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Research Article

On Nil-Symmetric Rings

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The concept of nil-symmetric rings has been introduced as a generalization of symmetric rings and a particular case of nil-semicommutative rings. A ring R is called right (left) nil-symmetric if, for $a, b, c \in R$, where a, b are nilpotent elements, $abc = 0$ ($cab = 0$) implies $acb = 0$. A ring is called nil-symmetric if it is both right and left nil-symmetric. It has been shown that the polynomial ring over a nil-symmetric ring may not be a right or a left nil-symmetric ring. Further, it is also proved that if R is right (left) nil-symmetric, then the polynomial ring $R[x]$ is a nil-Armendariz ring.

1. Introduction

Throughout this paper, all rings are associative with unity. Given a ring R, $\mathrm{nil}(R)$ and $R[x]$ denote the set of all nilpotent elements of R and the polynomial ring over R, respectively. A ring R is called reduced if it has no nonzero nilpotent elements; R is said to be Abelian if all idempotents of R are central; R is symmetric [1] if $abc = 0$ implies $acb = 0$ for all $a, b, c \in R$. An equivalent condition for a ring to be symmetric is that whenever product of any number of elements of the ring is zero, any permutation of the factors still gives the product zero [2]. R is reversible [3] if $ab = 0$ implies $ba = 0$ for all $a, b \in R$; R is called semicommutative [4] if $ab = 0$ implies $aRb = 0$ for all $a, b \in R$. In [5], Rege-Chhawchharia introduced the concept of an Armendariz ring. A ring R is called Armendariz if whenever polynomials $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, $g(x) = b_0 + b_1 x + \cdots + b_m x^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$ for each i, j. Liu-Zhao [6] and Antoine [7] further generalize the concept of an Armendariz ring by defining a weak-Armendariz and a nil-Armendariz ring, respectively. A ring R is called weak-Armendariz if whenever polynomials $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, $g(x) = b_0 + b_1 x + \cdots + b_m x^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j \in \mathrm{nil}(R)$ for each i, j. A ring R is called nil-Armendariz if whenever $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, $g(x) = b_0 + b_1 x + \cdots + b_m x^m \in R[x]$ satisfy $f(x)g(x) \in \mathrm{nil}(R)[x]$, then $a_i b_j \in \mathrm{nil}(R)$ for each i, j. Mohammadi et al. [8] initiated the notion of a nil-semicommutative ring as a generalization of a semicommutative ring. A ring R is nil-semicommutative if $ab = 0$ implies $aRb = 0$ for all $a, b \in \mathrm{nil}(R)$. In their paper it is shown that, in a nil-semicommutative ring R, $\mathrm{nil}(R)$ forms an ideal of R. Getting motivated by their paper we introduce the concept of a right (left) nil-symmetric ring which is a generalization of symmetric rings and a particular case of nil-semicommutative rings. Thus all the results valid for nil-semicommutative rings are valid for right (left) nil-symmetric rings also. We also prove that if a ring R is right (left) nil-symmetric and Armendariz, then $R[x]$ is right (left) nil-symmetric. In the context, there are also several other generalizations of symmetric rings (see [9, 10]).

2. Right (Left) Nil-Symmetric Rings

For a ring R, $M_n(R)$ and $T_n(R)$ denote the $n \times n$ full matrix ring and the upper triangular matrix ring over R, respectively. We observe that if R is a ring, then

$$\mathrm{nil}(T_n(R)) = \begin{pmatrix} \mathrm{nil}(R) & R & R & \cdots & R \\ 0 & \mathrm{nil}(R) & R & \cdots & R \\ 0 & 0 & \mathrm{nil}(R) & \cdots & R \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathrm{nil}(R) \end{pmatrix}. \quad (1)$$

Definition 1. A ring R is said to be *right (left) nil-symmetric* if whenever, for every $a, b \in \text{nil}(R)$ and for every $c \in R$, $abc = 0$ ($cab = 0$), then $acb = 0$. A ring R is nil-symmetric if it is both right and left nil-symmetric.

Example 2. let k be a field, and let R be the path algebra of the quiver

$$1 \xleftarrow{x} 2 \circlearrowleft^{y}, \tag{2}$$

over k, modulo the relation $y^2 = 0$. Let e_1 and e_2 be the paths of length 0 at vertices 1 and 2, respectively. Composing arrows from left to right, xy is a nonzero path, while yx is not.

Then any nilpotent element is a linear combination of x, y, and xy.

Let $(ax+by+cxy)$ and $(dx+ey+fxy)$ be two such elements and let $(ge_1 + he_2 + ix + jy + lxy)$ be an arbitrary element. We have

$$\begin{gathered}(ax + by + cxy)(dx + ey + fxy)(ge_1 + he_2 + ix + jy + lxy) \\ = (aeh)\,xy, \\ (ax + by + cxy)(ge_1 + he_2 + ix + jy + lxy)(dx + ey + fxy) \\ = (aeh)\,xy.\end{gathered} \tag{3}$$

Thus R is a right nil-symmetric ring. However, we have that $e_2xy = 0$, while $xe_2y = xy \neq 0$. Hence, R is not a left nil-symmetric ring.

Similarly by considering the opposite ring of R, one can have a left nil-symmetric ring which is not right nil-symmetric.

Clearly every symmetric ring is nil-symmetric but the converse is not true by Example 3 and that every subring of a right (left) nil-symmetric ring is right (left) nil-symmetric.

Example 3. For a reduced ring R, $T_2(R)$ is a nil-symmetric ring which is not symmetric. This can be verified as follows.

Let

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in \text{nil}(T_2(R)); \quad \text{let } \begin{pmatrix} c & d \\ 0 & e \end{pmatrix} \in T_2(R). \tag{4}$$

Then

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & e \end{pmatrix} = 0. \tag{5}$$

Also

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & e \end{pmatrix}\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = 0. \tag{6}$$

Thus $T_2(R)$ is a right nil-symmetric ring. Similarly it can be shown that $T_2(R)$ is a left nil-symmetric ring. But

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0 \tag{7}$$

whereas

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0. \tag{8}$$

Thus $T_2(R)$ is not symmetric.

From the above example we observe that a nil-symmetric ring need not be Abelian, as $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is an idempotent in $T_2(R)$, but

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}. \tag{9}$$

Remark 4. An Abelian ring also need not be either a right nil-symmetric or a left nil-symmetric ring as shown by the following example.

Example 5. We consider the ring in [11, Example 2.2]

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ a - d \equiv b \equiv c \equiv 0 \ (\text{mod } 2) \right\}. \tag{10}$$

R is an Abelian ring as $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ are the only idempotents. Again we have

$$\begin{gathered}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \in \text{nil}(R), \\ \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = 0 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix},\end{gathered} \tag{11}$$

but

$$\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \neq 0. \tag{12}$$

Hence, R is neither right nil-symmetric nor left nil-symmetric.

Proposition 6. *Let R be a reduced ring. Then*

$$S = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in R \right\} \tag{13}$$

is a nil-symmetric ring.

Proof. Let

$$\begin{pmatrix} 0 & b_1 & c_1 \\ 0 & 0 & d_1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b_2 & c_2 \\ 0 & 0 & d_2 \\ 0 & 0 & 0 \end{pmatrix} \in \text{nil}(S), \quad \begin{pmatrix} a_3 & b_3 & c_3 \\ 0 & a_3 & d_3 \\ 0 & 0 & a_3 \end{pmatrix} \in S \tag{14}$$

be such that

$$\begin{pmatrix} 0 & b_1 & c_1 \\ 0 & 0 & d_1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & b_2 & c_2 \\ 0 & 0 & d_2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a_3 & b_3 & c_3 \\ 0 & a_3 & d_3 \\ 0 & 0 & a_3 \end{pmatrix} = 0. \tag{15}$$

This implies

$$\begin{pmatrix} 0 & 0 & b_1d_2a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0, \quad \text{that is, } b_1d_2a_3 = 0. \tag{16}$$

Since R is reduced, $b_1 a_3 d_2 = 0$. Thus

$$\begin{pmatrix} 0 & b_1 & c_1 \\ 0 & 0 & d_1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_3 & b_3 & c_3 \\ 0 & a_3 & d_3 \\ 0 & 0 & a_3 \end{pmatrix} \begin{pmatrix} 0 & b_2 & c_2 \\ 0 & 0 & d_2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b_1 a_3 d_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0. \quad (17)$$

Hence, S is a right nil-symmetric ring. Similarly it can be shown that S is a left nil-symmetric ring. □

Let S be a reduced ring and we define a new ring as follows:

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} : a, a_{ij} \in S \right\}, \quad (18)$$

where $n \geq 2$. Based on Proposition 6, one may think that R_n may also be nil-symmetric for $n \geq 4$, but the following example nullifies that possibility.

Example 7. Let R be a reduced ring and let

$$R_4 = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix} : a, a_{ij} \in R \right\}. \quad (19)$$

Now

$$\begin{gathered} \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0, \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0 \end{gathered} \quad (20)$$

but

$$\begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0. \quad (21)$$

Thus R_4 is neither a right nil-symmetric ring nor a left nil-symmetric ring.

For a ring R, let

$$V(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix} : a, a_{ij} \in R \right\}. \quad (22)$$

Then $V(R)$ forms a subring of R_4.

Example 8. For every reduced ring R, $V(R)$ is nil-symmetric.

Let

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b_{12} & b_{13} & b_{14} \\ 0 & 0 & b_{23} & b_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \operatorname{nil}(V(R)) \quad (23)$$

and let

$$\begin{pmatrix} c & c_{12} & c_{13} & c_{14} \\ 0 & c & c_{23} & c_{24} \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix} \in V(R) \quad (24)$$

be such that

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & b_{12} & b_{13} & b_{14} \\ 0 & 0 & b_{23} & b_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} c & c_{12} & c_{13} & c_{14} \\ 0 & c & c_{23} & c_{24} \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix} = 0. \quad (25)$$

This gives

$$\begin{pmatrix} 0 & 0 & a_{12}b_{23}c & a_{12}b_{24}c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0. \quad (26)$$

Thus $a_{12}b_{23}c = 0$, $a_{12}b_{24}c = 0$. Since R is reduced, we have $a_{12}cb_{23} = 0$, $a_{12}cb_{24} = 0$. Therefore,

$$\begin{pmatrix} 0 & 0 & a_{12}cb_{23} & a_{12}cb_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} c & c_{12} & c_{13} & c_{14} \\ 0 & c & c_{23} & c_{24} \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix} \times \begin{pmatrix} 0 & b_{12} & b_{13} & b_{14} \\ 0 & 0 & b_{23} & b_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0. \quad (27)$$

Hence, $V(R)$ is a right nil-symmetric ring. Similarly, it can be shown that $V(R)$ is a left nil-symmetric ring.

We also observe that every right (left) nil-symmetric ring is nil-semicommutative.

Proposition 9. *Every right (left) nil-symmetric ring is nil-semicommutative.*

Proof. Let R be a right nil-symmetric ring and $a, b \in \text{nil}(R)$ such that $ab = 0$. Let $c \in R$ be arbitrary; then $abc = 0$. By right nil-symmetric property of R, $acb = 0$. Thus $aRb = 0$. Hence, R is nil-semicommutative. Proceeding similarly one can show that every left nil-symmetric ring is nil-semicommutative. □

Remark 10. The converse is however not true, as shown by the following example.

Example 11. For every reduced ring R, $T_3(R)$ is a nil-semicommutative ring which is neither a right nil-symmetric ring nor a left nil-symmetric ring. This can be verified as follows.

We have

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in \text{nil}\left(T_3\left(R\right)\right),$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \tag{28}$$

but

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \neq 0. \tag{29}$$

Thus $T_3(R)$ is neither a right nil-symmetric ring nor a left nil-symmetric ring. But $T_3(R)$ is nil-semicommutative by [8, Example 2.2].

Remark 12. Semicommutativity and nil-symmetry do not follow each other. In Example 3, $T_2(R)$ is a nil-symmetric ring but not Abelian (and so not semicommutative [12]). The following example [13, Example 2.8] shows that a semicommutative ring need not be a right or left nil-symmetric ring.

Example 13. Let $Q_8 = \{1, x_{-1}, x_i, x_{-i}, x_j, x_{-j}, x_k, x_{-k}\}$ be the quaternion group and let $\mathbb{Z}_2$ be the ring of integers modulo 2. Consider the group ring $R = \mathbb{Z}_2 Q_8$. By [14, Corollary 2.3], R is reversible and so semicommutative. Let $a = 1 + x_j$, $b = 1 + x_i$, $c = 1 + x_i + x_j + x_k$. Then $a, b \in \text{nil}(R)$ and $c \in R$ such that $abc = cab = 0$, but $acb \neq 0$. Hence, R is neither a right nil-symmetric ring nor a left nil-symmetric ring.

Proposition 14. *For a reduced ring R and for $n \geq 2$,*

$$V_n(R) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \cdots a_n \\ 0 & a_1 & a_2 & a_3 \cdots a_{n-1} \\ 0 & 0 & a_1 & a_2 \cdots a_{n-2} \\ \vdots & \vdots & \vdots & \ddots \vdots \\ 0 & 0 & 0 & 0 \cdots a_2 \\ 0 & 0 & 0 & 0 \cdots a_1 \end{pmatrix} : a_1, \ldots, a_n \in R \right\} \textit{ is a nil-symmetric ring.} \tag{30}$$

Proof. Let R be a reduced ring. Then by [9, Theorem 2.3], $R[x]/(x^n)$ is a symmetric ring and hence a nil-symmetric ring, where (x^n) is the ideal generated by x^n for any positive integer n. Also by [15], $R[x]/(x^n) \cong V_n(R)$ for $n \geq 2$. Hence, for $n \geq 2$, $V_n(R)$ is nil-symmetric. □

Since the class of nil-symmetric rings is contained in the class of nil-semicommutative rings, the results which are valid for nil-semicommutative rings are also valid for nil-symmetric rings. Mohammadi et al. [8, Example 2.8] have shown that $T_5(R)$ is not a nil-semicommutative ring, where R is a reduced ring. Thus $T_5(R)$ is not nil-symmetric. Now we give an example of a weak-Armendariz ring which is not nil-symmetric.

Example 15. Let R be a reduced ring and let

$$R_4 = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix} : a, a_{ij} \in R \right\}. \tag{31}$$

By [6, Example 2.4], R_4 is weak-Armendariz. By Example 7, R_4 is neither a right nor a left nil-symmetric ring.

Proposition 16. *Finite product of right (left) nil-symmetric rings is right (left) nil-symmetric.*

Proof. It comes from the fact that $\text{nil}(\prod_{i=1}^n R_i) = \prod_{i=1}^n \text{nil}(R_i)$ [8, Proposition 2.13]. Let $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in \text{nil}(\prod_{i=1}^n R_i)$ and $(c_1, c_2, \ldots, c_n) \in \prod_{i=1}^n R_i$ such that $(a_1, a_2, \ldots, a_n)(b_1, b_2, \ldots, b_n)(c_1, c_2, \ldots, c_n) = 0$. Thus, for each $i = 1, 2, \ldots, n$, $a_i b_i c_i = 0$. Since R_i is right nil-symmetric, $a_i c_i b_i = 0$ for each $i = 1, 2, \ldots, n$. So, we get $(a_1, a_2, \ldots, a_n)(c_1, c_2, \ldots, c_n)(b_1, b_2, \ldots, b_n) = 0$. The result can be similarly proved for left nil-symmetric rings. □

Proposition 17. *Let R be a ring and let Δ be a multiplicatively closed subset of R consisting of central nonzero-divisors. Then R is right (left) nil-symmetric if and only if $\Delta^{-1}R$ is right (left) nil-symmetric.*

Proof. It suffices to prove the necessary condition because subrings of right (left) nil-symmetric rings are also right (left) nil-symmetric. Let $\alpha\beta\gamma = 0$ with $\alpha = u^{-1}a$, $\beta = v^{-1}b \in \mathrm{nil}(\Delta^{-1}R)$, and $\gamma = w^{-1}c \in \Delta^{-1}R$; then $u, v, w \in \Delta$, $a, b \in \mathrm{nil}(R)$, and $c \in R$. Since Δ is contained in the center of R, we have $0 = \alpha\beta\gamma = u^{-1}av^{-1}bw^{-1}c = (uvw)^{-1}abc$ and so $abc = 0$. It follows that $acb = 0$, since R is right nil-symmetric. Thus $\alpha\gamma\beta = (uvw)^{-1}abc = 0$. Hence, $\Delta^{-1}R$ is right nil-symmetric. Similarly, $\Delta^{-1}R$ can be shown to be left nil-symmetric if R itself is a left nil-symmetric ring. □

Corollary 18. *For a ring R, $R[x]$ is a right (left) nil-symmetric ring if and only if $R[x; x^{-1}]$ is a right (left) nil-symmetric ring.*

Proof. It directly follows from Proposition 17. If $\Delta = \{1, x, x^2, \ldots\}$, then Δ is clearly a multiplicatively closed subset of $R[x]$ and $R[x; x^{-1}] = \Delta^{-1}R[x]$. □

Proposition 19. *Let R be a ring. Then eR and $(1 - e)R$ are right (left) nil-symmetric for some central idempotent e of R if and only if R is right (left) nil-symmetric.*

Proof. It suffices to prove the necessary condition because subrings of right (left) nil-symmetric rings are also right (left) nil-symmetric. Let eR and $(1 - e)R$ be right (left) nil-symmetric rings for some central idempotent e of R. Since, $R \cong eR \oplus (1 - e)R$, R is right (left) nil-symmetric by Proposition 16. □

Since the class of right (left) nil-symmetric rings is closed under subrings, therefore, for any right (left) nil-symmetric ring R and for any $e^2 = e \in R$, eRe is a right (left) nil-symmetric ring. The converse is, however, not true, in general as shown by the following example.

Example 20. Let S be any reduced ring. Then by Example 11, $R = T_3(S)$ is neither a right nil-symmetric nor a left nil-symmetric ring.

But for

$$e^2 = e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in R, \qquad eRe = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : a \in S \right\} \tag{32}$$

is a reduced ring and so a nil-symmetric ring.

For any nonempty subsets A, B, C of a ring R, ABC denotes the set of all finite sums of the elements of the type abc, where $a \in A, b \in B, c \in C$.

Proposition 21. *A ring R is right (left) nil-symmetric if and only if $ABC = 0$ implies $ACB = 0$ ($CAB = 0$ implies $ACB = 0$) for any two nonempty subsets A, B of $\mathrm{nil}(R)$ and any subset C of R.*

Proof. Let R be a right nil-symmetric ring and let A, B be nonempty subsets of $\mathrm{nil}(R)$; let C be a nonempty subset of R such that $ABC = 0$. Then $abc = 0$ for all $a \in A$, $b \in B$, $c \in C$. Right nil-symmetric property of R gives $acb = 0$ for all $a \in A$, $b \in B$, $c \in C$. Thus $ACB = 0$. Similar proof can be given for left nil-symmetric rings. The converse is straightforward. □

The following result shows that, for a semiprime ring, the properties of reduced, symmetric, reversible, semicommutative, nil-semicommutative, and nil-symmetric rings coincide. Note that a ring R is said to be semiprime if, for $a \in R$, $aRa = 0$ implies that $a = 0$.

Proposition 22. *For a semiprime ring R, the following statements are equivalent.*

(1) *R is reduced.*

(2) *R is symmetric.*

(3) *R is reversible.*

(4) *R is semicommutative.*

(5) *R is nil-semicommutative.*

(6) *R is right (left) nil-symmetric.*

Proof. (1)–(4) are equivalent by [16, Lemma 2.7]. (1)⇔(5) by [8, Proposition 2.18]. (2)⇒(6) is clear. (6)⇒ (1): let $a^2 = 0$ for $a \in R$. Then $a^2c = 0$ for any $c \in R$, and so $aca = 0$, since R is right nil-symmetric. Thus $a = 0$ by semiprimeness of R and, therefore, R is reduced. □

Given a ring R and a bimodule ${}_RM_R$, the trivial extension of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication:

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2). \tag{33}$$

This is isomorphic to the ring of all matrices:

$$\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}, \tag{34}$$

where $r \in R$ and $m \in M$ and the usual matrix operations are used.

Proposition 23. *For a reduced ring R, $T(R, R)$ is a nil-symmetric ring.*

Proof. Let R be a reduced ring. Since $T(R, R)$ is a subring of S in Proposition 6 and the class of right(left) nil-symmetric rings is closed under subrings, thus $T(R, R)$ is a nil-symmetric ring. □

Considering the above proposition one may conjecture that if a ring R is nil-symmetric, then $T(R, R)$ is nil-symmetric. However, the following example eliminates the possibility.

Example 24. Let $\mathbb{H}$ be the Hamilton quaternions over the real number field and let

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in \mathbb{H} \right\}. \tag{35}$$

Then by Proposition 6, R is a nil-symmetric ring. Let S be the trivial extension of R by itself. Then S is not a right nil-symmetric ring. Note that

$$\begin{pmatrix} \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix},$$

$$\begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & j \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} -i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & j \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix} \in \operatorname{nil}(S),$$

$$\begin{gathered} \begin{pmatrix} \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix} \\ \times \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & j \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} -i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & j \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix} \\ \times \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix} = 0. \end{gathered} \tag{36}$$

However we have

$$\begin{gathered} \begin{pmatrix} \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix} \\ \times \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix} \\ \times \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & j \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} -i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & j \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix} \\ = \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix} \neq 0. \end{gathered} \tag{37}$$

Thus $S = T(R, R)$ is not a right nil-symmetric ring.

Example 25. Let R be a ring and let I be an ideal of R such that R/I is nil-symmetric. Then R may not be nil-symmetric. This can be verified as follows. Let S be any reduced ring. Then by Example 11, $R = T_3(S)$ is not nil-symmetric but nil-semicommutative. Thus

$$I = \operatorname{nil}(R) = \left\{ \begin{pmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} : b, c, d \in S \right\} \tag{38}$$

is an ideal of R and R/I is reduced, so nil-symmetric.

Homomorphic image of a right (left) nil-symmetric ring need not be a right (left) nil-symmetric ring. This is discussed after Example 26.

3. Polynomial Extension of Nil-Symmetric Rings

Anderson-Camillo [17] proved that a ring R is Armendariz if and only if $R[x]$ is Armendariz; Huh et al. [12] have shown that polynomial rings over semicommutative rings need not be semicommutative; Kim-Lee [16] showed that polynomial rings over reversible rings need not be reversible. Recently Mohammadi et al. [8] have given an example of a nil-semicommutative ring R for which $R[x]$ is not nil-semicommutative. Based on the above findings, it is natural to check whether the polynomial ring over a nil-symmetric ring is nil-symmetric. However, the answer is given in the negative through the following example.

Example 26. Let $\mathbb{Z}_2$ be the field of integers modulo 2 and let $A = \mathbb{Z}_2[a_0, a_1, a_2, b_0, b_1, b_2, c]$ be the free algebra of polynomials with zero constant terms in noncommuting indeterminates a_0, a_1, a_2, b_0, b_1, b_2, and c over $\mathbb{Z}_2$. Consider an ideal of the ring $\mathbb{Z}_2 + A$, say I, generated by the following elements: a_0b_0, $a_0b_1 + a_1b_0$, $a_0b_2 + a_1b_1 + a_2b_0$, $a_1b_2 + a_2b_1$, a_2b_2, a_0rb_0, a_2rb_2, b_0a_0, $b_0a_1 + b_1a_0$, $b_0a_2 + b_1a_1 + b_2a_0$, $b_1a_2 + b_2a_1$, b_0ra_0, b_2ra_2, $(a_0+a_1+a_2)r(b_0+b_1+b_2)$, $(b_0+b_1+b_2)r(a_0+a_1+a_2)$, and $r_1r_2r_3r_4$, where $r, r_1, r_2, r_3, r_4 \in A$. Now $R = (\mathbb{Z}_2 + A)/I$ is symmetric by [9, Example 3.1] and so a nil-symmetric ring. By [8, Example 3.6], we have $a_0 + a_1x + a_2x^2$, $b_0 + b_1x + b_2x^2 \in \mathrm{nil}(R[x])$. Now $(a_0 + a_1x + a_2x^2)(b_0 + b_1x + b_2x^2)c$, $c(a_0 + a_1x + a_2x^2)(b_0 + b_1x + b_2x^2) \in I[x]$, but $(a_0 + a_1x + a_2x^2)c(b_0 + b_1x + b_2x^2) \notin I[x]$ because $a_0cb_1 + a_1cb_0 \notin I$. Hence $R[x]$ is neither a right nil-symmetric ring nor a left nil-symmetric ring.

Remark 27. The above example also helps in showing that homomorphic image of a right (left) nil-symmetric ring need not be a right (left) nil-symmetric ring. This is verified as follows.

Example 28. In Example 26, $(\mathbb{Z}_2 + A)[x]$ is a domain [16] and so a nil-symmetric ring. But the quotient ring $(\mathbb{Z}_2 + A)[x]/I[x] \cong R[x]$ is neither a right nil-symmetric ring nor a left nil-symmetric ring.

Now we study some conditions under which the answer may be given positively. Since every right (left) nil-symmetric ring is nil-semicommutative by Proposition 9, therefore, by [8, Theorem 3.3] for each right (left) nil-symmetric ring R, $\mathrm{nil}(R[x]) = \mathrm{nil}(R)[x]$. The converse is, however, not true, in general. Now we give an example of a ring R which satisfies $\mathrm{nil}(R[x]) = \mathrm{nil}(R)[x]$, but R is neither a right nil-symmetric ring nor a left nil-symmetric ring.

Example 29. We use the ring in [7, Example 4.8]. Let K be a field, $n \geq 2$ and $R = K\langle a, b \mid b^n = 0\rangle$. Then $\mathrm{nil}(R)$ is not an ideal of R. Thus R is neither a right nil-symmetric nor a left nil-symmetric ring by Proposition 9 and [8, Theorem 2.5]. But R is a nil-Armendariz ring and hence by [7, Corollary 5.2], $\mathrm{nil}(R[x]) = \mathrm{nil}(R)[x]$.

Proposition 30. *If R is a right (left) nil-symmetric and Armendariz ring, then the polynomial ring $R[x]$ is right (left) nil-symmetric.*

Proof. Let R be a right nil-symmetric and Armendariz ring and let $f(x) = \sum_{i=0}^{m} a_ix^i$, $g(x) = \sum_{j=0}^{n} b_jx^j \in \mathrm{nil}(R[x])$ and $h(x) = \sum_{k=0}^{p} a_kx^k \in R[x]$ such that $f(x)g(x)h(x) = 0$. Since R is right nil-symmetric, $\mathrm{nil}(R[x]) = \mathrm{nil}(R)[x]$ by Proposition 9 and [8, Theorem 3.3]. Thus $a_i, b_j \in \mathrm{nil}(R)$ for $i = 0, 1, 2, \ldots, m$; $j = 0, 1, 2, \ldots, n$. Since R is Armendariz, therefore, $a_ib_jc_k = 0$ by [17, Proposition 1]. Thus by right nil-symmetric property of R, $a_ic_kb_j = 0$. Therefore, $f(x)h(x)g(x) = 0$. Hence, $R[x]$ is a right nil-symmetric ring. Similarly it can be shown that $R[x]$ is a left nil-symmetric ring if R is a left nil-symmetric and Armendariz ring. □

Proposition 31. *If R is a right (left) nil-symmetric ring, then $R[x]$ is nil-Armendariz.*

Proof. Let R be a right (left) nil-symmetric ring. Thus by Proposition 9, R is nil-semicommutative. By [8, Corollary 2.9], R is a nil-Armendariz ring. Again by [8, Theorem 3.3], $\mathrm{nil}(R[x]) = \mathrm{nil}(R)[x]$. Thus by [7, Theorem 5.3], $R[x]$ is nil-Armendariz. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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