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*Research Article*

# Hermite-Hadamard and Simpson-Like Type Inequalities for Differentiable Harmonically Convex Functions

**İmdat İşcan**

*Department of Mathematics, Faculty of Arts and Sciences, Giresun University, 28100 Giresun, Turkey*

Correspondence should be addressed to İmdat İşcan; imdat.iscan@giresun.edu.tr

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A new identity for differentiable functions is derived. A consequence of the identity is that the author establishes some new general inequalities containing all of the Hermite-Hadamard and Simpson-like types for functions whose derivatives in absolute value at certain power are harmonically convex. Some applications to special means of real numbers are also given.

## 1. Introduction

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. Note that some of the classical inequalities for means can be derived from (1) for appropriate particular selections of the mapping $f$. Both inequalities hold in the reversed direction if $f$ is concave.

The following inequality is well known in the literature as Simpson inequality.

**Theorem 1.** *Let $f : [a,b] \to \mathbb{R}$ be a four times continuously differentiable mapping on $(a,b)$ and $\|f^{(4)}\|_\infty = \sup_{x\in(a,b)}|f^{(4)}(x)| < \infty$. Then the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4. \tag{2}$$

For some results which generalize, improve, and extend the Hermite-Hadamard and Simpson inequalities, one refers the reader to the recent papers (see [1–8]).

In [9], the author introduced the concept of harmonically convex functions and established some results connected with the right-hand side of new inequalities similar to inequality (1) for these classes of functions. Some applications to special means of positive real numbers were also given.

*Definition 2.* Let $I \subset \mathbb{R} \setminus \{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y) + (1-t)f(x) \tag{3}$$

for all $x, y \in I$ and $t \in [0,1]$. If inequality in (3) is reversed, then $f$ is said to be harmonically concave.

The following result of the Hermite-Hadamard type holds.

**Theorem 3.** *Let $f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$ then the following inequalities hold*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}. \tag{4}$$

*The above inequalities are sharp.*

Some results connected with the right part of (4) were given in [9] as follows.

**Theorem 4.** *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on $I^\circ$, $a,b \in I$ with $a < b$, and $f' \in L[a,b]$. If $|f'|^q$ is harmonically convex on $[a,b]$ for $q \geq 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)}{2}\lambda_1^{1-1/q}\left[\lambda_2\left|f'(a)\right|^q + \lambda_3\left|f'(b)\right|^q\right]^{1/q}, \tag{5}$$

*where*

$$\begin{aligned}\lambda_1 &= \frac{1}{ab} - \frac{2}{(b-a)^2}\ln\left(\frac{(a+b)^2}{4ab}\right),\\ \lambda_2 &= \frac{-1}{b(b-a)} + \frac{3a+b}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right),\\ \lambda_3 &= \frac{1}{a(b-a)} - \frac{3b+a}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right)\\ &= \lambda_1 - \lambda_2.\end{aligned} \tag{6}$$

**Theorem 5.** *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on $I^\circ$, $a,b \in I$ with $a < b$, and $f' \in L[a,b]$. If $|f'|^q$ is harmonically convex on $[a,b]$ for $q > 1$, $1/p + 1/q = 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{1/p}\left(\mu_1\left|f'(a)\right|^q + \mu_2\left|f'(b)\right|^q\right)^{1/q}, \tag{7}$$

*where*

$$\begin{aligned}\mu_1 &= \frac{\left[a^{2-2q} + b^{1-2q}\left[(b-a)(1-2q) - a\right]\right]}{2(b-a)^2(1-q)(1-2q)},\\ \mu_2 &= \frac{\left[b^{2-2q} - a^{1-2q}\left[(b-a)(1-2q) + b\right]\right]}{2(b-a)^2(1-q)(1-2q)}.\end{aligned} \tag{8}$$

In this paper, one gives some general integral inequalities connected with the left and right parts of (4); as a result of this, one obtains some new midpoint, trapezoid, and Simpson-like type inequalities for differentiable harmonically convex functions.

## 2. Main Results

In order to prove our main results we need the following lemma.

**Lemma 6.** *Let $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a differentiable function on $I^\circ$ and $a,b \in I$ with $a < b$. If $f' \in L[a,b]$ then for $\lambda \in [0,1]$ one has the equality*

$$\begin{aligned}&(1-\lambda)f\left(\frac{2ab}{a+b}\right) + \lambda\left(\frac{f(a)+f(b)}{2}\right) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\\ &= \frac{ab(b-a)}{2}\left[\int_0^{1/2}\frac{\lambda-2t}{A_t^2}f'\left(\frac{ab}{A_t}\right)dt + \int_{1/2}^1 \frac{2-\lambda-2t}{A_t^2}f'\left(\frac{ab}{A_t}\right)dt\right],\end{aligned} \tag{9}$$

*where $A_t = tb + (1-t)a$.*

*Proof.* It suffices to note that

$$\begin{aligned}I_1 &= ab(b-a)\int_0^{1/2}\frac{\lambda-2t}{A_t^2}f'\left(\frac{ab}{A_t}\right)dt\\ &= (2t-\lambda)f\left(\frac{ab}{A_t}\right)\Big|_0^{1/2} - 2\int_0^{1/2} f\left(\frac{ab}{A_t}\right)dt\\ &= (1-\lambda)f\left(\frac{2ab}{a+b}\right) + \lambda f(b) - 2\int_0^{1/2} f\left(\frac{ab}{A_t}\right)dt.\end{aligned} \tag{10}$$

Set $x = ab/A_t$ and $dx = (-ab(b-a)/A_t^2)dt$, which gives

$$I_1 = (1-\lambda)f\left(\frac{2ab}{a+b}\right) + \lambda f(b) - \frac{2ab}{b-a}\int_{2ab/(a+b)}^b \frac{f(x)}{x^2}dx. \tag{11}$$

Similarly, we can show that

$$\begin{aligned}I_2 &= ab(b-a)\int_{1/2}^1 \frac{2-\lambda-2t}{A_t^2}f'\left(\frac{ab}{A_t}\right)dt\\ &= \lambda f(a) + (1-\lambda)f\left(\frac{2ab}{a+b}\right)\\ &\quad - \frac{2ab}{b-a}\int_a^{2ab/(a+b)}\frac{f(x)}{x^2}dx.\end{aligned} \tag{12}$$

Thus,

$$\begin{aligned}\frac{I_1+I_2}{2} &= (1-\lambda)f\left(\frac{2ab}{a+b}\right) + \lambda\left(\frac{f(a)+f(b)}{2}\right)\\ &\quad - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\end{aligned} \tag{13}$$

which is required. □

**Theorem 7.** *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on $I^\circ$, $a,b \in I$ with $a < b$, and $f' \in L[a,b]$. If $|f'|^q$ is*

*harmonically convex on* $[a,b]$ *for* $q \geq 1$ *and then one has the following inequality for* $\lambda \in [0,1]$:

$$\begin{aligned}
&\left|(1-\lambda) f\left(\frac{2ab}{a+b}\right)+\lambda\left(\frac{f(a)+f(b)}{2}\right)\right. \\
&\quad \left.-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \\
&\quad \leq \frac{ab(b-a)}{2} \\
&\qquad \times\left\{C_1^{1-1/q}(\lambda;a,b)\right. \\
&\qquad \times\left[C_2(\lambda;a,b)\left|f'(a)\right|^q+C_3(\lambda;a,b)\left|f'(b)\right|^q\right]^{1/q} \\
&\qquad +C_1^{1-1/q}(\lambda;b,a) \\
&\qquad \left.\times\left[C_3(\lambda;b,a)\left|f'(a)\right|^q+C_2(\lambda;b,a)\left|f'(b)\right|^q\right]^{1/q}\right\},
\end{aligned} \tag{14}$$

*where*

$$\begin{aligned}
C_1(\lambda;u,\vartheta) &= \frac{1}{(\vartheta-u)^2} \\
&\quad \times\left[-4+\frac{[\lambda(\vartheta-u)+2u](3u+\vartheta)}{u(u+\vartheta)}\right. \\
&\quad \left.+2\ln\left(\frac{2u(u+\vartheta)}{(2u+\lambda(\vartheta-u))^2}\right)\right], \\
C_2(\lambda;u,\vartheta) &= \frac{1}{(\vartheta-u)^3} \\
&\quad \times\left\{[\lambda(\vartheta-u)+4u]\ln\left(\frac{[\lambda(\vartheta-u)+2u]^2}{2u(u+\vartheta)}\right)\right. \\
&\quad \left.-\frac{[\lambda(\vartheta-u)+2u](5u+3\vartheta)}{u+\vartheta}+7u+\vartheta\right\}, \\
C_3(\lambda;u,\vartheta) &= C_1(\lambda;u,\vartheta)-C_2(\lambda;u,\vartheta), \quad u,\vartheta>0.
\end{aligned} \tag{15}$$

*Proof.* Let $A_t = tb + (1-t)a$. From Lemma 6 and using the Hölder inequality, we have

$$\begin{aligned}
&\left|(1-\lambda) f\left(\frac{2ab}{a+b}\right)+\lambda\left(\frac{f(a)+f(b)}{2}\right)\right. \\
&\quad \left.-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \\
&\quad \leq \frac{ab(b-a)}{2}\left[\int_0^{1/2}\frac{|\lambda-2t|}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|dt\right. \\
&\qquad \left.+\int_{1/2}^1\frac{|2-\lambda-2t|}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|dt\right] \\
&\quad \leq \frac{ab(b-a)}{2} \\
&\qquad \times\left\{\left(\int_0^{1/2}\frac{|\lambda-2t|}{A_t^2}dt\right)^{1-1/q}\right. \\
&\qquad \times\left(\int_0^{1/2}\frac{|\lambda-2t|}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{1/q} \\
&\qquad +\left(\int_{1/2}^1\frac{|2-\lambda-2t|}{A_t^2}dt\right)^{1-1/q} \\
&\qquad \left.\times\left(\int_{1/2}^1\frac{|2-\lambda-2t|}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{1/q}\right\}.
\end{aligned} \tag{16}$$

Hence, by harmonically convexity of $|f'|^q$ on $[a,b]$, we have

$$\begin{aligned}
&\left|(1-\lambda) f\left(\frac{2ab}{a+b}\right)+\lambda\left(\frac{f(a)+f(b)}{2}\right)\right. \\
&\quad \left.-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \\
&\quad \leq \frac{ab(b-a)}{2} \\
&\qquad \times\left\{\left(\int_0^{1/2}\frac{|\lambda-2t|}{A_t^2}dt\right)^{1-1/q}\right. \\
&\qquad \times\left(\int_0^{1/2}\left(|\lambda-2t|\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]\right)\right. \\
&\qquad \left.\times\left(A_t^2\right)^{-1}dt\right)^{1/q} \\
&\qquad +\left(\int_{1/2}^1\frac{|2-\lambda-2t|}{A_t^2}dt\right)^{1-1/q} \\
&\qquad \times\left(\int_{1/2}^1\left(|2-\lambda-2t|\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]\right)\right. \\
&\qquad \left.\left.\times\left(A_t^2\right)^{-1}dt\right)^{1/q}\right\} \\
&\quad \leq \frac{ab(b-a)}{2}C_1^{1-1/q}(\lambda;a,b) \\
&\qquad \times\left\{\left[C_2(\lambda;a,b)\left|f'(a)\right|^q+C_3(\lambda;a,b)\left|f'(b)\right|^q\right]^{1/q}\right. \\
&\qquad \left.+\left[C_3(\lambda;b,a)\left|f'(a)\right|^q+C_2(\lambda;b,a)\left|f'(b)\right|^q\right]^{1/q}\right\}.
\end{aligned} \tag{17}$$

It is easily to check that

$$\begin{aligned}
&\int_0^{1/2} \frac{|\lambda-2t|}{A_t^2}dt \\
&\quad = C_1(\lambda;a,b) \\
&\quad = \frac{1}{(b-a)^2}\times\left[-4+\frac{[\lambda(b-a)+2a](3a+b)}{a(a+b)}\right. \\
&\qquad\qquad \left.+2\ln\left(\frac{2a(a+b)}{(2a+\lambda(b-a))^2}\right)\right], \\
&\int_0^{1/2} \frac{|\lambda-2t|\,t}{A_t^2}dt \\
&\quad = C_2(\lambda;a,b) \\
&\quad = \frac{1}{(b-a)^3} \\
&\qquad \times\left\{[\lambda(b-a)+4a]\ln\left(\frac{[\lambda(b-a)+2a]^2}{2a(a+b)}\right)\right. \\
&\qquad\qquad \left.-\frac{[\lambda(b-a)+2a](5a+3b)}{a+b}+7a+b\right\}, \\
&\int_0^{1/2} \frac{|\lambda-2t|(1-t)}{A_t^2}dt \\
&\quad = C_3(\lambda;a,b) = C_1(\lambda;a,b)-C_2(\lambda;a,b).
\end{aligned} \tag{18}$$

This concludes the proof. □

**Corollary 8.** *Under the assumptions of Theorem 7 with $\lambda = 0$, one has*

$$\begin{aligned}
&\left|f\left(\frac{2ab}{a+b}\right)-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \\
&\le \frac{ab(b-a)}{2} \\
&\quad \times\left\{C_1^{1-1/q}(0;a,b)\right. \\
&\qquad \times\left[C_2(0;a,b)\left|f'(a)\right|^q+C_3(0;a,b)\left|f'(b)\right|^q\right]^{1/q} \\
&\qquad +C_1^{1-1/q}(0;b,a) \\
&\qquad \left.\times\left[C_3(0;b,a)\left|f'(a)\right|^q+C_2(0;b,a)\left|f'(b)\right|^q\right]^{1/q}\right\},
\end{aligned} \tag{19}$$

*where*

$$\begin{aligned}
C_1(0;u,\vartheta) &= \frac{2}{(\vartheta-u)^2}\left[\ln\left(\frac{u+\vartheta}{2u}\right)-\frac{\vartheta-u}{u+\vartheta}\right], \\
C_2(0;u,\vartheta) &= \frac{1}{(\vartheta-u)^3}\left[\frac{(3u+\vartheta)(\vartheta-u)}{u+\vartheta}+4u\ln\left(\frac{2u}{u+\vartheta}\right)\right], \\
C_3(0;u,\vartheta) &= \frac{1}{(\vartheta-u)^2} \\
&\quad \times\left[\frac{2(u+\vartheta)}{\vartheta-u}\ln\left(\frac{u+\vartheta}{2u}\right)-\frac{u+3\vartheta}{u+\vartheta}\right], \\
&\qquad u,\vartheta>0.
\end{aligned} \tag{20}$$

**Corollary 9.** *Under the assumptions of Theorem 7 with $\lambda = 1$, one has*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \\
&\le \frac{ab(b-a)}{2} \\
&\quad \times\left\{C_1^{1-1/q}(1;a,b)\right. \\
&\qquad \times\left[C_2(1;a,b)\left|f'(a)\right|^q+C_3(1;a,b)\left|f'(b)\right|^q\right]^{1/q} \\
&\qquad +C_1^{1-1/q}(1;b,a) \\
&\qquad \left.\times\left[C_3(1;b,a)\left|f'(a)\right|^q+C_2(1;b,a)\left|f'(b)\right|^q\right]^{1/q}\right\},
\end{aligned} \tag{21}$$

*where*

$$\begin{aligned}
C_1(1;u,\vartheta) &= \frac{1}{(\vartheta-u)^2}\left[\frac{\vartheta-u}{u}+2\ln\left(\frac{2u}{u+\vartheta}\right)\right], \\
C_2(1;u,\vartheta) &= \frac{1}{(\vartheta-u)^3}\left[(3u+\vartheta)\ln\left(\frac{u+\vartheta}{2u}\right)-2(\vartheta-u)\right], \\
C_3(1;u,\vartheta) &= \frac{1}{(\vartheta-u)^2}\left[\frac{u+\vartheta}{u}-\frac{u+3\vartheta}{\vartheta-u}\ln\left(\frac{u+\vartheta}{2u}\right)\right], \\
&\qquad u,\vartheta>0.
\end{aligned} \tag{22}$$

**Corollary 10.** *Under the assumptions of Theorem 7 with $\lambda = 1/3$, one has*

$$\begin{aligned}
&\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{2ab}{a+b}\right)\right]-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \\
&\le \frac{ab(b-a)}{2} \\
&\quad \times\left\{C_1^{1-1/q}\left(\frac{1}{3};a,b\right)\right. \\
&\qquad \times\left[C_2\left(\frac{1}{3};a,b\right)\left|f'(a)\right|^q+C_3\left(\frac{1}{3};a,b\right)\left|f'(b)\right|^q\right]^{1/q} \\
&\qquad +C_1^{1-1/q}\left(\frac{1}{3};b,a\right) \\
&\qquad \left.\times\left[C_3\left(\frac{1}{3};b,a\right)\left|f'(a)\right|^q+C_2\left(\frac{1}{3};b,a\right)\left|f'(b)\right|^q\right]^{1/q}\right\},
\end{aligned} \tag{23}$$

*where*

$$C_1\left(\frac{1}{3};u,\vartheta\right) = \frac{1}{(\vartheta-u)^2}\left[\frac{(\vartheta-u)(\vartheta-3u)}{3u(u+\vartheta)}+2\ln\left(\frac{18u(u+\vartheta)}{(5u+\vartheta)^2}\right)\right],$$

$$C_2\left(\frac{1}{3};u,\vartheta\right) = \frac{1}{(\vartheta-u)^3}\left[\left(\frac{11u+\vartheta}{3}\right)\ln\left(\frac{(5u+\vartheta)^2}{18u(u+\vartheta)}\right)+\frac{4u(\vartheta-u)}{3(u+\vartheta)}\right],$$

$$C_3\left(\frac{1}{3};u,\vartheta\right) = \frac{1}{(\vartheta-u)^2}\left[\frac{\vartheta^2-4u\vartheta-u^2}{3u(u+\vartheta)}+\frac{5u+7\vartheta}{3(\vartheta-u)}\ln\left(\frac{18u(u+\vartheta)}{(5u+\vartheta)^2}\right)\right],$$

$$u,\vartheta>0. \tag{24}$$

**Theorem 11.** *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on $I^\circ$, $a,b \in I$ with $a<b$, and $f' \in L[a,b]$. If $|f'|^q$ is harmonically convex on $[a,b]$ for $q>1$ and then one has the following inequality for $\lambda \in [0,1]$ :*

$$\begin{aligned}&\left|(1-\lambda)f\left(\frac{2ab}{a+b}\right)+\lambda\left(\frac{f(a)+f(b)}{2}\right)-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\\ &\quad\le\frac{ab(b-a)}{4}\times\frac{C_4^{1/p}(\lambda,p)}{\left[(1-q)(1-2q)(b-a)^2\right]^{1/q}}\\ &\quad\times\left\{\left(C_5(q;a,b)\left|f'(a)\right|^q+C_6(q;a,b)\left|f'(b)\right|^q\right)^{1/q}\right.\\ &\quad\left.+\left(C_6(q;b,a)\left|f'(a)\right|^q+C_5(q;b,a)\left|f'(b)\right|^q\right)^{1/q}\right\},\end{aligned} \tag{25}$$

*where*

$$C_4(\lambda,p)=\frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{p+1},$$

$$C_5(q;u,\vartheta)=\left[\left(\frac{u+\vartheta}{2}\right)^{1-2q}\left[\frac{\vartheta-3u}{2}-q(\vartheta-u)\right]+u^{2-2q}\right],$$

$$\begin{aligned}C_6(q;u,\vartheta)=&\left[\left(\frac{u+\vartheta}{2}\right)^{1-2q}\left[\frac{3\vartheta-u}{2}-q(\vartheta-u)\right]\right.\\ &\left.+u^{1-2q}\left[u-2\vartheta+2q(\vartheta-u)\right]\right],\\ &u,\vartheta>0\end{aligned} \tag{26}$$

*and $1/p+1/q=1$.*

*Proof.* Let $A_t = tb+(1-t)a$. Using Lemma 6 and Hölder's integral inequality, we deduce

$$\begin{aligned}&\left|(1-\lambda)f\left(\frac{2ab}{a+b}\right)+\lambda\left(\frac{f(a)+f(b)}{2}\right)-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\\ &\quad\le\frac{ab(b-a)}{2}\left[\int_0^{1/2}\frac{|\lambda-2t|}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|dt+\int_{1/2}^1\frac{|2-\lambda-2t|}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|dt\right]\\ &\quad\le\frac{ab(b-a)}{2}\times\left\{\left(\int_0^{1/2}|\lambda-2t|^p dt\right)^{1/p}\times\left(\int_0^{1/2}\frac{1}{A_t^{2q}}\left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{1/q}\right.\\ &\quad\left.+\left(\int_{1/2}^1|2-\lambda-2t|^p dt\right)^{1/p}\times\left(\int_{1/2}^1\frac{1}{A_t^{2q}}\left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{1/q}\right\}.\end{aligned} \tag{27}$$

Using the harmonically convexity of $|f'|^q$, we obtain

$$\begin{aligned}&\int_0^{1/2}\frac{1}{A_t^{2q}}\left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\\ &\le\int_0^{1/2}\frac{t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q}{A_t^{2q}}dt\\ &=\frac{1}{2(1-q)(1-2q)(b-a)^2}\\ &\quad\times\left\{\left[\left(\frac{a+b}{2}\right)^{1-2q}\left[\frac{b-3a}{2}-q(b-a)\right]+a^{2-2q}\right]\left|f'(a)\right|^q\right.\\ &\quad\left.+\left[\left(\frac{a+b}{2}\right)^{1-2q}\left[\frac{3b-a}{2}-q(b-a)\right]+a^{1-2q}\left[a-2b+2q(b-a)\right]\right]\left|f'(b)\right|^q\right\},\end{aligned}$$

$$\int_0^{1/2} \frac{1}{A_t^{2q}} \left| f'\left(\frac{ab}{A_t}\right)\right|^q dt$$
$$\leq \int_{1/2}^1 \frac{t\left|f'(a)\right|^q + (1-t)\left|f'(b)\right|^q}{A_t^{2q}} dt$$
$$= \frac{1}{2(1-q)(1-2q)(b-a)^2}$$
$$\times \left\{\left[b^{1-2q}\left[b-2a-2q(b-a)\right] + \left(\frac{a+b}{2}\right)^{1-2q}\left[\frac{3a-b}{2}+q(b-a)\right]\right]\left|f'(a)\right|^q + \left[\left(\frac{a+b}{2}\right)^{1-2q}\left[\frac{a-3b}{2}+q(b-a)\right]+b^{2-2q}\right] \times \left|f'(b)\right|^q\right\}. \tag{28}$$

Further, we have

$$\int_0^{1/2} |\lambda - 2t|^p dt = \int_{1/2}^1 |2-\lambda-2t|^p dt = \frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{2(p+1)}. \tag{29}$$

A combination of (27)–(29) gives the required inequality (25). □

**Corollary 12.** *Under the assumptions of Theorem 11 with $\lambda = 0$, one has*

$$\left| f\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)}{4(p+1)^{1/p}} \times \frac{1}{\left[(1-q)(1-2q)(b-a)^2\right]^{1/q}} \times \left\{\left(C_5(q;a,b)\left|f'(a)\right|^q + C_6(q;a,b)\left|f'(b)\right|^q\right)^{1/q} + \left(C_6(q;b,a)\left|f'(a)\right|^q + C_5(q;b,a)\left|f'(b)\right|^q\right)^{1/q}\right\}. \tag{30}$$

**Corollary 13.** *Under the assumptions of Theorem 11 with $\lambda = 1$, one has*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)}{4(p+1)^{1/p}} \times \frac{1}{\left[(1-q)(1-2q)(b-a)^2\right]^{1/q}} \times \left\{\left(C_5(q;a,b)\left|f'(a)\right|^q + C_6(q;a,b)\left|f'(b)\right|^q\right)^{1/q} + \left(C_6(q;b,a)\left|f'(a)\right|^q + C_5(q;b,a)\left|f'(b)\right|^q\right)^{1/q}\right\}. \tag{31}$$

**Corollary 14.** *Under the assumptions of Theorem 11 with $\lambda = 1/3$, one has*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{2ab}{a+b}\right)\right] - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)}{4\left(3^{p+1}(p+1)\right)^{1/p}} \times \frac{1+2^{p+1}}{\left[(1-q)(1-2q)(b-a)^2\right]^{1/q}} \times \left\{\left(C_5(q;a,b)\left|f'(a)\right|^q + C_6(q;a,b)\left|f'(b)\right|^q\right)^{1/q} + \left(C_6(q;b,a)\left|f'(a)\right|^q + C_5(q;b,a)\left|f'(b)\right|^q\right)^{1/q}\right\}. \tag{32}$$

**Theorem 15.** *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on $I^\circ$, $a, b \in I$ with $a < b$, and $f' \in L[a,b]$. If $|f'|^q$ is harmonically convex on $[a,b]$ for $q > 1$ and then one has the following inequality for $\lambda \in [0,1]$:*

$$\left|(1-\lambda) f\left(\frac{2ab}{a+b}\right) + \lambda\left(\frac{f(a)+f(b)}{2}\right) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)^{1-1/p}}{2(2p-1)^{1/p}} \times \left[\frac{1}{4(q+1)(q+2)}\right]^{1/q} \times \left\{C_7^{1/p}(a,b;p) \times \left[C_9(\lambda,q)\left|f'(a)\right|^q + C_{10}(\lambda,q)\left|f'(b)\right|^q\right]^{1/q} + C_8^{1/p}(a,b;p) \times \left[C_{10}(\lambda,q)\left|f'(a)\right|^q + C_9(\lambda,q)\left|f'(b)\right|^q\right]^{1/q}\right\}, \tag{33}$$

*where*

$$\begin{aligned} C_7(a,b;p) &= a^{1-2p} - \left(\frac{a+b}{2}\right)^{1-2p}, \\ C_8(a,b;p) &= \left(\frac{a+b}{2}\right)^{1-2p} - b^{1-2p}, \\ C_9(\lambda,q) &= \lambda^{q+2} + (1-\lambda)^{q+1}(q+1+\lambda), \\ C_{10}(\lambda,q) &= \lambda^{q+1}(4+2q-\lambda) \\ &\quad + (1-\lambda)^{q+1}(3+q-\lambda), \end{aligned} \tag{34}$$

*and* $1/p + 1/q = 1$.

*Proof.* Let $A_t = tb + (1-t)a$. Using Lemma 6 and Hölder's integral inequality, we deduce

$$\begin{aligned} &\left|(1-\lambda) f\left(\frac{2ab}{a+b}\right) + \lambda\left(\frac{f(a)+f(b)}{2}\right) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \\ &\quad \le \frac{ab(b-a)}{2} \times \left\{\left(\int_0^{1/2} \frac{1}{A_t^{2p}}dt\right)^{1/p} \times \left(\int_0^{1/2} |\lambda-2t|^q \left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{1/q} \right. \\ &\quad \left. + \left(\int_{1/2}^1 \frac{1}{A_t^{2p}}dt\right)^{1/p} \times \left(\int_{1/2}^1 |2-\lambda-2t|^q \left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{1/q}\right\}. \end{aligned} \tag{35}$$

Using the harmonically convexity of $|f'|^q$, we obtain

$$\begin{aligned} &\int_0^{1/2} |\lambda-2t|^q \left|f'\left(\frac{ab}{A_t}\right)\right|^q dt \\ &\quad \le \int_0^{1/2} |\lambda-2t|^q \left[t\left|f'(a)\right|^q + (1-t)\left|f'(b)\right|^q\right] dt \\ &\quad = \frac{1}{4(q+1)(q+2)} \times \left\{\left[\lambda^{q+2} + (1-\lambda)^{q+1}(q+1+\lambda)\right]\left|f'(a)\right|^q \right. \\ &\qquad + \left[\lambda^{q+1}(4+2q-\lambda) + (1-\lambda)^{q+1}(3+q-\lambda)\right] \times \left|f'(b)\right|^q\Big\}, \\ &\int_{1/2}^1 |2-\lambda-2t|^q \left|f'\left(\frac{ab}{A_t}\right)\right|^q dt \\ &\quad \le \int_{1/2}^1 |2-\lambda-2t|^q \left[t\left|f'(a)\right|^q + (1-t)\left|f'(b)\right|^q\right] dt \\ &\quad = \frac{1}{4(q+1)(q+2)} \times \left\{\left[\lambda^{q+1}(4+2q-\lambda) + (1-\lambda)^{q+1}(3+q-\lambda)\right] \times \left|f'(a)\right|^q \right. \\ &\qquad \left. + \left[\lambda^{q+2} + (1-\lambda)^{q+1}(q+1+\lambda)\right]\left|f'(b)\right|^q\right\}. \end{aligned} \tag{36}$$

Further, we have

$$\begin{aligned} \int_0^{1/2} \frac{1}{A_t^{2p}}dt &= \frac{1}{(b-a)(2p-1)}\left[a^{1-2p} - \left(\frac{a+b}{2}\right)^{1-2p}\right], \\ \int_{1/2}^1 \frac{1}{A_t^{2p}}dt &= \frac{1}{(b-a)(2p-1)}\left[\left(\frac{a+b}{2}\right)^{1-2p} - b^{1-2p}\right]. \end{aligned} \tag{37}$$

A combination of (35)–(37) gives the required inequality (33). □

**Corollary 16.** *Under the assumptions of Theorem 15 with* $\lambda = 0$, *one has*

$$\begin{aligned} &\left|f\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \\ &\quad \le \frac{ab(b-a)^{1-1/p}}{2(2p-1)^{1/p}} \times \left[\frac{1}{4(q+1)(q+2)}\right]^{1/q} \\ &\quad \times \left\{C_7^{1/p}(a,b;p)\left[(q+1)\left|f'(a)\right|^q + (q+3)\left|f'(b)\right|^q\right]^{1/q} \right. \\ &\qquad + C_8^{1/p}(a,b;p) \times \left[(q+3)\left|f'(a)\right|^q + (q+1)\left|f'(b)\right|^q\right]^{1/q}\Big\}. \end{aligned} \tag{38}$$

**Corollary 17.** *Under the assumptions of Theorem 15 with* $\lambda = 1$, *one has*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \le \frac{ab(b-a)^{1-1/p}}{2(2p-1)^{1/p}}$$

$$\times \left[\frac{1}{4(q+1)(q+2)}\right]^{1/q}$$
$$\times \left\{C_7^{1/p}(a,b;p)\left[\left|f'(a)\right|^q + (2q+3)\left|f'(b)\right|^q\right]^{1/q}\right.$$
$$\left. + C_8^{1/p}(a,b;p)\left[(2q+3)\left|f'(a)\right|^q + \left|f'(b)\right|^q\right]^{1/q}\right\}. \tag{39}$$

**Corollary 18.** *Under the assumptions of Theorem 15 with $\lambda = 1/3$, one has*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{2ab}{a+b}\right)\right] - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right|$$
$$\leq \frac{ab(b-a)^{1-1/p}}{2(2p-1)^{1/p}}$$
$$\times \left[\frac{1}{4(q+1)(q+2)}\right]^{1/q}$$
$$\times \left\{C_7^{1/p}(a,b;p)\right.$$
$$\times \left[C_9\left(\frac{1}{3},q\right)\left|f'(a)\right|^q + C_{10}\left(\frac{1}{3},q\right)\left|f'(b)\right|^q\right]^{1/q}$$
$$+ C_8^{1/p}(a,b;p)\left[C_{10}\left(\frac{1}{3},q\right)\left|f'(a)\right|^q\right.$$
$$\left.\left. + C_9\left(\frac{1}{3},q\right)\left|f'(b)\right|^q\right]^{1/q}\right\}, \tag{40}$$

*where*

$$C_9\left(\frac{1}{3},q\right) = \frac{1}{3^{q+2}}\left(1 + 2^{q+1}(3q+4)\right),$$
$$C_{10}\left(\frac{1}{3},q\right) = \frac{1}{3^{q+2}}\left(11 + 6q + 2^{q+1}(8+3q)\right). \tag{41}$$

## 3. Some Applications for Special Means

Let us recall the following special means of two nonnegative number $a, b$ with $b > a$.

(1) The arithmetic mean

$$A = A(a,b) := \frac{a+b}{2}. \tag{42}$$

(2) The geometric mean

$$G = G(a,b) := \sqrt{ab}. \tag{43}$$

(3) The harmonic mean

$$H = H(a,b) := \frac{2ab}{a+b}. \tag{44}$$

(4) The logarithmic mean

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a}. \tag{45}$$

(5) The $p$-logarithmic mean

$$L_p = L_p(a,b) := \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{1/p}, \quad p \in \mathbb{R} \setminus \{-1, 0\}. \tag{46}$$

(6) The identric mean

$$I = I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}. \tag{47}$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature:

$$H \leq G \leq L \leq I \leq A. \tag{48}$$

It is also known that $L_p$ is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

**Proposition 19.** *Let $0 < a < b$ and $\lambda \in [0,1]$. Then one has the following inequality:*

$$\left|(1-\lambda)H + \lambda A - \frac{G^2}{L}\right| \leq \frac{ab(b-a)}{2}\left\{C_1(\lambda;a,b) + C_1(\lambda;b,a)\right\}, \tag{49}$$

*where $C_1$ is defined as in Theorem 7.*

*Proof.* The assertion follows from inequality (14) in Theorem 7, for $f : (0,\infty) \to \mathbb{R}$, $f(x) = x$. □

**Proposition 20.** *Let $0 < a < b$ and $\lambda \in [0,1]$. Then one has the following inequality:*

$$\left|(1-\lambda)H + \lambda A - \frac{G^2}{L}\right| \leq \frac{ab(b-a)C_4^{1/p}(\lambda,p)}{4\left[(1-q)(1-2q)(b-a)^2\right]^{1/q}}$$
$$\times \left\{\left(C_5(q;a,b) + C_6(q;a,b)\right)^{1/q}\right.$$
$$\left. + \left(C_6(q;b,a) + C_5(q;b,a)\right)^{1/q}\right\}, \tag{50}$$

*where $q > 1$, $1/p + 1/q = 1$, and $C_4$, $C_5$, and $C_6$ are defined as in Theorem 11.*

*Proof.* The assertion follows from inequality (25) in Theorem 11, for $f : (0,\infty) \to \mathbb{R}$, $f(x) = x$. □

**Proposition 21.** *Let $0 < a < b$ and $\lambda \in [0,1]$. Then one has the following inequality:*

$$\left|(1-\lambda)H + \lambda A - \frac{G^2}{L}\right| \le \frac{ab(b-a)^{1-1/p}}{2(2p-1)^{1/p}} \times \left[\frac{\lambda^{q+1} + (1-\lambda)^{q+1}}{2(q+1)}\right]^{1/q} \times \left[C_7^{1/p}(a,b;p) + C_8^{1/p}(a,b;p)\right], \tag{51}$$

*where $q > 1$, $1/p + 1/q = 1$, and $C_7$ and $C_8$ are defined as in Theorem 15.*

*Proof.* The assertion follows from inequality (33) in Theorem 15, for $f : (0,\infty) \to \mathbb{R}$, $f(x) = x$. □

**Proposition 22.** *Let $0 < a < b$, $\lambda \in [0,1]$ and $q \ge 1$. Then one has the following inequality:*

$$\begin{aligned}&\left|(1-\lambda)H^2 + \lambda A\left(a^2,b^2\right) - G^2\right| \\ &\le ab(b-a) \\ &\times \left\{C_1^{1-1/q}(\lambda;a,b)\left[C_2(\lambda;a,b)a^q + C_3(\lambda;a,b)b^q\right]^{1/q}\right. \\ &+ C_1^{1-1/q}(\lambda;b,a) \\ &\left.\times\left[C_3(\lambda;b,a)a^q + C_2(\lambda;b,a)b^q\right]^{1/q}\right\},\end{aligned} \tag{52}$$

*where $C_1, C_2$, and $C_3$ are defined as in Theorem 7.*

*Proof.* The assertion follows from inequality (14) in Theorem 7, for $f : (0,\infty) \to \mathbb{R}$, $f(x) = x^2$. □

**Proposition 23.** *Let $0 < a < b$ and $\lambda \in [0,1]$. Then one has the following inequality:*

$$\begin{aligned}&\left|(1-\lambda)H^2 + \lambda A\left(a^2,b^2\right) - G^2\right| \\ &\le \frac{ab(b-a)C_4^{1/p}(\lambda,p)}{2\left[(1-q)(1-2q)(b-a)^2\right]^{1/q}} \\ &\times \left\{\left(C_5(q;a,b)a^q + C_6(q;a,b)b^q\right)^{1/q}\right. \\ &\left.+\left(C_6(q;b,a)a^q + C_5(q;b,a)b^q\right)^{1/q}\right\},\end{aligned} \tag{53}$$

*where $q > 1$, $1/p + 1/q = 1$, and $C_4$, $C_5$, and $C_6$ are defined as in Theorem 11.*

*Proof.* The assertion follows from inequality (25) in Theorem 11, for $f : (0,\infty) \to \mathbb{R}$, $f(x) = x^2$. □

**Proposition 24.** *Let $0 < a < b$ and $\lambda \in [0,1]$. Then one has the following inequality:*

$$\begin{aligned}&\left|(1-\lambda)H^2 + \lambda A\left(a^2,b^2\right) - G^2\right| \\ &\le \frac{ab(b-a)^{1-1/p}}{(2p-1)^{1/p}} \times \left[\frac{1}{4(q+1)(q+2)}\right]^{1/q} \\ &\times \left\{C_7^{1/p}(a,b;p)\left[C_9(\lambda,q)a^q + C_{10}(\lambda,q)b^q\right]^{1/q}\right. \\ &\left.+ C_8^{1/p}(a,b;p)\left[C_{10}(\lambda,q)a^q + C_9(\lambda,q)b^q\right]^{1/q}\right\},\end{aligned} \tag{54}$$

*where $q > 1$, $1/p+1/q = 1$, and $C_7$, $C_8$, $C_9$, and $C_{10}$ are defined as in Theorem 15.*

*Proof.* The assertion follows from inequality (33) in Theorem 15, for $f : (0,\infty) \to \mathbb{R}$, $f(x) = x^2$. □

**Proposition 25.** *Let $0 < a < b$, $n \in (-1,\infty) \setminus \{0\}$, $\lambda \in [0,1]$, and $q \ge 1$. Then one has the following inequality:*

$$\begin{aligned}&\left|(1-\lambda)H^{n+2} + \lambda A\left(a^{n+2},b^{n+2}\right) - G^2 \cdot L_n^n\right| \\ &\le \frac{ab(b-a)(n+2)}{2} \\ &\times \left\{C_1^{1-1/q}(\lambda;a,b)\right. \\ &\times \left[C_2(\lambda;a,b)a^{(n+1)q} + C_3(\lambda;a,b)b^{(n+1)q}\right]^{1/q} \\ &+ C_1^{1-1/q}(\lambda;b,a) \\ &\left.\times\left[C_3(\lambda;b,a)a^{(n+1)q} + C_2(\lambda;b,a)b^{(n+1)q}\right]^{1/q}\right\},\end{aligned} \tag{55}$$

*where $C_1$, $C_2$, and $C_3$ are defined as in Theorem 7.*

*Proof.* The assertion follows from inequality (14) in Theorem 7, for $f : (0,\infty) \to \mathbb{R}$, $f(x) = x^{n+2}$, $n \in (-1,\infty) \setminus \{0\}$. □

**Proposition 26.** *Let $0 < a < b$, $\lambda \in [0,1]$, and $n \in (-1,\infty) \setminus \{0\}$. Then one has the following inequality:*

$$\begin{aligned}&\left|(1-\lambda)H^{n+2} + \lambda A\left(a^{n+2},b^{n+2}\right) - G^2 \cdot L_n^n\right| \\ &\le \frac{ab(b-a)(n+2)C_4^{1/p}(\lambda,p)}{4\left[(1-q)(1-2q)(b-a)^2\right]^{1/q}} \\ &\times \left\{\left(C_5(q;a,b)a^{(n+1)q} + C_6(q;a,b)b^{(n+1)q}\right)^{1/q}\right. \\ &\left.+\left(C_6(q;b,a)a^{(n+1)q} + C_5(q;b,a)b^{(n+1)q}\right)^{1/q}\right\},\end{aligned} \tag{56}$$

*where $q > 1$, $1/p + 1/q = 1$, and $C_5$ and $C_6$ are defined as in Theorem 11.*

*Proof.* The assertion follows from inequality (25) in Theorem 11, for $f : (0, \infty) \to \mathbb{R}$, $f(x) = x^{n+2}$, $n \in (-1, \infty) \setminus \{0\}$. □

**Proposition 27.** *Let $0 < a < b$, $\lambda \in [0, 1]$, and $n \in (-1, \infty) \setminus \{0\}$. Then one has the following inequality:*

$$\begin{aligned}
&\left|(1-\lambda) H^{n+2} + \lambda A\left(a^{n+2}, b^{n+2}\right) - G^2 \cdot L_n^n\right| \\
&\quad \leq \frac{ab(n+2)(b-a)^{1-1/p}}{2(2p-1)^{1/p}} \\
&\qquad \times \left[\frac{1}{4(q+1)(q+2)}\right]^{1/q} \\
&\qquad \times \Big\{C_7^{1/p}(a, b; p) \\
&\qquad\quad \times \left[C_9(\lambda, q) a^{(n+1)q} + C_{10}(\lambda, q) b^{(n+1)q}\right]^{1/q} \\
&\qquad\quad + C_8^{1/p}(a, b; p) \\
&\qquad\quad \times \left[C_{10}(\lambda, q) a^{(n+1)q} + C_9(\lambda, q) b^{(n+1)q}\right]^{1/q}\Big\},
\end{aligned} \tag{57}$$

*where $q > 1$, $1/p + 1/q = 1$, and $C_7$, $C_8$, $C_9$, and $C_{10}$ are defined as in Theorem 15.*

*Proof.* The assertion follows from inequality (33) in Theorem 15, for $f : (0, \infty) \to \mathbb{R}$, $f(x) = x^{n+2}$, $n \in (-1, \infty) \setminus \{0\}$. □

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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