Research Article

Basicity of the System of Exponents with a Linear Phase in Sobolev Weight Space

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In the present paper a criterion for basicity of exponential system with linear phase is obtained in Sobolev weight space $W_{p,\rho}^1(-\pi, \pi)$.

In solving mathematical physics problems by the Fourier method, there often arise the systems of exponents of the form

$$\{e^{i(nt+\alpha(t))}; e^{-i(nt+\beta(t))}\}_{n \geq 1}, \quad (1)$$

$$1 \cup \{e^{i(nt+\alpha(t))}; e^{-i(nt+\beta(t))}\}_{n \geq 1}, \quad (2)$$

where $\alpha(t)$ and $\beta(t)$ are continuous or piecewise-continuous functions. Substantiation of the method requires studying the basis properties of these systems in Lebesgue and Sobolev spaces of functions. In the case when $\alpha(t)$ and $\beta(t)$ are linear functions, the basis properties of these systems in $L_p(-\pi, \pi)$, $1 < p < +\infty$, were completely studied in the papers [1–9]. The weighted case of the space L_p was considered in the papers [10, 11]. The basis properties of these systems in Sobolev spaces were studied in [12–14]. It should be noted that the close problems were also considered in [15].

In the present paper we study basis properties of the systems (1) and (2) in Sobolev weight spaces when $\alpha(t) = \alpha t$, $\beta(t) = \alpha t$, where α is a real parameter. Therewith the issue of basicity of system (2) in Sobolev spaces is reduced to the issue of basicity of system (1) in respective Lebesgue spaces.

Let $L_{p,\rho}(-\pi, \pi)$ and $W_{p,\rho}^1(-\pi, \pi)$ be weight spaces with the norms

$$\|u\|_{L_{p,\rho}}^p = \int_{-\pi}^{\pi} |u(\theta)|^p \rho(\theta) d\theta, \quad (3)$$

$$\|u\|_{W_{p,\rho}^1} = \|u\|_{L_{p,\rho}} + \|u'\|_{L_{p,\rho}}, \quad (4)$$

respectively, where $\rho(t) = \prod_{i=1}^l (\sin |(t - \tau_i)/2|)^{\beta_i}$, $-\pi < \tau_1 < \tau_2 < \dots < \tau_l < \pi$. Denote by $\mathcal{L}_{p,\rho}(-\pi, \pi)$ the direct sum $\mathcal{L}_{p,\rho} = L_{p,\rho} \oplus \mathbb{C}$, where $\mathbb{C}$ is the complex plane. The norm in this space is defined by the expression $\|\hat{u}\|_{\mathcal{L}_{p,\rho}} = \|u\|_{L_{p,\rho}} + |\lambda|$, where $\hat{u} = (u; \lambda) \in \mathcal{L}_{p,\rho}$.

The following easily provable lemmas play an important role in obtaining the main results. It holds the following.

Lemma 1. *Let $\beta_i \in (-1, p-1)$, $i = \overline{1, l}$; $p \in (1, +\infty)$. Then the operator*

$$A(u; \lambda) = \lambda + \int_{-\pi}^t u(\theta) d\theta \quad (5)$$

realizes an isomorphism between the spaces $\mathcal{L}_{p,\rho}$ and $W_{p,\rho}^1(-\pi, \pi)$; that is, the spaces $\mathcal{L}_{p,\rho}$ and $W_{p,\rho}^1$ are isomorphic.

Proof. At first prove the boundedness of the operator A . We have

$$\begin{aligned} \|A\hat{u}\|_{W_{p,\rho}^1} &= \left\| \lambda + \int_{-\pi}^t u(\theta) d\theta \right\|_{L_{p,\rho}} + \|u\|_{L_{p,\rho}} \\ &\leq \|\lambda\|_{L_{p,\rho}} + \left\| \int_{-\pi}^t u(\theta) d\theta \right\|_{L_{p,\rho}} + \|u\|_{L_{p,\rho}} \\ &= \left(\int_{-\pi}^{\pi} |\lambda|^p \rho(\theta) d\theta \right)^{1/p} \end{aligned}$$

$$\begin{aligned}
& + \left(\int_{-\pi}^{\pi} \left| \int_{-\pi}^t u(\theta) d\theta \right|^p \rho(t) dt \right)^{1/p} \\
& + \|u\|_{L_{p,\rho}} \leq |\lambda| \|\rho\|_{L_1}^{1/p} \\
& + \left(\int_{-\pi}^{\pi} \left(\int_{-\pi}^t |u(\theta)| d\theta \right)^p \rho(t) dt \right)^{1/p} \\
& + \|u\|_{L_{p,\rho}} \\
& \leq |\lambda| \|\rho\|_{L_1}^{1/p} + \left(\int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} |u(\theta)| d\theta \right)^p \rho(t) dt \right)^{1/p} \\
& + \|u\|_{L_{p,\rho}} \\
& = |\lambda| \|\rho\|_{L_1}^{1/p} + \int_{-\pi}^{\pi} |u(\theta)| d\theta \|\rho\|_{L_1}^{1/p} + \|u\|_{L_{p,\rho}}.
\end{aligned} \tag{6}$$

Having applied the Holder inequality, hence we get

$$\begin{aligned}
\int_{-\pi}^{\pi} |u(\theta)| d\theta &= \int_{-\pi}^{\pi} |u(\theta)| \rho^{1/p} \rho^{-1/p} d\theta \\
&\leq \left(\int_{-\pi}^{\pi} |u(\theta)|^p \rho d\theta \right)^{1/p} \\
&\quad \times \left(\int_{-\pi}^{\pi} \rho^{-q/p}(\theta) d\theta \right)^{1/q} \\
&= \|u\|_{L_{p,\rho}} \|\rho^{1/(1-p)}\|_{L_1}^{1/q},
\end{aligned} \tag{7}$$

where

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{8}$$

Consequently

$$\begin{aligned}
\|A\hat{u}\|_{W_{p,\rho}^1} &\leq |\lambda| \|\rho\|_{L_1}^{1/p} + \|u\|_{L_{p,\rho}} \|\rho\|_{L_1}^{1/p} \|\rho^{1/(1-p)}\|_{L_1}^{1/q} \\
&\quad + \|u\|_{L_{p,\rho}} \\
&= |\lambda| \|\rho\|_{L_1}^{1/p} + \left(1 + \|\rho\|_{L_1}^{1/p} \|\rho^{1/(1-p)}\|_{L_1}^{1/q} + 1 \right) \|u\|_{L_{p,\rho}} \\
&\leq M \left(|\lambda| + \|u\|_{L_{p,\rho}} \right) \\
&= M \|\hat{u}\|_{\mathcal{L}_{p,\rho}},
\end{aligned} \tag{9}$$

where

$$M = \max \left\{ \|\rho\|_{L_1}^{1/p}; \|\rho\|_{L_1}^{1/p} \|\rho^{1/(1-p)}\|_{L_1}^{1/q} + 1 \right\}. \tag{10}$$

Let us show that $\text{Ker } A = \{0\}$. Put $A\hat{u} = 0$; that is,

$$\lambda + \int_{-\pi}^t u(\theta) d\theta = 0, \quad \forall t \in (-\pi, \pi), \tag{11}$$

where $\lambda \in \mathbb{C}$, $u \in \mathcal{L}_{p,\rho}$. By differentiating this equality, we get $u(\theta) = 0$, a.e. on $(-\pi, \pi)$. Hence it follows that $\lambda = 0$. From (11) it directly follows that $u = 0$ a.e. on $(-\pi, \pi)$, and so $\hat{u} = 0$. Show that $\text{Im } A = W_{p,\rho}^1(-\pi, \pi)$ ($\text{Im } A$ is the range of values of the operator A). Let $v \in W_{p,\rho}^1(-\pi, \pi)$ be an arbitrary function. Let $\hat{v} = (v'; v(0))$. It is clear that $A\hat{v} = v$ and $\hat{v} \in \mathcal{L}_{p,\rho}$. Then from the Banach theorem we get that the operator A is boundedly invertible.

The lemma is proved. $\square$

The following lemma is also valid.

Lemma 2. Let $p \in (1, +\infty)$ and $\beta_i \in (-1, p-1)$, $i = \overline{1, l}$. Then for all $p_0 \in (1, \alpha) : L_{p,\rho}(-\pi, \pi) \subset L_{p_0}(-\pi, \pi)$, where

$$\alpha = \min \left\{ 2; p; \frac{p}{\beta_1 + 1}; \frac{p}{\beta_2 + 1}; \dots; \frac{p}{\beta_l + 1} \right\}. \tag{12}$$

Proof. Let $f \in L_{p,\rho}(-\pi, \pi)$, $p_0 \in (1, \alpha)$. We have

$$\begin{aligned}
&\int_{-\pi}^{\pi} |f|^{p_0} dt \\
&= \int_{-\pi}^{\pi} |f|^{p_0} \rho^{p_0/p} \rho^{-p_0/p} dt \\
&\leq \left(\int_{-\pi}^{\pi} |f|^p \rho dt \right)^{p_0/p} \left(\int_{-\pi}^{\pi} \rho^{p_0/(p_0-p)} dt \right)^{(p-p_0)/p} \\
&= \|f\|_{L_{p,\rho}}^{p_0} \left(\int_{-\pi}^{\pi} \prod_{i=1}^l \left(\sin \left| \frac{t - \tau_i}{2} \right| \right)^{\beta_i p_0/(p_0-p)} dt \right)^{(p-p_0)/p}.
\end{aligned} \tag{13}$$

Since $p_0 < p/(\beta_i + 1)$ and $\beta_i > -1$, then $\beta_i p_0/(p_0 - p) > -1$, $i = \overline{1, l}$. It is easy to see that $f \in L_{p_0}(-\pi, \pi)$ and moreover $\|f\|_{L_{p_0}} \leq C \|f\|_{L_{p,\rho}}$.

The lemma is proved. $\square$

In obtaining the basic results we need the following main lemma.

Lemma 3. Let $f \in L_{p,\rho}(-\pi, \pi)$, $\rho(t) = \prod_{i=1}^l (\sin |(t - \tau_i)/2|)^{\beta_i}$ and $\beta_i \in (-1, p/q)$, $i = \overline{1, l}$, $\alpha \in \mathbb{R}$ be a real parameter, $1/p + 1/q = 1$. Let f have the expansion

$$f(t) = \sum_{n \neq 0} c_n e^{i(n+\alpha \text{sign } n)t} \tag{14}$$

in the space $L_{p,\rho}(-\pi, \pi)$. Then it is valid

$$\sum_{|n| > |\alpha|} \left| \frac{c_n}{n + \alpha \text{sign } n} \right| < +\infty. \tag{15}$$

Proof. As it follows from Lemma 2, $\exists p_0 \in (1, 2) : L_{p,\rho} \subset L_{p_0}$. At first consider the case when $\alpha > 1/2 p_0 - 1$, $1/p_0 + 1/q_0 = 1$. In this case the system $\{e^{i(n+\alpha \text{sign } n)t}\}_{n \neq 0}$ is minimal in $L_{p_0}(0, \pi)$ (see [4]). Then from the results of the paper [16],

the Hausdorff-Young inequality is valid for this system; that is,

$$\left(\sum_{n \neq 0}^{\infty} |c_n|^{q_0} \right)^{1/q_0} \leq M \|f\|_{L_{p_0}}. \quad (16)$$

Applying the Holder inequality, we obtain

$$\begin{aligned} \sum_{|n| > |\alpha|} \left| \frac{c_n}{n + \alpha \operatorname{sign} n} \right| &\leq \left(\sum_{|n| > |\alpha|} |c_n|^{q_0} \right)^{1/q_0} \\ &\times \left(\sum_{|n| > |\alpha|} \frac{1}{|n + \alpha \operatorname{sign} n|^{p_0}} \right)^{1/p_0}. \end{aligned} \quad (17)$$

If $\alpha < 1/2 p_0 - 1$, then $\exists k \in \mathbb{N}$, $\alpha' = \alpha + k > 1/2 p_0 - 1$. Then

$$\begin{aligned} f(t) &= \sum_{n \neq 0} c_n e^{i(n+\alpha \operatorname{sign} n)t} \\ &= \sum_{n=-k}^{-1} c_n e^{i(n-\alpha)t} + \sum_{n=1}^k c_n e^{i(n+\alpha)t} + \sum_{|n| \geq k+1} c_n e^{i(n+\alpha \operatorname{sign} n)t} \\ &= f_k(t) + \sum_{n \neq 0} c_{n+k \operatorname{sign} n} e^{i(n+(k+\alpha) \operatorname{sign} n)t} \\ &= f_k(t) + \sum_{n \neq 0} c_{n+k \operatorname{sign} n} e^{i(n+\alpha' \operatorname{sign} n)t}, \end{aligned} \quad (18)$$

where $f_k(t) = \sum_{n=-k}^{-1} c_n e^{i(n-\alpha)t} + \sum_{n=1}^k c_n e^{i(n+\alpha)t}$, $\alpha' = k + \alpha$. As a result, we have

$$f(t) - f_k(t) = \sum_{n \neq 0} c_{n+k \operatorname{sign} n} e^{i(n+\alpha' \operatorname{sign} n)t}. \quad (19)$$

Since $\beta' > 1/p_0 - 2$, then again from the results of the paper [16] it follows

$$\left(\sum_{n \neq 0} |c_{n+k \operatorname{sign} n}|^{q_0} \right)^{1/q_0} \leq M \|f - f_k\|_{L_{p_0}}. \quad (20)$$

In the same way we establish the convergence of the series $\sum_{|n| > |\alpha|} |c_n/(n + \alpha \operatorname{sign} n)|$.

Consider the case when $\alpha = 1/2 p_0 - 1$. Then $\exists p' \in (p_0, 2) : \alpha > 1/2 p' - 1$. In this case the system $\{e^{i(n+\alpha \operatorname{sign} n)t}\}_{n \neq 0}$ is minimal in $L_{p'}$, and, consequently, from the results of the paper [16] it holds the Hausdorff-Young inequality; that is,

$$\left(\sum_{n \neq 0} |c_n|^{q'} \right)^{1/q'} \leq M \|f\|_{L_{p'}}, \quad (21)$$

where $1/p' + 1/q' = 1$.

From the previous reasonings we get the absolute convergence of the series $\sum_{|n| > |\alpha|} |c_n/(n + \alpha \operatorname{sign} n)|$. The lemma is proved. $\square$

Theorem 4. Let $p \in (1, +\infty)$, $\rho(t) = \prod_{i=1}^l (\sin |(t - \tau_i)/2|)^{\beta_i}$ be a weight function, $\alpha(t) \equiv \alpha t$, $\beta(t) \equiv \alpha t$, $\alpha \in \mathbb{R}$ be a real parameter, and the inequalities $-1 < \beta_i < p/q$, $i = \overline{1, l}$; $1/2 p - 1/2 > \alpha > 1/2 p - 1$ hold. Then the following statements are equivalent:

- (1) system (2) forms a basis for $W_{p,\rho}(-\pi, \pi)$;
- (2) system (1) forms a basis for $L_{p,\rho}(-\pi, \pi)$.

Proof. At first, suppose that the system (1) forms a basis for $L_{p,\rho}(-\pi, \pi)$. Let us show that the system $\{\hat{u}_n\}_{n=0}^{\infty}$ forms a basis for $\mathcal{L}_{p,\rho}$, where $\hat{u}_0 = (0; 1)$, $\hat{u}_n = (i(n + \alpha \operatorname{sign} n)e^{i(n+\alpha \operatorname{sign} n)t}; e^{-i(n+\alpha \operatorname{sign} n)\pi})$, $n \neq 0$.

It is enough to prove that the arbitrary element $\hat{u} = (u; \lambda)$ of $\mathcal{L}_{p,\rho}$ has the unique expansion

$$\hat{u} = \sum_{n \neq 0} \lambda_n \hat{u}_n; \quad (22)$$

that is,

$$u(t) = \sum_{n \neq 0} i(n + \alpha \operatorname{sign} n) \lambda_n e^{i(n+\alpha \operatorname{sign} n)t}, \quad (23)$$

$$\lambda = \lambda_0 + \sum_{n \neq 0} \lambda_n. \quad (24)$$

Since the system (1) forms a basis for $L_{p,\rho}(-\pi, \pi)$, then the expansion (23) holds and the coefficients λ_n ($n \neq 0$) are uniquely determined. By Lemma 3 the series $\sum_{n \neq 0} \lambda_n$ absolutely converges. Then it is clear that the number λ_0 from (24) is uniquely determined. This means that the system $\{\hat{u}_n\}_{n=-\infty}^{\infty}$ forms a basis for $\mathcal{L}_{p,\rho}$. Consider the system $\{v_n\}_{n=-\infty}^{\infty}$, where $v_n = A\hat{u}_n$, $A[(u, \lambda)] = \lambda + \int_{-\pi}^t u(\tau) d\tau$, $\hat{u} = (u; \lambda) \in L_{p,\rho}$. It is not difficult to see that

$$v_0 = 1, \quad v_n = e^{i(n+\alpha \operatorname{sign} n)t}, \quad n \neq 0. \quad (25)$$

Now, let us prove the converse. Assume that the system (2) forms a basis for $W_p^1(-\pi, \pi)$. Consider the system $\hat{u}_n = A^{-1}v_n$, $n \in \mathbb{Z}$. It is easy to see that the inverse operator is determined as $A^{-1}v = (v'; v(-\pi))$. It is obvious that the system $\{\hat{u}_n\}_{n=-\infty}^{\infty}$ forms a basis for $\mathcal{L}_{p,\rho}$. We have

$$\begin{aligned} \hat{u}_0 &= (0; 1), \\ \hat{u}_n &= (i(n + \alpha \operatorname{sign} n) e^{i(n+\alpha \operatorname{sign} n)t}; e^{-i(n+\alpha \operatorname{sign} n)\pi}). \end{aligned} \quad (26)$$

Consequently, $\hat{u} = (u; \lambda) \in L_{p,\rho}$ has a unique expansion (22) in $\mathcal{L}_{p,\rho}$. As a result, we obtain that each $u \in L_{p,\rho}$ has a unique expansion of the form (23). Indeed, let there exist another expansion for $u(t)$ in $L_{p,\rho}$:

$$u(t) = \sum_{n \neq 0} i(n + \alpha \operatorname{sign} n) \mu_n e^{i(n+\alpha \operatorname{sign} n)t}. \quad (27)$$

The absolute convergence of the series $\sum_{n \neq 0} \mu_n$ follows from Lemma 3. Put $\mu_0 = \lambda - \sum_{n \neq 0} \mu_n$. It is clear that the biorthogonal coefficients of the element $\hat{u} = (u; \lambda)$ are $(\bar{\mu}; \mu_0)$, where $\bar{\mu} = \{\mu_n\}_{n \neq 0}$. From the basicity of the system $\{\hat{u}_n\}_{n=-\infty}^{\infty}$ in $\mathcal{L}_{p,\rho}$ we obtain that $\mu_n = \lambda_n$, $n \in \mathbb{Z}$. This contradicts our conjecture. The theorem is proved. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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