Research Article

Some Inequalities for the Derivative of Polynomials

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If $p(z) = \sum_{v=0}^n c_v z^v$ is a polynomial of degree n , having no zeros in $|z| < 1$, then Aziz (1989) proved $\max_{|z|=1} |p'(z)| \leq (n/2) (M_\alpha^2 + M_{\alpha+\pi}^2)^{1/2}$, where $M_\alpha = \max_{1 \leq k \leq n} |p(e^{i(\alpha+2k\pi)/n})|$. In this paper, we consider a class of polynomial P_n^μ of degree n , defined as $p(z) = a_0 + \sum_{v=\mu}^n a_v z^v$ and present certain generalizations of above inequality and some other well-known results.

1. Introduction and Statement of the Results

Let P_n denote the space of all complex polynomials of degree n . If $p \in P_n$, that is, $p(z) = \sum_{v=0}^n a_v z^v$, then according to the famous result of Bernstein (see [1])

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)| \quad (1)$$

and a simple deduction from the maximum modulus principle (see [2, page 346])

$$\max_{|z|=1} |p(Rz)| \leq R^n \max_{|z|=1} |p(z)|. \quad (2)$$

Both inequalities (1) and (2) are sharp and the equality holds if and only if $p(z)$ has all its zeros at the origin.

If we restrict ourselves to the class of polynomial having no zeros in $|z| < 1$, then (1) can be sharpened. In fact, P. Erdős conjectured and later Lax [3] proved that if $p(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \quad (3)$$

It was shown by Frappier et al. [4, Theorem 8] that if $p \in P_n$, then

$$\max_{|z|=1} |p'(z)| \leq n \max_{1 \leq \rho \leq 2n} |p(e^{i\rho\pi/n})| \quad (4)$$

and clearly (4) represents a refinement of (1), since the maximum of $|p(z)|$ on $|z| = 1$ may be larger than maximum of $|p(z)|$ taken over $(2n)$ th roots of unity as one can show by taking a simple example $p(z) = z^n + ia$, $a > 0$.

In this connection, Aziz [5] improved inequality (4) by showing that if $p \in P_n$, then for every real α

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} (M_\alpha + M_{\alpha+\pi}) \quad (5)$$

and inequality (2) for $R > 1$

$$\max_{|z|=1} |p(Rz) - p(z)| \leq \frac{R^n - 1}{2} (M_\alpha + M_{\alpha+\pi}), \quad (6)$$

where

$$M_\alpha = \max_{1 \leq \rho \leq n} |p(e^{i(\alpha+2\rho\pi)/n})|. \quad (7)$$

Inequality (3) has also been improved by Aziz [5] by using the same argument as in inequality (5) and he proved that if $p \in P_n$ and $p(z) \neq 0$ in $|z| < 1$, then for every real α

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} (M_\alpha^2 + M_{\alpha+\pi}^2)^{1/2} \quad (8)$$

and also improved inequality (6) for every real α and $R > 1$

$$\max_{|z|=1} |p(Rz) - p(z)| \leq \frac{R^n - 1}{2} (M_\alpha^2 + M_{\alpha+\pi}^2)^{1/2}, \quad (9)$$

where M_α is same as defined in (7).

Recently, Rather and Shah [6] considered the class of polynomials having no zeros in $|z| < k$, $k \geq 1$ and $m = \min_{|z|=1} |p(z)|$ and improved the inequalities (8) and (9) of Aziz [5] by showing the following.

Theorem A. If $p \in P_n$, $p(z) \neq 0$ in $|z| < k$, $k \geq 1$, and $m = \min_{|z|=k} |p(z)|$, then for every real α

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{\sqrt{2(1+k^2)}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2}, \quad (10)$$

where M_α is defined in (7).

Theorem B. If $p \in P_n$, $p(z) \neq 0$ in $|z| < k$, $k \geq 1$, and $m = \min_{|z|=k} |p(z)|$, then for every real α and $R > 1$

$$\max_{|z|=1} |p(Rz) - p(z)| \leq \frac{R^n - 1}{\sqrt{2(1+k^2)}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2}, \quad (11)$$

where M_α is defined in (7).

In this paper, we first present the following result, which is a generalization of inequality (10) on class of polynomial P_n^μ of degree n defined as $p(z) = a_0 + \sum_{v=\mu}^n a_v z^v$, $0 \leq \mu \leq n$ and having no zeros in $|z| < k$, $k \geq 1$.

Theorem 1. If $p \in P_n^\mu$, $p(z) \neq 0$ in $|z| < k$, $k \geq 1$, and $m = \min_{|z|=k} |p(z)|$, then for every real α

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{\sqrt{2(1+k^{2\mu})}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2}, \quad (12)$$

where M_α is defined in (7).

On taking $k = 1$ in Theorem 1, the following result has been obtained, which was shown by Rather and Shah [6].

Corollary 2. If $p \in P_n$, $p(z) \neq 0$ in $|z| < 1$ and $m = \min_{|z|=1} |p(z)|$, then for every real α

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2}, \quad (13)$$

where M_α is defined in (7).

If we consider that some zeros of the polynomial $p \in P_n^\mu$ are on $|z| = k$, then $m = 0$ and the following result has been obtained from Theorem 1.

Corollary 3. If $p \in P_n^\mu$ and $p(z) \neq 0$ in $|z| \leq k$, $k \geq 1$, then for every real α

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{\sqrt{2(1+k^{2\mu})}} (M_\alpha^2 + M_{\alpha+\pi}^2)^{1/2}, \quad (14)$$

where M_α is defined in (7).

Remark 4. For $k = 1$, Corollary 3 reduces to inequality (8) due to Aziz [5].

Taking $\mu = 1$ in inequality (14) of Corollary 3, we get the following result.

Corollary 5. If $p \in P_n$ and $p(z) \neq 0$ in $|z| \leq k$, $k \geq 1$, then for every real α

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{\sqrt{2(1+k^2)}} (M_\alpha^2 + M_{\alpha+\pi}^2)^{1/2}, \quad (15)$$

where M_α is defined in (7).

Next result is the generalization of inequality (11) of Rather and Shah [6] for the polynomial $p \in P_n^\mu$ and having no zeros in $|z| < k$, $k \geq 1$.

Theorem 6. If $p \in P_n^\mu$, $p(z) \neq 0$ in $|z| < k$, $k \geq 1$, and $m = \min_{|z|=k} |p(z)|$, then for every real α and $R \geq r \geq 1$

$$\begin{aligned} \max_{|z|=1} |p(Rz) - p(rz)| \\ \leq \frac{R^n - r^n}{\sqrt{2(1+k^{2\mu})}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2}, \end{aligned} \quad (16)$$

where M_α is defined in (7).

Remark 7. For taking $\mu = 1$ and $r = 1$, Theorem 6 reduces inequality (11) of Theorem B and on dividing both sides of inequality (16) by $R - r$ and taking $R \rightarrow r$, inequality (12) of Theorem 1 is obtained.

If we take $k = 1$ and $r = 1$ in inequality (16), we have the following inequality due to Rather and Shah [6].

Corollary 8. If $p \in P_n$, $p(z) \neq 0$ in $|z| < 1$ and $m = \min_{|z|=1} |p(z)|$, then for every real α and $R > 1$

$$\max_{|z|=1} |p(Rz) - p(z)| \leq \frac{R^n - 1}{2} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2}, \quad (17)$$

where M_α is defined in (7).

If we take $\mu = 1$ in inequality (16) of Theorem 6, we have the following result.

Corollary 9. If $p \in P_n$ and $p(z) \neq 0$ in $|z| < k$, $k \geq 1$, then for every α and $R \geq r \geq 1$

$$\max_{|z|=1} |p(Rz) - p(rz)| \leq \frac{R^n - r^n}{\sqrt{2(1+k^2)}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2}, \quad (18)$$

where M_α is defined in (7).

Remark 10. From inequality (18), it is clear that Corollary 9 is a generalization of inequality (11). Also on dividing inequality (18) by $R - r$ and making $R \rightarrow r$ we have inequality (13) of Corollary 2.

Another generalization of Theorem 6 has been obtained by considering the fact that some zeros of $p \in P_n^\mu$ are on $|z| = k$; that is, at $m = 0$ the following result has been found from Theorem 6.

Corollary 11. *If $p \in P_n^\mu$ and $p(z) \neq 0$ in $|z| \leq k$, $k \geq 1$, then for every real α and $R \geq r \geq 1$*

$$\max_{|z|=1} |p(Rz) - p(rz)| \leq \frac{R^n - r^n}{\sqrt{2(1+k^{2\mu})}} (M_\alpha^2 + M_{\alpha+\pi}^2)^{1/2}, \quad (19)$$

where M_α is defined in (7).

Remark 12. For $k = 1$ and $r = 1$, Corollary 11 reduces to inequality (9) due to Aziz [5] and dividing inequality (19) by $R-r$ and taking $R \rightarrow r$ we have inequality (14) of Corollary 3.

An immediate consequence of Corollary 11 has to be obtained by taking $\mu = 1$.

Corollary 13. *If $p \in P_n$ and $p(z) \neq 0$ in $|z| \leq k$, $k \geq 1$, then for every real α and $R \geq r \geq 1$*

$$\max_{|z|=1} |p(Rz) - p(rz)| \leq \frac{R^n - r^n}{\sqrt{2(1+k^2)}} (M_\alpha^2 + M_{\alpha+\pi}^2)^{1/2}, \quad (20)$$

where M_α is defined in (7).

The following result also gives a generalization of inequality (19) by taking $k = 1$ in Corollary 11.

Corollary 14. *If $p \in P_n$ and $p(z) \neq 0$ in $|z| \leq k$, $k \geq 1$, then for every real α and $R \geq r \geq 1$*

$$\max_{|z|=1} |p(Rz) - p(rz)| \leq \frac{R^n - r^n}{2} (M_\alpha^2 + M_{\alpha+\pi}^2)^{1/2}, \quad (21)$$

where M_α is defined in (7).

If we consider a class of polynomials $p(z) = z^s(a_0 + \sum_{\nu=\mu}^{n-s} a_\nu z^\nu)$, $1 \leq \mu \leq n-s$, $0 \leq s \leq n-1$, and having s -fold zeros at origin and remaining $n-s$ zeros in $|z| \geq k$, $k \geq 1$, then we prove the following.

Theorem 15. *If $p(z) = z^s(a_0 + \sum_{\nu=\mu}^{n-s} a_\nu z^\nu)$, $1 \leq \mu \leq n-s$, $0 \leq s \leq n-1$ is a polynomial of degree n and having s -fold zeros at origin and remaining $n-s$ zeros in $|z| \geq k$, $k \geq 1$, then for every α*

$$\max_{|z|=1} |p'(z)| \leq s \max_{|z|=1} |p(z)| + \frac{(n-s)}{\sqrt{2(1+k^{2\mu})}} (M_\alpha^{*2} + M_{\alpha+\pi}^{*2})^{1/2}, \quad (22)$$

where

$$M_\alpha^* = \max_{1 \leq p \leq n-s} |p(e^{i(\alpha+2p\pi)/(n-s)})|. \quad (23)$$

Remark 16. If there are no zeros at origin, that is if $s = 0$, then inequality (22) is reducing to inequality (14) of Corollary 3.

As an application of inequality (22), we obtained the following results by considering $\mu = 1$ and $k = 1$, respectively, which are generalization of inequality (15) of Corollary 5 and inequality (8) due to Aziz [5] on s -fold zeros, respectively.

Corollary 17. *If $p \in P_n$ and having s -fold zeros at origin and remaining $n-s$ zeros in $|z| \geq k$, $k \geq 1$, then for every α*

$$\max_{|z|=1} |p'(z)| \leq s \max_{|z|=1} |p(z)| + \frac{(n-s)}{\sqrt{2(1+k^2)}} (M_\alpha^{*2} + M_{\alpha+\pi}^{*2})^{1/2}, \quad (24)$$

where M_α^* is defined in (23).

Corollary 18. *If $p \in P_n$, having s -fold zeros at origin and remaining $n-s$ zeros in $|z| \geq 1$, then for every α*

$$\max_{|z|=1} |p'(z)| \leq s \max_{|z|=1} |p(z)| + \frac{n}{2} (M_\alpha^{*2} + M_{\alpha+\pi}^{*2})^{1/2}, \quad (25)$$

where M_α^* is defined in (23).

Inequality (22) of Theorem 15 can be improved by including $m = \min_{|z|=k} |p(z)|$. In this connection, instead of proving Theorem 15, we prove the following more general result.

Theorem 19. *If $p(z) = z^s(a_0 + \sum_{\nu=\mu}^{n-s} a_\nu z^\nu)$, $1 \leq \mu \leq n-s$, $0 \leq s \leq n-1$ is a polynomial of degree n and having s -fold zeros at origin and remaining $n-s$ zeros in $|z| \geq k$, $k \geq 1$, then for every real α and $m = \min_{|z|=k} |p(z)|$*

$$\max_{|z|=1} |p'(z)| \leq s \max_{|z|=1} |p(z)| + \frac{(n-s)}{\sqrt{2(1+k^{2\mu})}} \left\{ M_\alpha^{*2} + M_{\alpha+\pi}^{*2} - 2 \frac{m^2}{k^{2s}} \right\}^{1/2}, \quad (26)$$

where M_α^* is defined in (23).

By taking $\mu = 1$ in inequality (26) of Theorem 15, we obtained the following.

Corollary 20. *If $p \in P_n$, having s -fold zeros at origin and remaining $n-s$ zeros in $|z| \geq k$, $k \geq 1$, then for every real α and $m = \min_{|z|=k} |p(z)|$*

$$\max_{|z|=1} |p'(z)| \leq s \max_{|z|=1} |p(z)| + \frac{(n-s)}{\sqrt{2(1+k^2)}} \left\{ M_\alpha^{*2} + M_{\alpha+\pi}^{*2} - 2 \frac{m^2}{k^{2s}} \right\}^{1/2}, \quad (27)$$

where M_α^* is defined in (22).

And for $k = 1$ inequality (26) becomes as follows.

Corollary 21. If $p \in P_n$, having s -fold zeros at origin and remaining $n - s$ zeros in $|z| \geq 1$, then for every real α and $m = \min_{|z|=1} |p(z)|$

$$\max_{|z|=1} |p'(z)| \leq s \max_{|z|=1} |p(z)| + \frac{(n-s)}{2} (M_\alpha^{*2} + M_{\alpha+\pi}^{*2} - 2m^2)^{1/2}, \quad (28)$$

where M_α^* is defined in (22).

Remark 22. Corollaries 20 and 21 are generalization of inequalities (11) and (9) for s -fold zeros of $p \in P_n$ and for $s = 0$, Theorem 19 gives inequality (12) of Theorem 1.

2. Lemmas

For the proof of above results, the following lemmas are required. The very first lemma is due to Aziz [5].

Lemma 23. If $p \in P_n$, then for $|z| = 1$ and for every real α

$$|p'(z)|^2 + |np(z) - zp'(z)|^2 \leq \frac{n^2}{2} (M_\alpha^2 + M_{\alpha+\pi}^2), \quad (29)$$

where M_α is defined the same as in (7).

Next lemma is due to Aziz and Rather [7].

Lemma 24. If $p \in P_n^\mu$, $|p'(z)| \neq 0$ in $|z| < k$, $k \geq 1$, and $q(z) = z^n p(1/\bar{z})$, then for $|z| = 1$ and $R > 1$

$$k^\mu |p(Rz) - p(z)| \leq |q(Rz) - q(z)| - (R^n - 1)m, \quad (30)$$

where $m = \min_{|z|=k} |p(z)|$.

3. Proof of the Theorems

Proof of Theorem 1. By the hypothesis $p(z) \neq 0$ in $|z| < k$, $k \geq 1$, dividing (30) by $R - 1$ and taking $R \rightarrow 1$ in Lemma 24 we have, for $|z| = 1$,

$$(k^\mu |p'(z)| + nm)^2 \leq |q'(z)|^2. \quad (31)$$

Since

$$\begin{aligned} (k^\mu |p'(z)| + nm)^2 &= k^{2\mu} |p'(z)|^2 + n^2 m^2 + 2nmk^{2\mu} |p(z)| \\ &\geq k^{2\mu} |p'(z)|^2 + n^2 m^2 \end{aligned} \quad (32)$$

and for $|z| = 1$, $|q'(z)| = |np(z) - zp'(z)|$; therefore inequality (31) gives

$$k^{2\mu} |p'(z)|^2 + n^2 m^2 \leq |np(z) - zp'(z)|^2 \quad (33)$$

and using Lemma 23 in inequality (33), we have

$$(1 + k^{2\mu}) |p'(z)|^2 + n^2 m^2 \leq \frac{n^2}{2} (M_\alpha^2 + M_{\alpha+\pi}^2) \quad (34)$$

which completes the proof of Theorem 1. $\square$

Proof of Theorem 6. Applying the result (2) to the polynomial $p'(z)$, which is of degree $n - 1$, and using (12), we obtained for $t \geq 1$ and $0 \leq \theta < 2\pi$

$$\begin{aligned} |p'(te^{i\theta})| &\leq t^{n-1} \max_{|z|=1} |p'(z)| \\ &\leq t^{n-1} \frac{n}{\sqrt{2(1+k^{2\mu})}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2}. \end{aligned} \quad (35)$$

Hence for each θ , $0 \leq \theta < 2\pi$ and $R \geq r \geq 1$, we have

$$\begin{aligned} |p(Re^{i\theta}) - p(re^{i\theta})| &= \left| \int_r^R e^{i\theta} p'(te^{i\theta}) dt \right| \\ &\leq \int_r^R |p'(te^{i\theta})| dt \end{aligned} \quad (36)$$

and, from inequality (35), we have

$$\begin{aligned} |p(Re^{i\theta}) - p(re^{i\theta})| &\leq \frac{1}{\sqrt{2(1+k^{2\mu})}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2} \int_r^R nt^{n-1} dt \\ &\leq \frac{R^n - r^n}{\sqrt{2(1+k^{2\mu})}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2} \end{aligned} \quad (37)$$

which implies that, for $|z| = 1$ and $R \geq r \geq 1$,

$$|p(Rz) - p(rz)| \leq \frac{R^n - r^n}{\sqrt{2(1+k^{2\mu})}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2}. \quad (38)$$

Theorem 6 is completed. $\square$

Proof of Theorem 19. Let

$$p(z) = z^s \phi(z), \quad (39)$$

where $\phi(z) = a_0 + \sum_{\nu=\mu}^{n-s} a_\nu z^\nu$, $1 \leq \mu \leq n - s$, is a polynomial of degree $n - s$, having no zero in $|z| < k$, $k \geq 1$.

From (39) we have

$$\begin{aligned} zp'(z) &= sz^s \phi(z) + z^{s+1} \phi'(z) \\ &= sp(z) + z^{s+1} \phi'(z) \end{aligned} \quad (40)$$

and for $|z| = 1$,

$$|p'(z)| \leq s |p(z)| + |\phi'(z)| \quad (41)$$

and the above inequality holds for all point on $|z| = 1$.

Since $\phi(z)$ is polynomial of degree $n - s$ and from (39) it can easily be obtained that for $|z| = 1$, $|p(z)| = |\phi(z)|$ and

$$\min_{|z|=k} |\phi(z)| = \frac{1}{k^s} \min_{|z|=k} |p(z)| = \frac{m}{k^s} \quad (42)$$

and then from Theorem 15 (inequality (18)), we have

$$\max_{|z|=1} |\phi'(z)| \leq \frac{n-s}{\sqrt{2(1+k^{2\mu})}} \left\{ M_{\alpha}^{*2} + M_{\alpha+\pi}^{*2} - \frac{2m^2}{k^{2s}} \right\}^{1/2} \quad (43)$$

on combining inequality (41) and (43); the proof of Theorem 19 is completed. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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