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Research Article

Higher Integrability of Weak Solutions to a Class of Double Obstacle Systems

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We first introduce double obstacle systems associated with the second-order quasilinear elliptic differential equation $\operatorname{div}(A(x,\nabla u)) = \operatorname{div} f(x,u)$, where $A(x,\nabla u)$, $f(x,u)$ are two $n \times N$ matrices satisfying certain conditions presented in the context, then investigate the local and global higher integrability of weak solutions to the double obstacle systems, and finally generalize the results of the double obstacle problems to the double obstacle systems.

1. Introduction

Let $\Omega \subset R^n$ be a bounded domain. We consider the following quasilinear elliptic systems:

$$D_i A_\alpha^i(x,\nabla u) = D_i f_\alpha^i(x,u), \quad \alpha = 1,2,\ldots,N, \tag{1}$$

where $A_\alpha^i(x,h)$, $f_\alpha^i(x,u)$ satisfy the conditions given in the following context. If we denote $A(x,h) = (A_\alpha^i(x,h))$, $f(x,u) = (f_\alpha^i(x,u))(n \times N$ matrices), then (1) turns into

$$\operatorname{div}\ A(x,\nabla u) = \operatorname{div}\ f(x,u). \tag{2}$$

Our aim is to generalize the integrability results of double obstacle problems ($N = 1$) to systems ($N > 1$). In order to do that, first, we have to define the obstacle problems corresponding to systems (2), and then we investigate the integrability of the weak solutions to the double obstacle systems.

In order to narrate our assumptions and our results, we give the following notations.

Let $f(x) = (f_1(x), f_2(x),\ldots,f_N(x))$, $g(x) = (g_1(x),\ldots,g_N(x))$ be two vector-valued functions defined on Ω, then we say that $f(x) \le g(x)$ if and only if $f_\alpha(x) \le g_\alpha(x)$ a.e. $x \in \Omega$, $\forall 1 \le \alpha \le N$, and define $\max\{f(x),g(x)\} = (\max\{f_1(x),g_1(x)\},\ldots,\max\{f_N(x),g_N(x)\})$, $\min\{f(x),g(x)\} = -\max\{-f(x),-g(x)\}$, $\theta^+(x) = (\max\{\theta_1(x),0\},\ldots,\max\{\theta_N(x),0\})$, $\theta^-(x) = (\min\{\theta_1(x),0\},\ldots,\min\{\theta_N(x),0\})$.

Let $W^{1,p}(\Omega)$, and let $W_0^{1,p}(\Omega)$, $W_{\mathrm{loc}}^{1,p}(\Omega)$ be usual Sobolev spaces, then define

$$\begin{aligned}
W^{1,p}\left(\Omega,R^N\right) &= \left\{f(x) \mid f(x) = \left(f_1(x),\ldots,f_N(x)\right),\right.\\
&\qquad \left. f_\alpha(x) \in W^{1,p}(\Omega), \alpha = 1,\ldots,N\right\},\\
W_0^{1,p}\left(\Omega,R^N\right) &= \left\{f(x) \mid f(x) = \left(f_1(x),\ldots,f_N(x)\right),\right.\\
&\qquad \left. f_\alpha(x) \in W_0^{1,p}(\Omega), \alpha = 1,\ldots,N\right\},\\
W_{\mathrm{loc}}^{1,p}\left(\Omega,R^N\right) &= \left\{f(x) \mid f(x) = \left(f_1(x),\ldots,f_N(x)\right),\right.\\
&\qquad \left. f_\alpha(x) \in W_{\mathrm{loc}}^{1,p}(\Omega), \alpha = 1,\ldots,N\right\}.
\end{aligned} \tag{3}$$

Let $\theta \in W^{1,p}(\Omega,R^N)$ and $\varphi,\psi : \Omega \to R^N$, then we denote

$$\begin{aligned}
&K_{\varphi,\psi}^{\theta,p}\left(\Omega,R^N\right)\\
&\quad = \left\{u \in W^{1,p}\left(\Omega,R^N\right) : u-\theta \in W_0^{1,p}\left(\Omega,R^N\right),\right.\\
&\qquad \left.\varphi \le u \le \psi \text{ a.e. in } \Omega\right\},
\end{aligned} \tag{4}$$

and we call φ, ψ obstacles and θ boundary value.

Let $A = (a_\alpha^i)$, $B = (b_\alpha^i)$ be two $n \times N$ matrixes, then define $A \bullet B = a_\alpha^i b_\alpha^i$; here and in the following we use the convention that repeated indices are summed: here α goes from 1 to N and i from 1 to n.

We consider the higher integrability of weak solutions to $K_{\varphi,\psi}^{\theta,p}(A)$-double obstacle systems corresponding to (2).

Definition 1. We call a function $u \in K_{\varphi,\psi}^{\theta,p}(\Omega, R^N)$ a weak solution to $K_{\varphi,\psi}^{\theta,p}(A)$-double obstacle systems if

$$\int_\Omega \left(A(x, \nabla u) - f(x, u)\right) \bullet \nabla(v - u)\, dx \geq 0, \tag{5}$$

holds for all $v \in K_{\varphi,\psi}^{\theta,p}(\Omega, R^N)$, here $\nabla u = (\nabla u_1, \nabla u_2, \ldots, \nabla u_1)^T$.

Obstacle problems naturally appear in the nonlinear potential theory and variational inequalities (see [1, 2] and references therein). It can be applied to phase transitions in materials science, flame propagation, combustion theory, crystal growth, optimal control problems, elasto-plastic problems, or financial problems [3, 4]. Reference [5] obtained higher integrability and stability results of weak solutions to $K_{\varphi,\psi}^{\theta,p}(A)$-obstacle problems under the conditions $N = 1$, $f = 0$, and A satisfies homogeneous conditions. In this paper, we investigate the local and global higher integrability of weak solutions associated with $K_{\varphi,\psi}^{\theta,p}(A)$-double obstacle systems. This kind of higher integrability has been previously studied in [6] for single obstacle problems ($N = 1$). Our notation is standard.

2. Main Results

Let $1 < p < \infty$, and $s > p$. We assume that our mappings $A : \Omega \times R^{nN} \to R^{nN}$, $f : \Omega \times R^N \to R^{nN}$ are Caratheodory functions and satisfy the following conditions for fixed $0 < \alpha < \beta < \infty$, $0 < \lambda < \infty$:

(A1) for all $h \in R^{nN}$ and a.e. $x \in \Omega$,

$$A(x, h) \bullet h \geq \alpha |h|^p, \quad |A(x, h)| \leq \beta |h|^{p-1}; \tag{6}$$

(A2) for all $u \in R^N$ and a.e. $x \in \Omega$,

$$\left|f(x, u)\right| \leq \phi(x) + \lambda |u|^{(p-1)\gamma}. \tag{7}$$

Fix $x_0 \in \Omega$, let Q_r be a cube with center x_0 and side length r, and let $Q_{\lambda r}$ ($\lambda > 0$) be the cube parallel to Q_r with the same center as Q_r and side length λr. We denote

$$f_r \triangleq \fint_{Q_r} f dx \triangleq \frac{1}{|Q_r|} \int_{Q_r} f dx, \tag{8}$$

where $|Q_r|$ denotes the Lebesgue Measure of Q_r.

Theorem 2. *Suppose that $\varphi, \psi, \theta \in W^{1,s}(\Omega, R^N)$ ($s > p$), and let $u \in K_{\varphi,\psi}^{\theta,p}(\Omega, R^N)$ be a weak solution to $K_{\varphi,\psi}^{\theta,p}(A)$-double obstacle systems under conditions (A1) and (A2) with $1 \leq \gamma < n/(n-p)$, $\phi(x) \in L^{s/(p-1)}(\Omega, R^N)$, then there exists a constant $0 < \epsilon_0 = \epsilon_0(n, N, p, s, \alpha, \beta, \gamma, \lambda, \mathrm{diam}(\Omega)) < s - p$ such that, for each $\epsilon \in [0, \epsilon_0)$, one has $u \in W_{\mathrm{loc}}^{1,p+\epsilon}(\Omega, R^N)$. Furthermore, for every $x_0 \in \Omega$ and every cube $Q_r \subset \Omega$ ($r < r_0$ small enough) centered at x_0 such that $Q_{2r} \subset\subset \Omega$, one has*

$$\left[\fint_{Q_r} \left(|\nabla u| + |u|^\gamma\right)^{p+\epsilon} dx\right]^{1/(p+\epsilon)} \leq C\left\{\left[\fint_{Q_{2r}} \left(|\nabla u| + |u|^\gamma\right)^p dx\right]^{1/p} + \left[\fint_{Q_{2r}} H^s dx\right]^{1/s}\right\}, \tag{9}$$

where $H = |\nabla\varphi| + |\nabla\psi| + |\phi|^{1/(p-1)}$ and $C = C(n, N, p, s, \alpha, \beta, \gamma, \lambda, \mathrm{diam}(\Omega)) < \infty$.

In order to obtain the global higher integrability of weak solutions to $K_{\varphi,\psi}^{\theta,p}(A)$-double obstacle systems, it seems that we need to impose some regularity condition for $\partial\Omega$, the boundary of Ω. We say that $\partial\Omega$ is p-Poincaré thick if there exists $0 < a < \infty$ such that, for all open cubes $Q_r \subset R^n$ with side length $r > 0$, there holds

$$\left(\int_{Q_{2r}} |u|^p dx\right)^{1/p} \leq ax\left(\int_{Q_{2r}} |\nabla u|^{pn/(p+n)} dx\right)^{(p+n)/pn}, \tag{10}$$

whenever $u \in W^{1,p}(Q_{2r})$, $u = 0$, a.e. on $(R^n \setminus \Omega) \cap Q_{2r}$ and $Q_{3r/2} \cap \Omega^C \neq \emptyset$. Theorem 2.3 and Corollary 2.7 in [7] have given some simple conditions such that (10) holds for $p \geq n/(n-1)$.

Theorem 3. *Suppose that the boundary $\partial\Omega$ of Ω is p-Poincaré thick with $p > n/(n-1)$, and $\varphi, \psi, \theta \in W^{1,s}(\Omega, R^N)$ ($s > p$). If $u \in K_{\varphi,\psi}^{\theta,p}(\Omega, R^N)$ is a weak solution to $K_{\varphi,\psi}^{\theta,p}(A)$-double obstacle systems under conditions (A1) and (A2) with $1 \leq \gamma < n/(n-p)$, $\phi(x) \in L^{s/(p-1)}(\Omega, R^N)$, then there exists a constant $0 < \epsilon_0 = \epsilon_0(n, N, p, s, a, \alpha, \beta, \gamma, \lambda, \mathrm{diam}(\Omega)) < s - p$ such that for each $\epsilon \in [0, \epsilon_0)$, one has $u \in W^{1,p+\epsilon}(\Omega, R^N)$. Furthermore, we have*

$$\fint_\Omega \left(|\nabla u| + |u|^\gamma\right)^{p+\epsilon} dx\Big]^{1/(p+\epsilon)} \leq C\left\{\left[\fint_\Omega \left(|\nabla u| + |u|^\gamma\right)^p dx\right]^{1/p} + \left[\fint_\Omega H^s dx\right]^{1/s}\right\}, \tag{11}$$

where $H = |\nabla\varphi| + |\nabla\psi| + |\nabla\theta| + |\phi|^{1/(p-1)}$ and $C = C(n, N, p, s, a, \alpha, \beta, \gamma, \lambda, \mathrm{diam}(\Omega)) < \infty$.

3. Proofs of Main Results

The following lemma is due to Giaquinta and Modica [8].

Lemma 4 (Reverse Hölder's inequality). *Let Q be an n-cube and g, G be two nonnegative functions defined on Q. Suppose that for each $x_0 \in Q$ and each $r < \min\{(1/2)\,\mathrm{dist}(x_0, \partial Q), r_0\}$*

$$\fint_{Q_r} g^p dx \leq b\left(\fint_{Q_{2r}} g dx\right)^p + \tau \fint_{Q_{2r}} g^p dx + \fint_{Q_{2r}} G^p dx, \tag{12}$$

where constants $b > 1$, $r_0 > 0$, $0 \le \tau < 1$. Then $g \in L^q_{\mathrm{loc}}(Q)$ for $q \in [p, p+\epsilon)$ and

$$\left(\fint_{Q_r} g^q dx\right)^{1/q} \le C\left(\fint_{Q_{2r}} g^p dx\right)^{1/p} + \left(\fint_{Q_{2r}} G^q dx\right)^{1/q}, \quad (13)$$

for $Q_{2r} \subset\subset Q$, $r < r_0$, where C and ϵ are positive constants depending only on b, τ, p, n.

The proofs of Theorems 2 and 3 are stimulated by [5]. The general constant C denotes a constant whose value may change even on the same line.

Proof of Theorem 2. For any fixed $x_0 \in \Omega$ and cube Q_r centered at x_0 with side length r such that $Q_{2r} \subset\subset \Omega$, let $\eta \in C_0^\infty(Q_{2r})$ be a cutoff function such that $0 \le \eta \le 1$, $|\nabla\eta| \le C/r$ and $\eta \equiv 1$ on Q_r.

Let $v = (1-\eta^p)(u-u_{2r}) + \eta^p w$, where $w = (\psi - u_{2r})^- + \min((\varphi - u_{2r})^+, (\psi - u_{2r})^+)$. Due to the boundedness of Ω, we have $\varphi, \psi \in W^{1,p}(\Omega, R^N)$. Moreover, $v + u_{2r} \in K^{\theta,p}_{\varphi,\psi}(\Omega, R^N)$ because

$$w = \begin{cases} (\varphi - u_{2r})^+, & \psi \ge u_{2r}, \\ \psi - u_{2r}, & \psi < u_{2r}, \end{cases} \quad (14)$$

and this yields $\varphi \le v + u_{2r} \le \psi$ a.e. in Ω. Hence we have, by (5),

$$\int_\Omega \left(A(x, \nabla u) - f(x, u)\right) \bullet \nabla(v-u)\, dx \ge 0. \quad (15)$$

By the choice of v, we get

$$v - u = -u_{2r} - \eta^p (u - u_{2r}) + \eta^p w, \quad (16)$$

and hence

$$\nabla(v-u) = -\eta^p \nabla u + \eta^p \nabla w + p\eta^{p-1} \nabla\eta \otimes \left[w - (u - u_{2r})\right]. \quad (17)$$

This and (15), together with the structure assumptions (A1), (A2), yield

$$\begin{aligned} \alpha \int_{Q_{2r}} \eta^p |\nabla u|^p dx &\le \int_{Q_{2r}} \eta^p A(x, \nabla u) \bullet \nabla u dx \\ &\le \beta \int_{Q_{2r}} \eta^p |\nabla u|^{p-1} |\nabla w|\, dx + \int_{Q_{2r}} \eta^p |\phi| |\nabla u|\, dx \\ &\quad + \lambda \int_{Q_{2r}} \eta^p |u|^{(p-1)\gamma} |\nabla u|\, dx \\ &\quad + p\beta \int_{Q_{2r}} \eta^{p-1} \left[|w| + |u - u_{2r}|\right] |\nabla\eta| |\nabla u|^{p-1} dx \\ &\quad + \int_{Q_{2r}} \eta^p |\phi| |\nabla w|\, dx \\ &\quad + \lambda \int_{Q_{2r}} \eta^p |u|^{(p-1)\gamma} |\nabla w|\, dx \\ &\quad + p \int_{Q_{2r}} \eta^{p-1} |\phi| \left[|w| + |u - u_{2r}|\right] |\nabla\eta|\, dx \\ &\quad + p\lambda \int_{Q_{2r}} \eta^{p-1} |u|^{(p-1)\gamma} \left[|w| + |u - u_{2r}|\right] |\nabla\eta|\, dx. \end{aligned} \quad (18)$$

Using Hölder's inequality, Young's inequality, and Minikowski's inequality, we can estimate each term in (18) as follows:

$$\begin{aligned} &\beta \int_{Q_{2r}} \eta^p |\nabla u|^{p-1} |\nabla w|\, dx \\ &\quad \le \beta \left(\int_{Q_{2r}} \eta^p |\nabla u|^p dx\right)^{(p-1)/p} \left(\int_{Q_{2r}} \eta^p |\nabla w|^p dx\right)^{1/p} \\ &\quad \le \frac{\alpha}{8} \int_{Q_{2r}} \eta^p |\nabla u|^p dx + C \int_{Q_{2r}} \eta^p |\nabla w|^p dx, \\ &\int_{Q_{2r}} \eta^p |\phi| |\nabla u|\, dx \\ &\quad \le \frac{\alpha}{8} \int_{Q_{2r}} \eta^p |\nabla u|^p dx + C \int_{Q_{2r}} \eta^p |\phi|^{p/(p-1)} dx, \\ &\lambda \int_{Q_{2r}} \eta^p |u|^{(p-1)\gamma} |\nabla u|\, dx \\ &\quad \le \frac{\alpha}{8} \int_{Q_{2r}} \eta^p |\nabla u|^p dx + C \int_{Q_{2r}} \eta^p |u|^{p\gamma} dx, \\ &p\beta \int_{Q_{2r}} \eta^{p-1} \left[|w| + |u - u_{2r}|\right] |\nabla\eta| |\nabla u|^{p-1} dx \\ &\quad \le \frac{\alpha}{8} \int_{Q_{2r}} \eta^p |\nabla u|^p dx \\ &\qquad + Cr^{-p} \int_{Q_{2r}} \left[|w|^p + |u - u_{2r}|\right]^p dx. \\ &\int_{Q_{2r}} \eta^p |\phi| |\nabla w|\, dx \\ &\quad \le C \int_{Q_{2r}} \eta^p |\nabla w|^p dx \\ &\qquad + C \int_{Q_{2r}} \eta^p |\phi|^{p/(p-1)} dx, \end{aligned}$$

$$\begin{aligned}
&\lambda \int_{Q_{2r}} \eta^p |u|^{(p-1)\gamma} |\nabla w|\, dx \\
&\quad \le C \int_{Q_{2r}} \eta^p |\nabla w|^p dx \\
&\qquad + C \int_{Q_{2r}} \eta^p |u|^{p\gamma} dx, \\
&p \int_{Q_{2r}} \eta^{p-1} \left[|w| + |u - u_{2r}|\right] |\phi|\, |\nabla \eta|\, dx \\
&\quad \le C \int_{Q_{2r}} \eta^p |\phi|^{p/(p-1)} dx \\
&\qquad + C r^{-p} \int_{Q_{2r}} \left[|w|^p + |u - u_{2r}|^p\right] dx, \\
&p\lambda \int_{Q_{2r}} \eta^{p-1} |u|^{(p-1)\gamma} \left[|w| + |u - u_{2r}|\right] |\nabla \eta|\, dx \\
&\quad \le C \int_{Q_{2r}} \eta^p |u|^{p\gamma} dx \\
&\qquad + C r^{-p} \int_{Q_{2r}} \left[|w|^p + |u - u_{2r}|^p\right] dx.
\end{aligned} \tag{19}$$

We deduce from (14) that

$$\begin{aligned}
|w| &\le \begin{cases} |\varphi - \varphi_{2r}|, & \psi \ge u_{2r}, \\ |\psi - \psi_{2r}|, & \psi < u_{2r}, \end{cases} \\
|\nabla w| &\le |\nabla \varphi| + |\nabla \psi|.
\end{aligned} \tag{20}$$

Hence

$$\begin{aligned}
&\fint_{Q_{2r}} \eta^p |\nabla u|^p dx \\
&\quad \le C \fint_{Q_{2r}} \eta^p \left(|\nabla \varphi| + |\nabla \psi| + |\phi|^{1/(p-1)}\right)^p dx \\
&\qquad + C \fint_{Q_{2r}} \eta^p |u|^{p\gamma} dx \\
&\qquad + C r^{-p} \fint_{Q_{2r}} \left(|\varphi - \varphi_{2r}|^p + |\psi - \psi_{2r}|^p + |u - u_{2r}|^p\right) dx.
\end{aligned} \tag{21}$$

Now choosing $\max\{1, np/(n+p)\} \le t < p$ and using Poincaré's inequality and Sobolev-Poincaré's inequality, we get

$$r^{-p} \fint_{Q_{2r}} |\varphi - \varphi_{2r}|^p dx \le C \fint_{Q_{2r}} |\nabla \varphi|^p dx, \tag{22}$$

$$r^{-p} \fint_{Q_{2r}} |\psi - \psi_{2r}|^p dx \le C \fint_{Q_{2r}} |\nabla \psi|^p dx, \tag{23}$$

$$r^{-p} \fint_{Q_{2r}} |u - u_{2r}|^p dx \le C \left(\fint_{Q_{2r}} |\nabla u|^t dx \right)^{p/t}, \tag{24}$$

$$\begin{aligned}
\fint_{Q_{2r}} \eta^p |u|^{p\gamma} dx &\le C r^{-n} \int_{Q_{2r}} |u - u_{2r}|^{p\gamma} dx \\
&\quad + C \fint_{Q_{2r}} |u_{2r}|^{p\gamma} dx \\
&\le C r^{[n((1/p\gamma)-(1/p))+1]p\gamma} \left(\int_{Q_{2r}} |\nabla u|^p dx \right)^{\gamma - 1} \\
&\quad \times \fint_{Q_{2r}} |\nabla u|^p dx + C \left(\fint_{Q_{2r}} |u|^{\gamma t} dx \right)^{p/t} \\
&\triangleq \xi(r) \fint_{Q_{2r}} |\nabla u|^p dx + C \left(\fint_{Q_{2r}} |u|^{\gamma t} dx \right)^{p/t},
\end{aligned} \tag{25}$$

where $\xi(r) \to 0$ as $r \to 0$ because of the absolute continuity of integrals and the fact that $\gamma \ge 1$.

The above five inequalities imply that

$$\begin{aligned}
&\fint_{Q_r} |\nabla u|^p dx \\
&\quad \le \xi(r) \fint_{Q_{2r}} |\nabla u|^p dx + C \left(\fint_{Q_{2r}} |u|^{\gamma t} dx \right)^{p/t} \\
&\qquad + C \fint_{Q_{2r}} \left(|\nabla \varphi| + |\nabla \psi| + |\phi|^{1/(p-1)}\right)^p dx \\
&\qquad + C \left(\fint_{Q_{2r}} |\nabla u|^t dx \right)^{p/t}.
\end{aligned} \tag{26}$$

To complete our proof, adding $\fint_{Q_{2r}} |u|^{p\gamma} dx$ to each side of (26), using (25), and setting

$$\begin{aligned}
g &= \left(|\nabla u| + |u|^\gamma\right)^t, \qquad k = \frac{p}{t}, \\
H &= |\nabla \varphi| + |\nabla \psi| + |\phi|^{1/(p-1)}, \\
G &= H^t.
\end{aligned} \tag{27}$$

Equation (26) can be rewritten as

$$\begin{aligned}
&\fint_{Q_r} g^k dx \\
&\quad \le C \xi(r) \fint_{Q_r} g^k dx + C \left[\left(\fint_{Q_{2r}} g dx \right)^k + \fint_{Q_{2r}} G^k dx \right].
\end{aligned} \tag{28}$$

For r ($r < r_0$) small enough, we have $\tau = C\xi(r) < 1$, and then Lemma 4 implies that there exists $0 < \epsilon_0 = \epsilon_0(n, N, p, s, \alpha, \beta, \gamma, \lambda, \mathrm{diam}(\Omega)) < s - p$ such that, for

$0 \le \epsilon < \epsilon_0$, we have $u \in W^{1,p+\epsilon}_{\text{loc}}(\Omega, R^N)$, and for every cube Q_r $(r < r_0)$ such that $Q_{2r} \subset \Omega$, we have

$$\left[\fint_{Q_r} \left(|\nabla u| + |u|^\gamma\right)^{p+\epsilon} dx\right]^{1/(p+\epsilon)} \le C\left\{\left[\fint_{Q_{2r}} \left(|\nabla u| + |u|^\gamma\right)^p dx\right]^{1/p} + \left[\fint_{Q_{2r}} G^s dx\right]^{1/s}\right\}, \tag{29}$$

where $C = C(n, N, p, s, \alpha, \beta, \gamma, \lambda, \text{diam}(\Omega)) < \infty$. □

Proof of Theorem 3. Choose a cube $Q_0 = Q_{2r_0}$ such that $\Omega \subset Q_0$. For an arbitrary cube $Q_{2r} \subset Q_0$, there are two possibilities to consider: (I) $Q_{3r/2} \subset \Omega$, or (II) $Q_{3r/2} \cap \Omega^C \neq \emptyset$.

In the case (I), following the proof of Theorem 2, we have

$$\fint_{Q_r} g^k dx \le C\xi(r) \fint_{Q_{2r}} g^k dx + C\left[\left(\fint_{Q_{2r}} g dx\right)^k + \fint_{Q_{2r}} G^k dx\right]. \tag{30}$$

with $g = (|\nabla u| + |u|^\gamma)^t$, $H = |\nabla\varphi| + |\nabla\psi| + |\phi|^{1/(p-1)}$, $G = H^t$ in $Q_{2r} \cap \Omega$, and $g = G = 0$ in $Q_{2r} \setminus \Omega$, $k = p/t$, where $\max\{1, np/(n+p)\} \le t < p$, and $\xi(r) \to 0$ as $r \to 0$.

In case (II), observing that replacing θ by $\theta_1 = \min\{\psi, \max\{\varphi, \theta\}\}$, we may as well assume that the boundary function θ satisfies $\varphi \le \theta \le \psi$ in Ω. Indeed, $\theta_1 = (\varphi - \theta)^+ - (\psi - \theta)^- + \theta$, and since $0 \le (\varphi - \theta)^+ \le (u - \theta)^+ \in W^{1,p}_0(\Omega, R^N)$, $0 \le -(\psi - \theta)^- \le (u - \theta)^- \in W^{1,p}_0(\Omega, R^N)$, the functions $(\varphi - \theta)^+, -(\psi - \theta)^-$, and hence $u - \theta_1$ belongs to $W^{1,p}_0(\Omega, R^N)$. Next consider the function $v = u - \eta^p(u - \theta)$ in Ω, where $\eta \in C^\infty_0(Q_{2r})$ is a standard test function as in the proof of Theorem 2, then $v \in K^{\theta,p}_{\varphi,\psi}(\Omega, R^N)$. Indeed, because $v - \theta \in W^{1,p}_0$ and $\varphi \le u \le \psi$, $\varphi \le \theta \le \psi$ a.e. in Ω, we have

$$\begin{aligned} v &= \left(1 - \eta^p\right) u + \eta^p \theta \ge \varphi, \\ v &= \left(1 - \eta^p\right) u + \eta^p \theta \le \psi \quad \text{a.e. in } \Omega. \end{aligned} \tag{31}$$

Since

$$\nabla v - \nabla u = -\eta^p \nabla u - p\eta^{p-1}(u - \theta)\nabla\eta + \eta^p \nabla\theta, \tag{32}$$

we have, by (5) and assumptions (A1) and (A2)

$$\begin{aligned} \alpha \int_\Omega \eta^p |\nabla u|^p dx &\le \int_\Omega \eta^p A(x, \nabla u) \bullet \nabla u dx \\ &\le \int_\Omega \left[\left(A(x, \nabla u) - f(x, u)\right) \bullet \left(\eta^p \nabla\theta - p\eta^{p-1}\nabla\eta \otimes (u - \theta)\right) + \eta^p f(x, u) \bullet \nabla u\right] dx \\ &\le \beta \int_\Omega \eta^p |\nabla u|^{p-1} |\nabla\theta| dx + \int_\Omega \eta^p |\phi| |\nabla\theta| dx + \int_\Omega \eta^p |\phi| |\nabla u| dx \\ &\quad + \lambda \int_\Omega \eta^p |u|^{(p-1)\gamma} |\nabla\theta| dx + \lambda \int_\Omega \eta^p |u|^{(p-1)\gamma} |\nabla u| dx \\ &\quad + p\beta \int_\Omega \eta^{p-1} |u - \theta| |\nabla\eta| |\nabla u|^{p-1} dx + p\beta \int_\Omega \eta^{p-1} |\phi| |u - \theta| |\nabla\eta| dx \\ &\quad + p\beta \int_\Omega \eta^{p-1} |u|^{(p-1)\gamma} |u - \theta| |\nabla\eta| dx \\ &\le \frac{\alpha}{2} \int_\Omega \eta^p |\nabla u|^p dx + C \int_\Omega \eta^p |\nabla\theta|^p dx + C \int_\Omega \eta^p |\phi|^{p/(p-1)} dx \\ &\quad + C \int_\Omega \eta^p |u|^{p\gamma} dx + C \int_\Omega |u - \theta|^p |\nabla\eta|^p dx, \end{aligned} \tag{33}$$

where we have used Hölder's inequality and Young's inequality several times. Hence

$$\begin{aligned} \int_\Omega \eta^p |\nabla u|^p dx &\le C\left[\int_\Omega \eta^p \left(|\nabla\theta|^p + |\phi|^{p/(p-1)}\right) dx + \int_\Omega \eta^p |u|^{p\gamma} dx + \int_\Omega |u - \theta|^p |\nabla\eta|^p dx\right], \end{aligned} \tag{34}$$

where the generic constant C is depending only on $n, N, p, \alpha, \beta, \lambda$.

To estimate the last term in (34), we employ the p-Poincaré thickness of $\partial\Omega$. Indeed, the function $u - \theta$ can be extended continuously to be 0 to $C\Omega$, and therefore

$$\int_\Omega |u - \theta|^p |\nabla\eta|^p dx \le Cr^{-p} x\left[\int_{Q_{2r} \cap \Omega} |\nabla(u - \theta)|^{np/(n+p)} d\right]^{(n+p)/n}. \tag{35}$$

Using Minikowski's inequality and Hölder's inequality, we obtain the following:

$$\begin{aligned} r^{-p}&\left[\int_{Q_{2r}\cap\Omega}|\nabla(u-\theta)|^{np/(n+p)}dx\right]^{(n+p)/n} \\ &\le Cr^{-p}\left[\left(\int_{Q_{2r}\cap\Omega}|\nabla\theta|^{np/(n+p)}dx\right)^{(n+p)/np}\right. \\ &\qquad\left.+\left(\int_{Q_{2r}\cap\Omega}|\nabla u|^{np/(n+p)}dx\right)^{(n+p)/np}\right]^{p} \\ &\le C\int_{Q_{2r}\cap\Omega}|\nabla\theta|^{p}dx \\ &\quad+Cr^{n-(np/t)}\left(\int_{Q_{2r}\cap\Omega}|\nabla u|^{t}dx\right)^{p/t}. \end{aligned} \tag{36}$$

Hence we derive from (25), (34), (35), and (36) that

$$\begin{aligned} &\fint_{Q_r\cap\Omega}|\nabla u|^{p}dx \\ &\le C\xi(r)\fint_{Q_{2r}\cap\Omega}|\nabla u|^{p}dx \\ &\quad+C\fint_{Q_{2r}\cap\Omega}\left(|\nabla\theta|+|\phi|^{1/(p-1)}\right)^{p}dx \\ &\quad+C\left[\fint_{Q_{2r}\cap\Omega}\left(|u|^{\gamma t}+|\nabla u|^{t}\right)dx\right]^{p/t}. \end{aligned} \tag{37}$$

Adding $\fint_{Q_r}|u|^{p\gamma}dx$ to each side of (37) and using (25), setting $g=(|\nabla u|+|u|^{\gamma})^{t}$, $H=|\nabla\varphi|+|\nabla\psi|+|\nabla\theta|+|\phi|^{1/(p-1)}$, $G=H^{t}$ in $Q_{2r}\cap\Omega$, $g=G=0$ in $Q_{2r}\setminus\Omega$, $k=p/t$, where $\max\{1,np/(n+p)\}\le t<p$, we obtain

$$\begin{aligned} &\fint_{Q_r\cap\Omega}g^{k}dx \\ &\le C\xi(r)\fint_{Q_{2r}\cap\Omega}g^{k}dx \\ &\quad+C\left[\left(\fint_{Q_{2r}\cap\Omega}gdx\right)^{k}+\fint_{Q_{2r}\cap\Omega}G^{k}dx\right], \end{aligned} \tag{38}$$

where $\xi(r)\to 0$ as $r\to 0$ and $C=C(n,N,p,s,a,\alpha,\beta,\gamma,\lambda,\operatorname{diam}(\Omega))>0$.

For r ($r<r_0$) small enough, we have $\tau=C\xi(r)<1$, and then Lemma 4 implies that there exists $0<\epsilon_0=\epsilon_0(n,N,p,s,a,\alpha,\beta,\gamma,\lambda,\operatorname{diam}(\Omega))<s-p$ such that, for $0\le\epsilon<\epsilon_0$, we have $u\in W^{1,p+\epsilon}(\Omega,R^{N})$, and (11) holds. Hence the theorem follows. □

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