\} \\ &\subseteq \{1, 2, 3, \dots, N-1\} \end{aligned} \quad (18)$$

and the ideal $\mathcal{F}$ is admissible, we have $A \in \mathcal{F}$. This shows that $\mathcal{F}_\theta^F$ -lim $x_1, x_2, \dots, x_{n-1}, x_k = L$.

(ii) Let $\varepsilon > 0$, $\lambda \in (0, 1)$ and nonzero $x_1, x_2, \dots, x_{n-1} \in X$. Choose $\eta \in (0, 1)$ such that $(1 - \eta) * (1 - \eta) > (1 - \lambda)$. Since $\mathcal{F}_\theta^F$ -lim $x_1, x_2, \dots, x_{n-1}, x_k = L_1$ and $\mathcal{F}_\theta^F$ -lim $x_1, x_2, \dots, x_{n-1}, y_k = L_2$, the sets

$$\begin{aligned} A &= \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} (x_1, x_2, \dots, x_{n-1}, x_k \notin \mathcal{N}_{L_1}(\frac{\varepsilon}{2}, \lambda)) \right\}, \\ B &= \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} (x_1, x_2, \dots, x_{n-1}, y_k \notin \mathcal{N}_{L_2}(\frac{\varepsilon}{2}, \lambda)) \right\} \end{aligned} \quad (19)$$

belong to $\mathcal{F}$. Let $C = \{r \in \mathbb{N} : (1/h_r) \sum_{k \in I_r} (x_1, x_2, \dots, x_{n-1}, x_k + y_k \notin \mathcal{N}_{L_1+L_2}(\varepsilon, \lambda))\}$. Since $\mathcal{F}$ is an ideal, it is sufficient to show that $C \subset A \cup B$. This is equivalent to show that $C^c \supset A^c \cap B^c$ where A^c and B^c belong to $\mathcal{F}(\mathcal{F})$. Let $r \in A^c \cap B^c$, that is, $r \in A^c$ and $r \in B^c$, and we have

$$\begin{aligned} &F_{x_1, x_2, \dots, x_{n-1}, (x_k + y_k) - (L_1 + L_2)}(\varepsilon) \\ &\geq F_{x_1, x_2, \dots, x_{n-1}, x_k - L_1}(\frac{\varepsilon}{2}) * F_{x_1, x_2, \dots, x_{n-1}, y_k - L_2}(\frac{\varepsilon}{2}) \\ &> (1 - \eta) * (1 - \eta) \\ &> (1 - \lambda). \end{aligned} \quad (20)$$

Since $r \in C^c \supset A^c \cap B^c \in \mathcal{F}(\mathcal{F})$, we have $C \subset A \cup B \in \mathcal{F}$.

(iii) It is trivial for $\alpha = 0$. Now, let $\alpha \neq 0$, $\varepsilon > 0$, $\lambda \in (0, 1)$, and nonzero $x_1, x_2, \dots, x_{n-1} \in X$. Since $\mathcal{F}_\theta^F$ -lim $x_1, x_2, \dots, x_{n-1}, x_k = L$, we have

$$A = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} (x_1, x_2, \dots, x_{n-1}, x_k \notin \mathcal{N}_L(\varepsilon, \lambda)) \right\} \in \mathcal{F}. \quad (21)$$

This implies that

$$\begin{aligned} A^c &= \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} (x_1, x_2, \dots, x_{n-1}, x_k \in \mathcal{N}_L(\varepsilon, \lambda)) \right\} \\ &\in \mathcal{F}(\mathcal{F}). \end{aligned} \quad (22)$$

Let $r \in A^c$. Then, we have

$$\begin{aligned} &F_{x_1, x_2, \dots, x_{n-1}, \alpha x_k - \alpha L}(\varepsilon) \\ &= F_{x_1, x_2, \dots, x_{n-1}, x_k - L}(\frac{\varepsilon}{|\alpha|}) \\ &\geq F_{x_1, x_2, \dots, x_{n-1}, x_k - L}(\varepsilon) * F_0\left(\frac{\varepsilon}{|\alpha|} - \varepsilon\right) \\ &> (1 - \lambda) * 1 \\ &= (1 - \lambda). \end{aligned} \quad (23)$$

So,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} (x_1, x_2, \dots, x_{n-1}, \alpha x_k \notin \mathcal{N}_{\alpha L}(\varepsilon, \lambda)) \right\} \in \mathcal{F}. \quad (24)$$

Hence, $\mathcal{F}_\theta^F\text{-}\lim x_1, x_2, \dots, x_{n-1}, \alpha x_k = \alpha L$.

(iv) The result follows from (ii) and (iii). $\square$

We introduce the concept of $\mathcal{F}_\theta^{F^*}$ -convergence closely related to $\mathcal{F}_\theta^F$ -convergence of sequence in random n -normed space and show that $\mathcal{F}_\theta^{F^*}$ -convergence implies $\mathcal{F}_\theta^F$ -convergence but not conversely.

Definition 16. Let $(X, F, *)$ be RnN space, and let $\mathcal{F}$ be an admissible ideal. We say that a sequence $x = (x_k)$ in X is said to be lacunary $\mathcal{F}^{F^*}$ -convergent to $L \in X$ with respect to the random n -norm F if there is a set $K = \{m_1 < m_2 < \dots < m_k < \dots\} \subseteq \mathbb{N}$ such that $K' = \{r \in \mathbb{N} : m_k \in I_r\} \in \mathcal{F}(\mathcal{F})$ and $F^\theta\text{-}\lim_{k \in K} x_1, x_2, \dots, x_{n-1}, x_{m_k} = L$ for each nonzero $x_1, x_2, \dots, x_{n-1} \in X$.

In this case, we write $\mathcal{F}_\theta^{F^*}\text{-}\lim x_1, x_2, \dots, x_{n-1}, x_k = L$, and L is called the $\mathcal{F}_\theta^{F^*}$ -limit of the sequence $x = (x_k)$.

Theorem 17. Let $(X, F, *)$ be RnN space, and let $\mathcal{F}$ be an admissible ideal. If $\mathcal{F}_\theta^{F^*}\text{-}\lim x_1, x_2, \dots, x_{n-1}, x_k = L$, then $\mathcal{F}_\theta^F\text{-}\lim x_1, x_2, \dots, x_{n-1}, x_k = L$.

Proof. Suppose that $\mathcal{F}_\theta^{F^*}\text{-}\lim x_1, x_2, \dots, x_{n-1}, x_k = L$. Then, by definition, there exists

$$K = \{m_1 < m_2 < \dots < m_k < \dots\} \subseteq \mathbb{N} \quad (25)$$

such that $K' \in \mathcal{F}(\mathcal{F})$ and $F^\theta\text{-}\lim_{k \in K} x_1, x_2, \dots, x_{n-1}, x_{m_k} = L$ for each nonzero $x_1, x_2, \dots, x_{n-1} \in X$. Let $\varepsilon > 0$, $\lambda \in (0, 1)$ and nonzero $x_1, x_2, \dots, x_{n-1} \in X$ be given. Since $F^\theta\text{-}\lim_{k \in K} x_1, x_2, \dots, x_{n-1}, x_{m_k} = L$, there exists $N \in \mathbb{N}$ such that

$$\frac{1}{h_r} \sum_{k \in I_r} (x_1, x_2, \dots, x_{n-1}, x_{m_k} \in \mathcal{N}_L(\varepsilon, \lambda)) \quad (26)$$

for every $m \geq N$. Since

$$A = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} (x_1, x_2, \dots, x_{n-1}, x_{m_k} \notin \mathcal{N}_L(\varepsilon, \lambda)) \right\} \quad (27)$$

is contained in

$$B = \{m_1, m_2, \dots, m_{N-1}\} \quad (28)$$

and the ideal $\mathcal{F}$ is admissible, we have $A \in \mathcal{F}$. Hence,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} (x_1, x_2, \dots, x_{n-1}, x_k \notin \mathcal{N}_L(\varepsilon, \lambda)) \right\} \subseteq K \cup B \in \mathcal{F} \quad (29)$$

for $\varepsilon > 0$, $\lambda \in (0, 1)$ and nonzero $x_1, x_2, \dots, x_{n-1} \in X$. Therefore, we conclude that $\mathcal{F}_\theta^F\text{-}\lim x_1, x_2, \dots, x_{n-1}, x_k = L$. $\square$

The following example shows that the converse of Theorem 17 needs not to be true.

Example 18. Consider $X = \mathbb{R}^n$ with

$$\|x_1, x_2, \dots, x_n\| = abs \left(\begin{pmatrix} x_{11} \dots x_{1n} \\ \vdots \\ x_{n1} \dots x_{nn} \end{pmatrix} \right), \quad (30)$$

where $x_i = (x_{i1}, \dots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \dots, n$, and let $a * b = ab$ for all $a, b \in S$. For all $(y_1, y_2, \dots, y_{n-1}, x) \in \mathbb{R}^n$ and $t > 0$, consider

$$F_{y_1, y_2, \dots, y_{n-1}, x}(t) = \frac{t}{t + \|y_1, y_2, \dots, y_{n-1}, x\|}. \quad (31)$$

Then, $(\mathbb{R}^n, F, *)$ is RnN space. Consider a decomposition of $\mathbb{N}$ as $\mathbb{N} = \cup_i \Delta_i$ such that for any $m \in \mathbb{N}$, each Δ_i contains infinitely many (i) 's, where $i \geq m$ and $\Delta_i \cap \Delta_m = \emptyset$ for $i \neq m$. Let $\mathcal{F}$ be the class of all subsets of $\mathbb{N}$ which intersect almost a finite number of Δ_i 's. Then, $\mathcal{F}$ is an admissible ideal. We define a sequence (x_m) as follows: $x_m = (1/i, 0, 0, \dots, 0) \in \mathbb{R}^n$, $i = 1, 2, \dots$, if $m \in \Delta_i$. Then, for nonzero $y_1, y_2, \dots, y_{n-1} \in X$, we have

$$F_{x_m, y_1, y_2, \dots, y_{n-1}}(t) = \frac{t}{t + \|y_1, y_2, \dots, y_{n-1}, x_m\|} \rightarrow 1 \quad (32)$$

as $m \rightarrow \infty$. Hence, $\mathcal{F}_\theta^F\text{-}\lim_m y_1, y_2, \dots, y_{n-1}, x_m = 0$.

Now, we show that $\mathcal{F}_\theta^{F^*}\text{-}\lim_m y_1, y_2, \dots, y_{n-1}, x_m \neq 0$. Suppose that

$$\mathcal{F}_\theta^{F^*}\text{-}\lim_m y_1, y_2, \dots, y_{n-1}, x_m = 0. \quad (33)$$

Then, by definition, there exists a subset

$$K = \{m_j : m_1 < m_2 < \dots\} \subset \mathbb{N} \quad (34)$$

such that $K' \in \mathcal{F}(\mathcal{F})$ and $F^\theta\text{-}\lim_{j \in K} y_1, y_2, \dots, y_{n-1}, x_{m_j} = 0$.

Since $K' \in \mathcal{F}(\mathcal{F})$, there exists $H \in \mathcal{F}$ such that $K = \mathbb{N} \setminus H$. Then, there exist positive integers p such that

$$H \subset \left(\bigcup_{m=1}^p \Delta_m \right). \quad (35)$$

Thus, $\Delta_{p+1} \subset K$, and so $x_{m_j} = 1/(p+1) > 0$ for infinitely many values m_j 's in K . This contradicts the assumption that $F^\theta\text{-}\lim_{j \in K} y_1, y_2, \dots, y_{n-1}, x_{m_j} = 0$. Hence, $\mathcal{F}_\theta^{F^*}\text{-}\lim_m y_1, y_2, \dots, y_{n-1}, x_m \neq 0$.

Hence, the converse of the theorem needs not to be true.

The following theorem shows that the converse holds if the ideal $\mathcal{F}$ satisfies condition (AP).

Definition 19 (see [22]). An admissible ideal $\mathcal{I} \subset P$ is said to satisfy the condition (AP) if for every sequence $(A_n)_{n \in \mathbb{N}}$ of pairwise disjoint sets from $\mathcal{I}$ there are sets $B_n \subset \mathbb{N}$, $n \in \mathbb{N}$, such that the symmetric difference $A_n \Delta B_n$ is a finite set for every n and $\cup_{n \in \mathbb{N}} B_n \in \mathcal{I}$.

Theorem 20. Let $(X, F, *)$ be RnN space, and let the ideal $\mathcal{I}$ satisfy the condition (AP). If $x = (x_k)$ is a sequence in X such that $\mathcal{I}_\theta^F\text{-}\lim y_1, y_2, \dots, y_{n-1}, x = L$, then $\mathcal{I}_\theta^{F*}\text{-}\lim y_1, y_2, \dots, y_{n-1}, x = L$.

Proof. Since $\mathcal{I}_\theta^F\text{-}\lim y_1, y_2, \dots, y_{n-1}, x = L$, so for every $\varepsilon > 0$, $\lambda \in (0, 1)$ and nonzero $y_1, y_2, \dots, y_{n-1} \in X$, the set

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} (y_1, y_2, \dots, y_{n-1}, x_k \notin \mathcal{N}_L(\varepsilon, \lambda)) \right\} \in \mathcal{I}. \quad (36)$$

We define the set A_p for $p \in \mathbb{N}$ as

$$A_p = \left\{ k \in \mathbb{N} : 1 - \frac{1}{p} \leq F_{y_1, y_2, \dots, y_{n-1}, x_k - L} < 1 - \frac{1}{p+1} \right\}. \quad (37)$$

Then, it is clear that $\{A_1, A_2, \dots\}$ is a countable family of mutually disjoint sets belonging to $\mathcal{I}$, and so by the condition (AP), there is a countable family of sets $\{B_1, B_2, \dots\} \in \mathcal{I}$ such that the symmetric difference $A_i \Delta B_i$ is a finite set for each $i \in \mathbb{N}$ and $B = \cup_{i=1}^\infty B_i \in \mathcal{I}$. Since $B \in \mathcal{I}$, there is a set $K \in \mathcal{F}(\mathcal{I})$ such that $K = \mathbb{N} \setminus B$. Now, we prove that the subsequence $(x_k)_{k \in K}$ is convergent to L with respect to the random n -norm F . Let $\eta \in (0, 1)$, $\varepsilon > 0$ and nonzero $y_1, y_2, \dots, y_{n-1} \in X$. Choose a positive q such that $q^{-1} < \eta$. Then,

$$\begin{aligned} & \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} (y_1, y_2, \dots, y_{n-1}, x_k \notin \mathcal{N}_L(\varepsilon, \eta)) \right\} \\ & \subset \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left(y_1, y_2, \dots, y_{n-1}, x_k \notin \mathcal{N}_L\left(\varepsilon, \frac{1}{q}\right) \right) \right\} \\ & \subset \bigcup_{i=1}^{q-1} A_i. \end{aligned} \quad (38)$$

Since $A_i \Delta B_i$ is a finite set for each $i = 1, 2, \dots, q-1$, there exists $j_0 \in \mathbb{N}$ such that

$$\begin{aligned} & \left(\bigcup_{i=1}^{q-1} B_i \right) \cap \{k \in \mathbb{N} : k \geq k_0\} \\ & = \left(\bigcup_{i=1}^{q-1} A_i \right) \cap \{k \in \mathbb{N} : k \geq k_0\}. \end{aligned} \quad (39)$$

If $k \geq k_0$ and $k \in K$, then $k \notin \cup_{i=1}^{q-1} B_i$ and $k \notin \cup_{i=1}^{q-1} A_i$. Hence, for every $k \geq k_0$ and $k \in K$, we have

$$\frac{1}{h_r} \sum_{k \in I_r} (y_1, y_2, \dots, y_{n-1}, x_k \notin \mathcal{N}_L(\varepsilon, \eta)). \quad (40)$$

Since this holds for every $\varepsilon > 0$, $\eta \in (0, 1)$ and nonzero $y_1, y_2, \dots, y_{n-1} \in X$, so we have $\mathcal{I}_\theta^{F*}\text{-}\lim y_1, y_2, \dots, y_{n-1}, x = L$. This completes the proof of the theorem. $\square$

3. $\mathcal{I}_\theta^F$ and $\mathcal{I}_\theta^{F*}$ -Cauchy Sequences in RnN Spaces

In this section, we study the concepts $\mathcal{I}_\theta^F$ -Cauchy and $\mathcal{I}_\theta^{F*}$ -Cauchy of a sequence in $(X, F, *)$. Also, we will study the relations between these concepts.

Definition 21. Let $(X, F, *)$ be RnN space, and let $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$. Then, a sequence $x = (x_k)$ of elements in X is called lacunary $\mathcal{I}^F$ -Cauchy sequence in X if for every $\varepsilon > 0$, $\lambda \in (0, 1)$ and nonzero $x_1, x_2, \dots, x_{n-1} \in X$, there exists $m \in \mathbb{N}$ satisfying

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} (x_1, x_2, \dots, x_{n-1}, x_k - x_m \notin \mathcal{N}_\phi(\varepsilon, \lambda)) \right\} \in \mathcal{I}. \quad (41)$$

Definition 22. Let $(X, F, *)$ be RnN space, and let $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$. We say that a sequence $x = (x_k)$ of elements in X is called lacunary $\mathcal{I}^{F*}$ -Cauchy sequence in X if for every $\varepsilon > 0$, $\lambda \in (0, 1)$ and nonzero $x_1, x_2, \dots, x_{n-1} \in X$, there exists a set

$$K = \{m_1 < m_2 < \dots < m_k < \dots\} \subset \mathbb{N} \quad (42)$$

such that $K' = \{r \in \mathbb{N} : m_k \in I_r\} \in \mathcal{F}(\mathcal{I})$, and (x_{m_k}) is a lacunary Cauchy with respect to the random n -norm F .

The next theorem gives that $\mathcal{I}_\theta^{F*}$ -Cauchy sequence implies $\mathcal{I}_\theta^F$ -Cauchy sequence.

Theorem 23. Let $(X, F, *)$ be RnN space, and let $\mathcal{I}$ be a nontrivial ideal of $\mathbb{N}$. If $x = (x_k)$ is a $\mathcal{I}_\theta^{F*}$ -Cauchy sequence, then $x = (x_k)$ is a $\mathcal{I}_\theta^F$ -Cauchy sequence too.

Proof. Let (x_k) be a $\mathcal{I}_\theta^{F*}$ -Cauchy sequence. Then, for $\varepsilon > 0$, $\lambda \in (0, 1)$ and nonzero $x_1, x_2, \dots, x_{n-1} \in X$, there exist $K' \in \mathcal{F}(\mathcal{I})$ and a number $N \in \mathbb{N}$ such that

$$\frac{1}{h_r} \sum_{k \in I_r} (x_1, x_2, \dots, x_{n-1}, x_{m_k} - x_{m_p} \in \mathcal{N}_\phi(\varepsilon, \lambda)) \quad (43)$$

for every $k, p \geq N$. Now, fix $p = m_{N+1}$. Then, for every $\varepsilon > 0$, $\lambda \in (0, 1)$ and nonzero $x_1, x_2, \dots, x_{n-1} \in X$, we have

$$\frac{1}{h_r} \sum_{k \in I_r} (x_1, x_2, \dots, x_{n-1}, x_{m_k} - x_p \in \mathcal{N}_\phi(\varepsilon, \lambda)) \quad (44)$$

for every $k \geq N$.

Let $H = \mathbb{N} \setminus K$. It is obvious that $H \in \mathcal{F}$ and

$$\begin{aligned} A(\varepsilon, \lambda) &= \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k,p \in I_r} (x_1, x_2, \dots, x_{n-1}, x_k - x_p \notin \mathcal{N}_\phi(\varepsilon, \lambda)) \right\} \\ &\subset H \cup \{m_1 < m_2 < \dots < m_N\} \in \mathcal{F}. \end{aligned} \quad (45)$$

Therefore, for every $\varepsilon > 0$, $\lambda \in (0, 1)$ and nonzero $x_1, x_2, \dots, x_{n-1} \in X$, we can find $p \in \mathbb{N}$ such that $A(\varepsilon, \lambda) \in \mathcal{F}$, that is, (x_k) is a $\mathcal{F}_\theta^F$ -Cauchy sequence. $\square$

Now, we will prove that $\mathcal{F}_\theta^{F*}$ -convergence implies $\mathcal{F}_\theta^F$ -Cauchy condition in random n -normed space.

Theorem 24. Let $(X, F, *)$ be RnN space, and let $\mathcal{F}$ be an admissible ideal of $\mathbb{N}$. If a sequence $x = (x_k)$ is $\mathcal{F}_\theta^{F*}$ -convergent, then it is a $\mathcal{F}_\theta^F$ -Cauchy sequence.

Proof. By assumption, there exists a set

$$K = \{m_k : m_1 < m_2 < \dots\} \subset \mathbb{N} \quad (46)$$

such that $K' = \{r \in \mathbb{N} : m_k \in I_r\} \in \mathcal{F}(\mathcal{F})$, and (x_{m_k}) is a lacunary Cauchy with respect to the random n -norm F . Choose $\eta \in (0, 1)$ such that $(1 - \eta) * (1 - \eta) > (1 - \lambda)$. Since

$$\begin{aligned} &F_{x_1, x_2, \dots, x_{n-1}, x_{m_k} - x_{m_p}}(\varepsilon) \\ &\geq F_{x_1, x_2, \dots, x_{n-1}, x_{m_k} - L} \left(\frac{\varepsilon}{2} \right) * F_{x_1, x_2, \dots, x_{n-1}, x_{m_p} - L} \left(\frac{\varepsilon}{2} \right) \\ &> (1 - \eta) * (1 - \eta) \\ &> 1 - \lambda \end{aligned} \quad (47)$$

for every $\varepsilon > 0$, $\lambda \in (0, 1)$, each nonzero $x_1, x_2, \dots, x_{n-1}$ in X and $k > N$, $p > N$, we have $(1/h_r) \sum_{k \in I_r} (x_1, x_2, \dots, x_{n-1}, x_{m_k} - x_{m_p} \notin \mathcal{N}_L(\varepsilon, \lambda))$ for every $k, p > N$ and each nonzero $x_1, x_2, \dots, x_{n-1} \in X$, that is, (x_k) in X is an $\mathcal{F}_\theta^{F*}$ -Cauchy sequence in X . Then, by Theorem 23 (x_k) is a $\mathcal{F}_\theta^F$ -Cauchy sequence in RnN space. $\square$

Theorem 25. Let $(X, F, *)$ be RnN space, and let $\mathcal{F}$ be an admissible ideal of $\mathbb{N}$. If a sequence $x = (x_k)$ of elements in X is $\mathcal{F}_\theta^F$ -convergent, then it is $\mathcal{F}_\theta^F$ -Cauchy sequence.

Proof. Suppose that (x_k) is $\mathcal{F}_\theta^F$ -convergent to $L \in X$. Let $\varepsilon > 0$, $\lambda \in (0, 1)$ and nonzero $x_1, x_2, \dots, x_{n-1} \in X$ be given. Then, we have

$$\begin{aligned} A &= \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} (x_1, x_2, \dots, x_{n-1}, x_k \notin \mathcal{N}_L(\frac{\varepsilon}{2}, \lambda)) \right\} \\ &\in \mathcal{F}. \end{aligned} \quad (48)$$

This implies that

$$\begin{aligned} A^c &= \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} (x_1, x_2, \dots, x_{n-1}, x_k \in \mathcal{N}_L(\frac{\varepsilon}{2}, \lambda)) \right\} \\ &\in \mathcal{F}(\mathcal{F}). \end{aligned} \quad (49)$$

Choose $\eta \in (0, 1)$ such that $(1 - \eta) * (1 - \eta) > (1 - \lambda)$. Then, for every $r \in A^c$,

$$\begin{aligned} &F_{x_1, x_2, \dots, x_{n-1}, x_k - x_s}(\varepsilon) \\ &\geq F_{x_1, x_2, \dots, x_{n-1}, x_k - L} \left(\frac{\varepsilon}{2} \right) * F_{x_1, x_2, \dots, x_{n-1}, x_s - L} \left(\frac{\varepsilon}{2} \right) \\ &> (1 - \eta) * (1 - \eta) \\ &> (1 - \lambda). \end{aligned} \quad (50)$$

Hence $\{r \in \mathbb{N} : (1/h_r) \sum_{k \in I_r} (x_1, x_2, \dots, x_{n-1}, x_k - x_s \in \mathcal{N}_\phi(\varepsilon, \lambda))\} \in \mathcal{F}(\mathcal{F})$. This implies that

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} (x_1, x_2, \dots, x_{n-1}, x_k - x_s \notin \mathcal{N}_\phi(\varepsilon, \lambda)) \right\} \in \mathcal{F}. \quad (51)$$

that is, (x_k) is a $\mathcal{F}_\theta^F$ -Cauchy sequence. $\square$

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