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Research Article

Co-Cohen-Macaulay Modules and Local Cohomology

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Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring and let M be a finitely generated R-module of dimension d. Then the following statements hold: (a) if width $(H^i_{\mathfrak{m}}(M)) \geq i-1$ for all i with $2 \leq i < d$, then $H^d_{\mathfrak{m}}(M)$ is co-Cohen-Macaulay of Noetherian dimension d; (b) if M is an unmixed R-module and depth $M \geq d-1$, then $H^d_{\mathfrak{m}}(M)$ is co-Cohen-Macaulay of Noetherian dimension d if and only if $H^{d-1}_{\mathfrak{m}}(M)$ is either zero or co-Cohen-Macaulay of Noetherian dimension $d-2$. As consequence, if $H^i_{\mathfrak{m}}(M)$ is co-Cohen-Macaulay of Noetherian dimension i for all i with $0 \leq i < d$, then $H^d_{\mathfrak{m}}(M)$ is co-Cohen-Macaulay of Noetherian dimension d.

1. Introduction

Throughout this paper, let $(R, \mathfrak{m})$ be a commutative Noetherian local ring and let M be a finitely generated R-module of dimension d. We denote the ith local cohomology module of M with respect to $\mathfrak{m}$ by $H^i_{\mathfrak{m}}(M)$. It is well known that $H^i_{\mathfrak{m}}(M)$ is Artinian for all i (cf. [1]).

The *Noetherian dimension* of an Artinian R-module A, denoted by N-dim A, is defined inductively as follows: when $A = 0$, put N-dim $A = -1$. Then by induction, for any integer $r \geq 0$, put N-dim $A = r$ if N-dim $A < r$ is false, and for any ascending chain $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ of submodules of A there exists an integer n_0 such that $N\text{-dim}(A_{n+1}/A_n) < r$ for all $n > n_0$. Therefore N-dim $A = 0$ if and only if A is a nonzero Noetherian module. Moreover, if $0 \to A' \to A \to A'' \to 0$ is an exact sequence of Artinian modules, then $N\text{-dim}\, A = \max\{N\text{-dim}\, A', N\text{-dim}\, A''\}$. Let $x_1, \ldots, x_n \in \mathfrak{m}$. $x_1, \ldots, x_n$ is an A-coregular sequence if $0:_A(x_1, \ldots, x_{i-1})R \xrightarrow{x_i} 0:_A(x_1, \ldots, x_{i-1})R$ is surjective for $i = 1, \ldots, n$ and $0:_A(x_1, \ldots, x_n)R \neq 0$. The width of A, denoted by width (A), is the length of any maximal A-coregular sequence in $\mathfrak{m}$. For any A-coregular element $x \in \mathfrak{m}$, we have that $N\text{-dim}(0:_A x) = N\text{-dim}\, A - 1$ and $\text{width}(0:_A x) = \text{width}\, A - 1$. Details about N-dim A and width A can be found in Roberts [2], Kirby [3], and Ooishi [4]; there is a general fact: for any Artinian R-module A width $A \leq N$-dim $A < \infty$ holds and A is co-Cohen-Macaulay if and only if width A = N-dim A holds (cf. [5–7]). Tang [8] has shown that if either $d \leq 2$ or M is Cohen-Macaulay, then $H^d_{\mathfrak{m}}(M)$ is co-Cohen-Macaulay (see also [9]). Following Nagata [10], M is unmixed if $\dim(\widehat{R}/\mathfrak{p}) = d$ for all $\mathfrak{p} \in \text{Ass}_{\widehat{R}}\widehat{M}$.

The main aim of this paper is to prove the following theorem.

Theorem 1. *The following statements are true.*

(a) *If width $(H^i_{\mathfrak{m}}(M)) \geq i-1$ for all i with $2 \leq i < d$, then $H^d_{\mathfrak{m}}(M)$ is co-Cohen-Macaulay of Noetherian dimension d.*

(b) *If M is an unmixed R-module and depth $M \geq d-1$, then $H^d_{\mathfrak{m}}(M)$ is co-Cohen-Macaulay of Noetherian dimension d if and only if $H^{d-1}_{\mathfrak{m}}(M)$ is either zero or co-Cohen-Macaulay of Noetherian dimension $d-2$.*

2. The Results

Following Macdonald [11], every Artinian R-module A has minimal secondary representation $A = A_1 + \cdots + A_n$, where A_i is $\mathfrak{p}_i$ secondary. The set $\{p_1, \ldots, p_n\}$ is independent of the choice of the minimal secondary representation of A. This set is called *the set of attached prime ideals* of A and denoted by Att A. The set of all minimal elements of Att A is exactly the set of all minimal elements of $\mathrm{Var}(\mathrm{Ann}\,A)$. A sequence $(x_1,\dots,x_n)$ of elements in $\mathfrak{m}$ is called a *strict f-sequence* of M if $x_{j+1}\notin\mathfrak{p}$ for all $\mathfrak{p}\in\cup_{i=1}^{d-j}\mathrm{Att}(H^i_{\mathfrak{m}}(M/(x_1,\dots,x_j)M)\setminus\{\mathfrak{m}\}$ for all $j=0,1,\dots,n-1$. This notion was introduced in [12].

Lemma 2 (see [9]). *For all integer $0\le i<d$, one has N-dim $(H^i_{\mathfrak{m}}(M))\le i$ and N-dim $(H^d_{\mathfrak{m}}(M))=d$.*

Lemma 3. *Let $x\in\mathfrak{m}$ be a strict f-sequence of M. Then the following statements are true.*

(i) *Suppose that $d\ge 3$ and $H^d_{\mathfrak{m}}(M)$ is co-Cohen-Macaulay of Noetherian dimension d. Then $H^{d-1}_{\mathfrak{m}}(M/xM)$ is also co-Cohen-Macaulay of Noetherian dimension $d-1$.*

(ii) *Suppose that $H^{d-1}_{\mathfrak{m}}(M/xM)$ is co-Cohen-Macaulay of Noetherian dimension $d-1$ and $\mathrm{width}(H^d_{\mathfrak{m}}(M))\ge 3$. Then $H^d_{\mathfrak{m}}(M)$ is co-Cohen-Macaulay of Noetherian dimension d.*

Proof. (i) By our hypothesis and using [1, Exercise 11.3.9], we have $\ell(0:_M x)<\infty$. Hence from the exact sequences

$$0\longrightarrow(0:_M x)\longrightarrow M\longrightarrow\frac{M}{0:_M x}\longrightarrow 0,$$
$$0\longrightarrow\frac{M}{(0:_M x)}\xrightarrow{x}M\longrightarrow\frac{M}{xM}\longrightarrow 0,\tag{1}$$

we get the exact sequence

$$0\longrightarrow\frac{H^i_{\mathfrak{m}}(M)}{xH^i_{\mathfrak{m}}(M)}\longrightarrow H^i_{\mathfrak{m}}\left(\frac{M}{xM}\right)\longrightarrow\left(0:_{H^{i+1}_{\mathfrak{m}}(M)}x\right)\longrightarrow 0\,(\dagger),\tag{2}$$

for all $i=1,2,\dots,d-1$. Thus in case $i=d-1$ we have that $(0:_{H^d_{\mathfrak{m}}(M)}x)$ is co-Cohen-Macaulay of Noetherian dimension $d-1$. By the choice of $x\in\mathfrak{m}$ the module $H^{d-1}_{\mathfrak{m}}(M)/xH^{d-1}_{\mathfrak{m}}(M)$ is an R-module of finite length. Moreover, since $\mathrm{width}(H^{d-1}_{\mathfrak{m}}(M/xM))\ge\min\{2,d-1\}>0$ by [8, Proposition 2.4], we have $H^{\mathfrak{m}}_0(H^{d-1}_{\mathfrak{m}}(M/xM))=0$ by [13, Theorem 4.11]. Thus, by [13, Corollary 3.7], the long exact local homology sequence with respect to $i=d-1$ over the exact sequence ($\dagger$) provides $H^{d-1}_{\mathfrak{m}}(M)/xH^{d-1}_{\mathfrak{m}}(M)=0$. Hence there is an isomorphism $H^{d-1}_{\mathfrak{m}}(M/xM)\cong(0:_{H^d_{\mathfrak{m}}(M)}x)$ and so $H^{d-1}_{\mathfrak{m}}(M/xM)$ is a co-Cohen-Macaulay module of Noetherian dimension $d-1$.

The proof of (ii) follows by the same arguments as in the proof of (i). □

Brodmann and Sharp [14], for all integer $i\ge 0$, defined the set $\{\mathfrak{p}\in\mathrm{Spec}(R): H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})\ne 0\}$, the ith *pseudo support* of M, and denoted by $\mathrm{Psupp}^i(M)$. Note that if R is complete with respect to $\mathfrak{m}$-adic topology, then by [15, Theorem 3.1] $\mathrm{Var}(\mathrm{Ann}\,(H^i_{\mathfrak{m}}(M)))=\mathrm{Psupp}^i(M)$. The module M satisfies Serre's condition S_n, where n is nonnegative integer, provided $\mathrm{depth}\,M_{\mathfrak{p}}\ge\min\{n,\dim M_{\mathfrak{p}}\}$ for all $\mathfrak{p}\in\mathrm{Supp}(M)$. Note that M satisfies the condition Serre S_1 if and only if M has no imbedded primes, that is, $\mathrm{Ass}\,M=\min\,\mathrm{Ass}\,M$ is unmixed.

Lemma 4. *Let M be unmixed and $d\le 2$. Then $H^{d-1}_{\mathfrak{m}}(M)$ is either zero or co-Cohen-Macaulay of Noetherian dimension $d-2$.*

Proof. When $d=0$, it is trivial. We assume that $d=1$. Since M is unmixed then $\mathrm{depth}\,M\ge 1$ and so we have $H^0_{\mathfrak{m}}(M)=0$. Thus the result has been proved in this case. Now assume that $d=2$ and $H^1_{\mathfrak{m}}(M)\ne 0$. By using [6, Theorem 1.4], we can assume that R is complete with respect to $\mathfrak{m}$-adic topology. Let $\mathfrak{p}\in\mathrm{Psupp}^1(M)$. Then $H^{1-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})\ne 0$. Therefore $1\le\mathrm{depth}\,M_{\mathfrak{p}}\le 1-\dim R/\mathfrak{p}$ and so $\dim R/\mathfrak{p}=0$. Hence $\mathrm{Psupp}^1(M)=\{\mathfrak{m}\}$ and so by [15, Theorem 3.1] $\mathrm{Att}(H^1_{\mathfrak{m}}(M))\subseteq\{\mathfrak{m}\}$. This implies that $H^1_{\mathfrak{m}}(M)$ is of finite length (see [1, Corollary 7.2.12]). Hence $H^1_{\mathfrak{m}}(M)$ is co-Cohen-Macaulay of Noetherian dimension zero. □

Theorem 5. *Let M be an unmixed R-module. If $\mathrm{depth}\,M\ge d-1$, then the following statements are equivalent:*

(i) *the module $H^d_{\mathfrak{m}}(M)$ is co-Cohen-Macaulay of Noetherian dimension d;*

(ii) *the module $H^{d-1}_{\mathfrak{m}}(M)$ is either zero or co-Cohen-Macaulay of Noetherian dimension $d-2$.*

Proof. (i) $\Rightarrow$ (ii). We use induction on d. The case $d\le 2$ follows by Lemma 4. Let $d\ge 3$. Let $x\in\mathfrak{m}$ be a strict f-sequence on M. By [1, Exercise 11.3.9], $x\notin\mathfrak{p}$ for all $\mathfrak{p}\in\mathrm{Ass}\,M\setminus\{\mathfrak{m}\}$. Note that $\mathfrak{m}\notin\mathrm{Ass}\,M$, since $\mathrm{depth}\,M>0$. Therefore x is M regular. Thus, by Lemma 3, $H^{d-1}_{\mathfrak{m}}(M/xM)$ is co-Cohen-Macaulay of Noetherian dimension $d-1$. Since M/xM unmixed and $\mathrm{depth}\,M/xM\ge d-2$, it follows from the inductive hypothesis that $H^{d-2}_{\mathfrak{m}}(M/xM)$ is either zero or co-Cohen-Macaulay of Noetherian dimension $d-3$. Hence, from the exact sequence

$$0\longrightarrow\frac{H^{d-2}_{\mathfrak{m}}(M)}{xH^{d-2}_{\mathfrak{m}}(M)}\longrightarrow H^{d-2}_{\mathfrak{m}}\left(\frac{M}{xM}\right)\longrightarrow\left(0:_{H^{d-1}_{\mathfrak{m}}(M)}x\right)\longrightarrow 0,\tag{3}$$

and our assumption we get $H^{d-2}_{\mathfrak{m}}(M/xM)\cong(0:_{H^{d-1}_{\mathfrak{m}}(M)}x)$ and so $(0:_{H^{d-1}_{\mathfrak{m}}(M)}x)$ is either zero or co-Cohen-Macaulay of Noetherian dimension $d-3$. Since x is a coregular sequence on $H^{d-1}_{\mathfrak{m}}(M)$, we have $H^{d-1}_{\mathfrak{m}}(M)$ being either zero or co-Cohen-Macaulay of Noetherian dimension $d-2$, as required.

(ii) $\Rightarrow$ (i). We prove by induction on d. By [8, Corollary 2.5], we can assume that $d\ge 3$. By our hypothesis there exists $x\in\mathfrak{m}\setminus(\cup_{\mathfrak{p}\in\mathrm{Ass}(M)}\mathfrak{p})\cup(\cup_{\mathfrak{q}\in\mathrm{Att}(H^{d-1}_{\mathfrak{m}}(M))}\mathfrak{q})$. Hence, from the exact sequence

$$0\longrightarrow\frac{H^i_{\mathfrak{m}}(M)}{xH^i_{\mathfrak{m}}(M)}\longrightarrow H^i_{\mathfrak{m}}\left(\frac{M}{xM}\right)\longrightarrow\left(0:_{H^{i+1}_{\mathfrak{m}}(M)}x\right)\longrightarrow 0,\tag{4}$$

we have the isomorphism $H^i_{\mathfrak{m}}(M/xM)\cong(0:_{H^{i+1}_{\mathfrak{m}}(M)}x)(\ddagger)$ for $i\in\{d-2,d-1\}$. Since x is a coregular sequence on

$H_{\mathfrak{m}}^{d-1}(M)$, $(0:_{H_{\mathfrak{m}}^{d-1}(M)}x)$ is either zero or co-Cohen-Macaulay of Noetherian dimension $d-3$ and so is $H_{\mathfrak{m}}^{d-2}(M/xM)$. Hence by induction hypothesis $H_{\mathfrak{m}}^{d-1}(M/xM)$ is co-Cohen-Macaulay of Noetherian dimension $d-1$. Since x is a coregular sequence on $H_{\mathfrak{m}}^{d}(M)$, it follows, by (‡), that $H_{\mathfrak{m}}^{d}(M)$ is co-Cohen-Macaulay of Noetherian dimension d. This complete the proof. □

The following consequence follows by Theorem 5.

Corollary 6. *Let M be a Cohen-Macaulay module. Then $H_{\mathfrak{m}}^{d}(M)$ is co-Cohen-Macaulay of Noetherian dimension d.*

The following theorem extends [16, Corollary 3.6].

Theorem 7. *Let width $(H_{\mathfrak{m}}^{i}(M)) \geq i-1$ for all i with $2 \leq i < d$. Then $H_{\mathfrak{m}}^{d}(M)$ is co-Cohen-Macaulay of Noetherian dimension d.*

Proof. We use induction on d. Let $d = 3$. Then, by [1, Corollary 2.1.7] and our assumption, there exists $x \in \mathfrak{m} \setminus (\cup_{\mathfrak{p}\in \mathrm{Ass}(M)}\mathfrak{p}) \cup (\cup_{\mathfrak{q}\in \mathrm{Att}(H_{\mathfrak{m}}^{2}(M))}\mathfrak{q})$. Hence from the exact sequence

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow \frac{M}{xM} \longrightarrow 0\,(*), \tag{5}$$

we get the exact sequence

$$0 \longrightarrow H_{\mathfrak{m}}^{2}\left(\frac{M}{xM}\right) \longrightarrow H_{\mathfrak{m}}^{3}(M) \xrightarrow{x} H_{\mathfrak{m}}^{3}(M) \longrightarrow 0. \tag{6}$$

Therefore $H_{\mathfrak{m}}^{2}(M/xM) \cong (0:_{H_{\mathfrak{m}}^{3}(M)}x)$ and so $(0:_{H_{\mathfrak{m}}^{3}(M)}x)$ is co-Cohen-Macaulay of Noetherian dimension 2. Thus $H_{\mathfrak{m}}^{3}(M)$ is co-Cohen-Macaulay of Noetherian dimension 3. The result has been proved in this case. Now suppose that $d > 3$ and assume that our assertion is true for $d-1$. There exists $x \in \mathfrak{m} \setminus (\cup_{\mathfrak{p}\in \mathrm{Ass}(M)}\mathfrak{p}) \cup (\cup_{i=2}^{d-1}\cup_{\mathfrak{q}\in \mathrm{Att}(H_{\mathfrak{m}}^{i}(M))}\mathfrak{q})$ and so from the exact sequence $(*)$ we have the following long exact sequence

$$\cdots \longrightarrow H_{\mathfrak{m}}^{i}\left(\frac{M}{xM}\right) \longrightarrow H_{\mathfrak{m}}^{i}(M) \xrightarrow{x} H_{\mathfrak{m}}^{i}(M) \longrightarrow \cdots. \tag{7}$$

Thus there is an isomorphism $H_{\mathfrak{m}}^{i}(M/xM) \cong (0:_{H_{\mathfrak{m}}^{i+1}(M)}x)(\star)$ for all i with $2 \leq i < d$. Hence width $(H_{\mathfrak{m}}^{i}(M/xM)) \geq i-1$ for all i with $2 \leq i < d-1$ and so by the induction hypothesis $H_{\mathfrak{m}}^{d-1}(M/xM)$ is co-Cohen-Macaulay of Noetherian dimension $d-1$. Therefore, in view of $(\star)$, the module $H_{\mathfrak{m}}^{d}(M)$ is co-Cohen-Macaulay of Noetherian dimension d, as required. □

The following corollary immediately follows by Theorem 7 and [8, Corollary 2.5].

Corollary 8. *Let $H_{\mathfrak{m}}^{i}(M)$ be co-Cohen-Macaulay of Noetherian dimension i for all i with $0 \leq i < d$. Then $H_{\mathfrak{m}}^{d}(M)$ is co-Cohen-Macaulay of Noetherian dimension d.*

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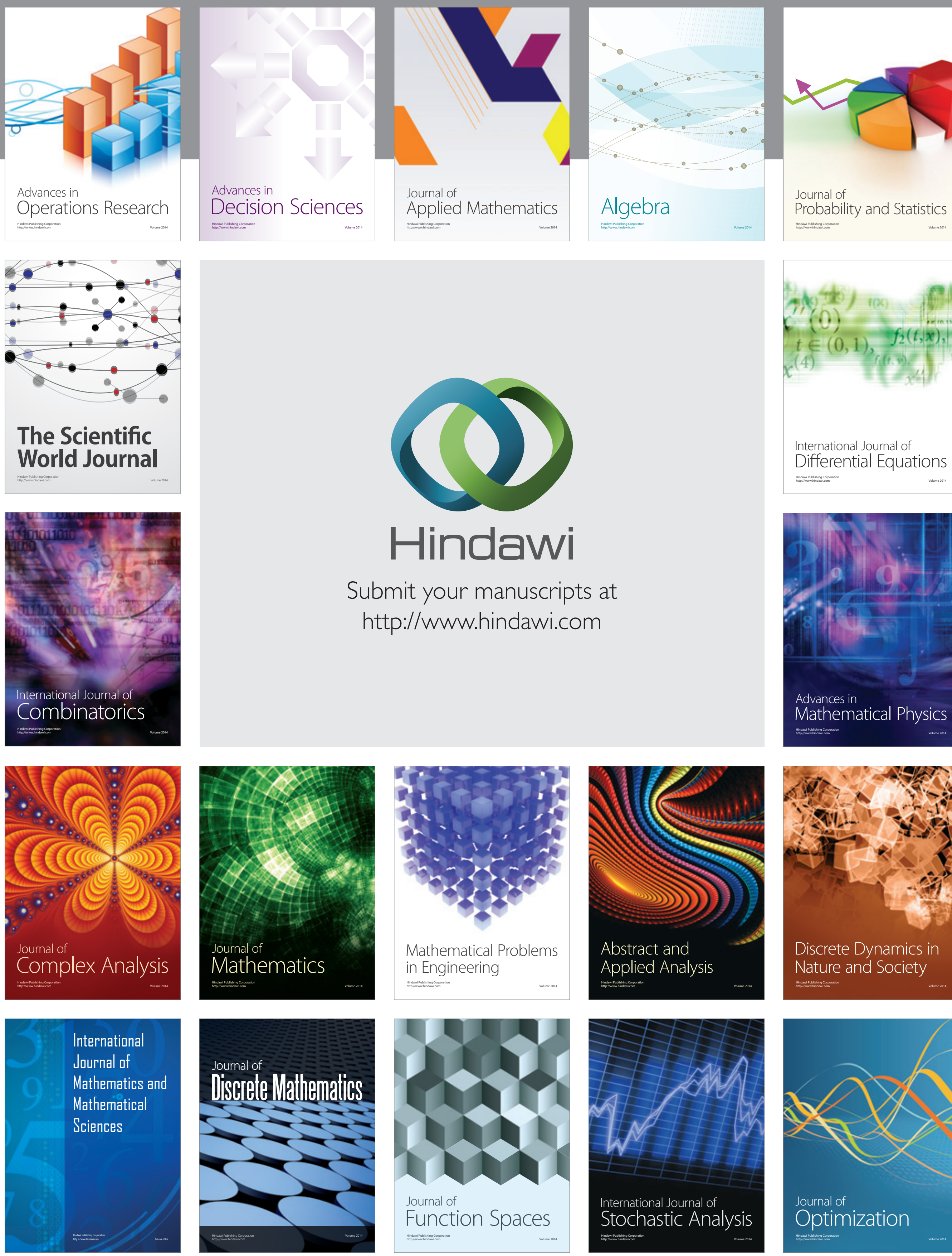
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