Research Article

On Some Grüss-Type Fractional Inequalities Using Saigo Fractional Integral Operator

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The main aim of this paper is to establish some new fractional integral inequalities of Grüss-type via the Saigo fractional integral operator.

1. Introduction

Grüss inequality is an inequality which establishes a connection between the integral of the product of two functions and the product of the integrals of the two functions. In 1935, Grüss proved the following well-known classical integral inequality; see [1, 2].

Theorem 1 (see [2]). *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be two integrable functions such that $\phi \leq f(x) \leq \Phi$ and $\gamma \leq g(x) \leq \Gamma$ for all $x \in [a, b]$; ϕ, Φ, γ , and Γ are constant; then*

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \frac{1}{b-a} \int_a^b f(x) dx \cdot \frac{1}{b-a} \int_a^b g(x) dx \right| \leq \frac{1}{4} (\Gamma - \gamma) (\Phi - \phi), \quad (1)$$

where the constant $1/4$ is sharp.

In the past decade, many researchers had numerous generalizations, variants, and extensions of Grüss inequality done in literature; to mention a few, see [1–12] and the references cited therein.

Recently, many authors have studied the fractional integral inequalities via Caputo, Riemann-Liouville, and q -fractional integral; see [6, 13–21]. Some authors have studied the Saigo fractional integral operator; for example, we refer the reader to [22–29] and references cited therein. In [19], Dahmani et al. gave the following fractional integral inequality using Riemann-Liouville fractional integral.

Theorem 2 (see [19]). *Let f and g be two integrable functions on $[0, \infty)$ satisfying the condition*

$$\begin{aligned} m &\leq f(x) \leq M, \\ p &\leq g(x) \leq P; \\ m, M, p, P &\in \mathbb{R}, \end{aligned} \quad (2)$$

on $[0, \infty)$. Then, for all $t > 0$ and $\alpha > 0$, one has

$$\begin{aligned} &\left| \frac{t^\alpha}{\Gamma(\alpha+1)} J^\alpha f g(t) - J^\alpha f(t) J^\alpha g(t) \right| \\ &\leq \left(\frac{t^\alpha}{2\Gamma(\alpha+1)} \right)^2 (M-m)(P-p). \end{aligned} \quad (3)$$

In literature few results have been obtained on some fractional integral inequalities using Hadamard fractional integral and Saigo fractional integral operator in [25, 26, 30–32]. Our purpose in this paper is to establish some new results using Saigo fractional integral.

2. Preliminaries

We give some definitions and properties which will be used later. For more details, see [28, 33].

Definition 3. A real-valued function $f(x)$ ($x > 1$) is said to be in space C_μ ($\mu \in \mathbb{R}$), if there exists a real number $p > \mu$ such that $f(x) = x^p \phi(x)$, where $\phi(x) \in C(0, \infty)$.

Definition 4 (see [27, 28]). Let $\alpha > 0$, $\beta, \eta \in \mathbb{R}$; then the Saigo fractional integral $I_{0,x}^{\alpha,\beta,\eta}[f(x)]$ of order α for a real-valued continuous function $f(x)$ is defined by

$$\begin{aligned} I_{0,x}^{\alpha,\beta,\eta}[f(x)] &= \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \\ &\cdot \int_0^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}\right) f(t) dt, \end{aligned} \quad (4)$$

where the function ${}_2F_1(-)$ in the right-hand side of (4) is the Gaussian hypergeometric function defined by

$${}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}. \quad (5)$$

And $(a)_n$ is the Pochhammer symbol

$$\begin{aligned} (a)_n &= a(a+1) \cdots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \\ (a)_0 &= 1. \end{aligned} \quad (6)$$

For $f(x) = x^\mu$ in (4), we have the known result [28] as

$$I_{0,x}^{\alpha,\beta,\eta}[x^\mu] = \frac{\Gamma(\mu+1) \Gamma(\mu+1-\beta+\eta)}{\Gamma(\mu+1-\beta) \Gamma(\mu+1+\alpha+\eta)} x^{\mu-\beta} \quad (7)$$

($\alpha > 0$, $\min(\mu, \mu - \beta + \eta) > -1$, $x > 0$).

3. Grüss-Type Fractional Integral Inequality

In this section, we establish Grüss-type fractional integral inequality involving Saigo fractional integral operator (4), for which we assume the following.

(A₁) There exist two integrable functions $\Phi_1(x)$ and $\Phi_2(x)$ on $[0, \infty[$, such that

$$\Phi_1(x) \leq u(x) \leq \Phi_2(x), \quad \forall x \in [0, \infty[. \quad (8)$$

(A₂) There exist two integrable functions $\Psi_1(x)$ and $\Psi_2(x)$ on $[0, \infty[$, such that

$$\Psi_1(x) \leq v(x) \leq \Psi_2(x), \quad \forall x \in [0, \infty[. \quad (9)$$

Lemma 5. Suppose that u is an integrable function on $[0, \infty)$ and $\Phi_1(x)$ and $\Phi_2(x)$ are two integrable functions on $[0, \infty)$. Assume that the condition (A₁) holds.

Then, for all $x > 0$, $\alpha > \max\{0, -\beta\}$, $\beta < 1$, $\beta - 1 < \eta < 0$, one has

$$\begin{aligned} &\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta) \Gamma(1+\alpha+\eta)} x^\beta I_{0,x}^{\alpha,\beta,\eta}[u^2(x)] - \left(I_{0,x}^{\alpha,\beta,\eta}[u(x)]\right)^2 \\ &= \left(I_{0,x}^{\alpha,\beta,\eta}[\Phi_2(x)] - I_{0,x}^{\alpha,\beta,\eta}[u(x)]\right) \\ &\quad \cdot \left(I_{0,x}^{\alpha,\beta,\eta}[u(x)] - I_{0,x}^{\alpha,\beta,\eta}[\Phi_1(x)]\right) \\ &\quad - \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta) \Gamma(1+\alpha+\eta)} x^\beta I_{0,x}^{\alpha,\beta,\eta} \\ &\quad \cdot [(\Phi_2(x) - u(x))(u(x) - \Phi_1(x))] \\ &\quad + \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta) \Gamma(1+\alpha+\eta)} x^\beta I_{0,x}^{\alpha,\beta,\eta}[\Phi_1 u(x)] \\ &\quad - I_{0,x}^{\alpha,\beta,\eta}[\Phi_1(x)] I_{0,x}^{\alpha,\beta,\eta}[u(x)] \\ &\quad + \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta) \Gamma(1+\alpha+\eta)} x^\beta I_{0,x}^{\alpha,\beta,\eta}[\Phi_2 u(x)] \\ &\quad - I_{0,x}^{\alpha,\beta,\eta}[\Phi_2(x)] I_{0,x}^{\alpha,\beta,\eta}[u(x)] \\ &\quad + I_{0,x}^{\alpha,\beta,\eta}[\Phi_1(x)] I_{0,x}^{\alpha,\beta,\eta}[\Phi_2(x)] \\ &\quad - \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta) \Gamma(1+\alpha+\eta)} x^\beta I_{0,x}^{\alpha,\beta,\eta}[\Phi_1 \Phi_2(x)]. \end{aligned} \quad (10)$$

Proof. Let u be an integrable function on $[0, \infty)$. For all $\rho, \sigma > 0$, we have

$$\begin{aligned} &(\Phi_2(\sigma) - u(\sigma))(u(\rho) - \Phi_1(\rho)) \\ &\quad + (\Phi_2(\rho) - u(\rho))(u(\sigma) - \Phi_1(\sigma)) \\ &\quad - (\Phi_2(\rho) - u(\rho))(u(\rho) - \Phi_1(\rho)) \\ &\quad - (\Phi_2(\sigma) - u(\sigma))(u(\sigma) - \Phi_1(\sigma)) \\ &= u^2(\rho) + u^2(\sigma) - 2u(\sigma)u(\rho) \\ &\quad + \Phi_2(\sigma)u(\rho) + \Phi_1(\rho)u(\sigma) - \Phi_1(\rho)\Phi_2(\sigma) \\ &\quad + \Phi_1(\rho)u(\sigma) + \Phi_1(\sigma)u(\rho) - \Phi_1(\sigma)\Phi_2(\rho) \\ &\quad - \Phi_2(\rho)u(\rho) + \Phi_1(\rho)\Phi_2(\rho) - \Phi_1(\rho)u(\rho) \\ &\quad - \Phi_2(\sigma)u(\sigma) + \Phi_1(\sigma)\Phi_2(\sigma) - \Phi_1(\sigma)u(\sigma). \end{aligned} \quad (11)$$

Consider

$$\begin{aligned} G(x, \rho) &= \frac{x^{-\alpha-\beta}(x-\rho)^{\alpha-1}}{\Gamma(\alpha)} \\ &\quad \cdot {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\rho}{x}\right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\Gamma(\alpha)} \frac{(x-\rho)^{\alpha-1}}{x^{\alpha+\beta}} + \frac{(\alpha+\beta)(-\eta)(x-\rho)^\alpha}{\Gamma(\alpha+1)} \frac{1}{x^{\alpha+\beta+1}} \\
&\quad + \frac{(\alpha+\beta)(\alpha+\beta+1)(-\eta)(-\eta+1)(x-\rho)^{\alpha+1}}{\Gamma(\alpha+2)} \frac{1}{x^{\alpha+\beta+2}} \\
&\quad + \cdots, \quad (\rho \in (0, x); x > 0).
\end{aligned} \tag{12}$$

Clearly, function $G(x, \rho)$ positive since for all $\rho \in (0, x) (x > 0)$. Multiplying both sides of (11) by $G(x, \rho)$, then integrating resulting identity with respect to ρ from 0 to x , we get

$$\begin{aligned}
&(\Phi_2(\sigma) - u(\sigma)) \left(I_{0,x}^{\alpha,\beta,\eta} [u(x)] - I_{0,x}^{\alpha,\beta,\eta} [\Phi_1(x)] \right) \\
&\quad + \left(I_{0,x}^{\alpha,\beta,\eta} [\Phi_2(x)] - I_{0,x}^{\alpha,\beta,\eta} [u(x)] \right) (u(\sigma) - \Phi_1(\sigma)) \\
&\quad - I_{0,x}^{\alpha,\beta,\eta} [(\Phi_2(x) - u(x))(u(x) - \Phi_1(x))] \\
&\quad - \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^\beta} \\
&\quad \cdot (\Phi_2(\sigma) - u(\sigma))(u(\sigma) - \Phi_1(\sigma)) \\
&= I_{0,x}^{\alpha,\beta,\eta} [u^2(x)] + u^2(\sigma) \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^\beta} \\
&\quad - 2u(\sigma) I_{0,x}^{\alpha,\beta,\eta} [u(x)] + \Phi_2(\sigma) I_{0,x}^{\alpha,\beta,\eta} [u(x)] \\
&\quad + u(\sigma) I_{0,x}^{\alpha,\beta,\eta} [\Phi_1(x)] - \Phi_2(\sigma) I_{0,x}^{\alpha,\beta,\eta} [\Phi_1(x)] \\
&\quad + u(\sigma) I_{0,x}^{\alpha,\beta,\eta} [\Phi_2(x)] + \Phi_1(\sigma) I_{0,x}^{\alpha,\beta,\eta} [u(x)] \\
&\quad - \Phi_1(\sigma) I_{0,x}^{\alpha,\beta,\eta} [\Phi_2(x)] - I_{0,x}^{\alpha,\beta,\eta} [\Phi_2 u(x)] \\
&\quad + I_{0,x}^{\alpha,\beta,\eta} [\Phi_1 \Phi_2(x)] - I_{0,x}^{\alpha,\beta,\eta} [\Phi_1 u(x)] \\
&\quad - \Phi_2(\sigma) u(\sigma) \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^\beta} \\
&\quad + \Phi_1(\sigma) \Phi_2(\sigma) \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^\beta} \\
&\quad - \Phi_1(\sigma) u(\sigma) \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^\beta}.
\end{aligned} \tag{13}$$

Multiplying both sides of (13) by $G(x, \sigma)$ and is positive since for all $\sigma \in (0, x) (x > 0)$, and defined as in (12). Then integrating resulting identity with respect to σ from 0 to x , we have

$$\begin{aligned}
&(I_{0,x}^{\alpha,\beta,\eta} [\Phi_2(x)] - I_{0,x}^{\alpha,\beta,\eta} [u(x)]) \\
&\quad \cdot (I_{0,x}^{\alpha,\beta,\eta} [u(x)] - I_{0,x}^{\alpha,\beta,\eta} [\Phi_1(x)]) \\
&\quad + (I_{0,x}^{\alpha,\beta,\eta} [\Phi_2(x)] - I_{0,x}^{\alpha,\beta,\eta} [u(x)]) \\
&\quad \cdot (I_{0,x}^{\alpha,\beta,\eta} [u(x)] - I_{0,x}^{\alpha,\beta,\eta} [\Phi_1(x)])
\end{aligned}$$

$$\begin{aligned}
&- \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^\beta} I_{0,x}^{\alpha,\beta,\eta} \\
&\quad \cdot [(\Phi_2(x) - u(x))(u(x) - \Phi_1(x))] \\
&- \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^\beta} I_{0,x}^{\alpha,\beta,\eta} \\
&\quad \cdot [(\Phi_2(x) - u(x))(u(x) - \Phi_1(x))] \\
&= I_{0,x}^{\alpha,\beta,\eta} [u^2(x)] \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^\beta} \\
&\quad + I_{0,x}^{\alpha,\beta,\eta} [u^2(x)] \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^\beta} \\
&\quad - 2I_{0,x}^{\alpha,\beta,\eta} [u(x)] I_{0,x}^{\alpha,\beta,\eta} [u(x)] + I_{0,x}^{\alpha,\beta,\eta} [\Phi_2(x)] I_{0,x}^{\alpha,\beta,\eta} [u(x)] \\
&\quad + I_{0,x}^{\alpha,\beta,\eta} [u(x)] I_{0,x}^{\alpha,\beta,\eta} [\Phi_1(x)] \\
&\quad - I_{0,x}^{\alpha,\beta,\eta} [\Phi_2(x)] I_{0,x}^{\alpha,\beta,\eta} [\Phi_1(x)] \\
&\quad + I_{0,x}^{\alpha,\beta,\eta} [u(x)] I_{0,x}^{\alpha,\beta,\eta} [\Phi_2(x)] + I_{0,x}^{\alpha,\beta,\eta} [\Phi_1(x)] I_{0,x}^{\alpha,\beta,\eta} [u(x)] \\
&\quad - I_{0,x}^{\alpha,\beta,\eta} [\Phi_1(x)] I_{0,x}^{\alpha,\beta,\eta} [\Phi_2(x)] \\
&\quad - I_{0,x}^{\alpha,\beta,\eta} [\Phi_2 u(x)] \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^\beta} \\
&\quad + I_{0,x}^{\alpha,\beta,\eta} [\Phi_1 \Phi_2(x)] \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^\beta} \\
&\quad - I_{0,x}^{\alpha,\beta,\eta} [\Phi_1 u(x)] \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^\beta} \\
&\quad - I_{0,x}^{\alpha,\beta,\eta} [\Phi_2 u(x)] \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^\beta} \\
&\quad + I_{0,x}^{\alpha,\beta,\eta} [\Phi_1 \Phi_2(x)] \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^\beta} \\
&\quad - I_{0,x}^{\alpha,\beta,\eta} [\Phi_1 u(x)] \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^\beta}.
\end{aligned}$$

(14)

It follows that

$$\begin{aligned}
&2(I_{0,x}^{\alpha,\beta,\eta} [\Phi_2(x)] - I_{0,x}^{\alpha,\beta,\eta} [u(x)]) \\
&\quad \cdot (I_{0,x}^{\alpha,\beta,\eta} [u(x)] - I_{0,x}^{\alpha,\beta,\eta} [\Phi_1(x)]) \\
&\quad - 2 \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^\beta} I_{0,x}^{\alpha,\beta,\eta} \\
&\quad \cdot [(\Phi_2(x) - u(x))(u(x) - \Phi_1(x))]
\end{aligned}$$

$$\begin{aligned}
&= 2I_{0,x}^{\alpha,\beta,\eta} [u^2(x)] \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^\beta} \\
&\quad - 2(I_{0,x}^{\alpha,\beta,\eta} [u(x)])^2 + 2I_{0,x}^{\alpha,\beta,\eta} [\Phi_2(x)] I_{0,x}^{\alpha,\beta,\eta} [u(x)] \\
&\quad + 2I_{0,x}^{\alpha,\beta,\eta} [u(x)] I_{0,x}^{\alpha,\beta,\eta} [\Phi_1(x)] \\
&\quad - 2I_{0,x}^{\alpha,\beta,\eta} [\Phi_2(x)] I_{0,x}^{\alpha,\beta,\eta} [\Phi_1(x)] \\
&\quad - 2I_{0,x}^{\alpha,\beta,\eta} [\Phi_2 u(x)] \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^\beta} \\
&\quad + 2I_{0,x}^{\alpha,\beta,\eta} [\Phi_1 \Phi_2(x)] \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^\beta} \\
&\quad - 2I_{0,x}^{\alpha,\beta,\eta} [\Phi_1 u(x)] \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^\beta}, \tag{15}
\end{aligned}$$

which gives (10). $\square$

If $\Phi_1(x) = p$ and $\Phi_2(x) = P$, $p, P \in \mathbb{R}$ for all $x \in [0, \infty[$, then inequality (10) reduces to following lemma.

Lemma 6. Let $p, P \in \mathbb{R}$, and $u(x)$ be an integrable function on $[0, \infty)$ satisfying the condition $p \leq u(x) \leq P$. Then for all $x > 0$, $\alpha > \max\{0, -\beta\}$, $\beta < 1$, $\beta - 1 < \eta < 0$, one has

$$\begin{aligned}
&\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^\beta} I_{0,x}^{\alpha,\beta,\eta} [u^2(x)] - (I_{0,x}^{\alpha,\beta,\eta} [u(x)])^2 \\
&= \left(P \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^\beta} - I_{0,x}^{\alpha,\beta,\eta} [u(x)] \right) \\
&\quad \cdot \left(I_{0,x}^{\alpha,\beta,\eta} [u(x) - p] \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^\beta} \right) \\
&\quad - \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^\beta} \\
&\quad \times I_{0,x}^{\alpha,\beta,\eta} [(P - u(x))(u(x) - p)]. \tag{16}
\end{aligned}$$

Theorem 7. Let u and v be two integrable functions on $[0, \infty)$, and $\Phi_1(x)$, $\Phi_2(x)$, $\Psi_1(x)$, and $\Psi_2(x)$ are four integrable functions on $[0, \infty)$ satisfying the conditions A_1 and A_2 on $[0, \infty)$. Then for all $x > 0$, $\alpha > \max\{0, -\beta\}$, $\beta < 1$, $\beta - 1 < \eta < 0$, one has

$$\begin{aligned}
&\left| \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^\beta} I_{0,x}^{\alpha,\beta,\eta} [uv(x)] \right. \\
&\quad \left. - I_{0,x}^{\alpha,\beta,\eta} [u(x)] I_{0,x}^{\alpha,\beta,\eta} [v(x)] \right| \tag{17} \\
&\leq \sqrt{T(u, \Phi_1(x), \Phi_2(x)) T(v, \Psi_1(x), \Psi_2(x))},
\end{aligned}$$

where $T(a, b, c)$ is defined by

$$\begin{aligned}
&T(a, b, c) \\
&= (I_{0,x}^{\alpha,\beta,\eta} [c(x)] - I_{0,x}^{\alpha,\beta,\eta} [a(x)])(I_{0,x}^{\alpha,\beta,\eta} [a(x)] - I_{0,x}^{\alpha,\beta,\eta} [b(x)]) \\
&\quad + \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^\beta} I_{0,x}^{\alpha,\beta,\eta} [ba(x)] \\
&\quad - I_{0,x}^{\alpha,\beta,\eta} [b(x)] I_{0,x}^{\alpha,\beta,\eta} [a(x)] \\
&\quad + \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^\beta} I_{0,x}^{\alpha,\beta,\eta} [ca(x)] \\
&\quad - I_{0,x}^{\alpha,\beta,\eta} [c(x)] I_{0,x}^{\alpha,\beta,\eta} [a(x)] + I_{0,x}^{\alpha,\beta,\eta} [b(x)] I_{0,x}^{\alpha,\beta,\eta} [c(x)] \\
&\quad + \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^\beta} I_{0,x}^{\alpha,\beta,\eta} [bc(x)]. \tag{18}
\end{aligned}$$

Proof. Let u and v be two functions defined on $[0, \infty[$ satisfying the conditions A_1 and A_2 . Define

$$\begin{aligned}
&H(\rho, \sigma) := (u(\rho) - u(\sigma))(v(\rho) - v(\sigma)); \\
&\rho, \sigma \in (0, x), \quad x > 0. \tag{19}
\end{aligned}$$

It follows that

$$\begin{aligned}
&H(\rho, \sigma) := u(\rho)v(\rho) - u(\rho)v(\sigma) - u(\sigma)v(\rho) \\
&\quad + u(\sigma)v(\sigma). \tag{20}
\end{aligned}$$

Then, multiplying both sides of (20) by $G(x, \rho)$ which is defined by (12) and is positive because for all $\rho \in (0, x)$ ($x > 0$). Then integrating resulting identity with respect to ρ from 0 to x , we have

$$\begin{aligned}
&\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-\rho)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\rho}{x}\right) H(\rho, \sigma) d\rho \\
&= I_{0,x}^{\alpha,\beta,\eta} [uv(x)] - u(\sigma) I_{0,x}^{\alpha,\beta,\eta} [u(x)] \\
&\quad - u(\sigma) I_{0,x}^{\alpha,\beta,\eta} [v(x)] + u(\sigma)v(\sigma) I_{0,x}^{\alpha,\beta,\eta}. \tag{21}
\end{aligned}$$

Again, multiplying both sides of (21) by $G(x, \sigma)$ which is defined by (12) and is positive because for all $\sigma \in (0, x)$ ($x > 0$) since each term of (12) is positive. Then integrating resulting identity with respect to σ from 0 to x , we have

$$\begin{aligned}
&\frac{x^{-\alpha-\beta} x^{-\alpha-\beta}}{\Gamma^2(\alpha)} \\
&\quad \cdot \int_0^x \int_0^x (x-\rho)^{\alpha-1} (x-\sigma)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\rho}{x}\right) \\
&\quad \cdot {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\sigma}{x}\right) H(\rho, \sigma) d\rho d\sigma
\end{aligned}$$

$$= 2 \left(\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} x^\beta I_{0,x}^{\alpha,\beta,\eta} [uv(x)] - I_{0,x}^{\alpha,\beta,\eta} [u(x)] I_{0,x}^{\alpha,\beta,\eta} [v(x)] \right). \quad (22)$$

Applying the Cauchy-Schwarz inequality to (22), we have

$$\begin{aligned} & \left(\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} x^\beta I_{0,x}^{\alpha,\beta,\eta} I_{0,x}^{\alpha,\beta,\eta} [uv(x)] - I_{0,x}^{\alpha,\beta,\eta} [u(x)] I_{0,x}^{\alpha,\beta,\eta} [v(x)] \right)^2 \\ & \leq \left(\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} x^\beta I_{0,x}^{\alpha,\beta,\eta} I_{0,x}^{\alpha,\beta,\eta} [u^2(x)] - (I_{0,x}^{\alpha,\beta,\eta} [u(x)])^2 \right) \\ & \quad \times \left(\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} x^\beta I_{0,x}^{\alpha,\beta,\eta} I_{0,x}^{\alpha,\beta,\eta} [v^2(x)] - (I_{0,x}^{\alpha,\beta,\eta} [v(x)])^2 \right). \end{aligned} \quad (23)$$

Since $(\Phi_2(x) - u(t))(u(t) - \Phi_1(x)) \geq 0$ and $(\Psi_2(x) - v(t))(v(t) - \Psi_1(x)) \geq 0$, we have

$$\begin{aligned} & \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} x^\beta I_{0,x}^{\alpha,\beta,\eta} \\ & \cdot [(\Phi_2(x) - u(t))(u(t) - \Phi_1(x))] \geq 0, \\ & \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} x^\beta I_{0,x}^{\alpha,\beta,\eta} \\ & \cdot [(\Psi_2(x) - v(t))(v(t) - \Psi_1(x))] \geq 0. \end{aligned} \quad (24)$$

Thus we have

$$\begin{aligned} & \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} x^\beta I_{0,x}^{\alpha,\beta,\eta} [u^2(x)] - (I_{0,x}^{\alpha,\beta,\eta} [u(x)])^2 \\ & \leq (I_{0,x}^{\alpha,\beta,\eta} [\Phi_2(x)] - I_{0,x}^{\alpha,\beta,\eta} [u(x)]) \\ & \quad \cdot (I_{0,x}^{\alpha,\beta,\eta} [u(x)] - I_{0,x}^{\alpha,\beta,\eta} [\Phi_1(x)]) \\ & \quad + \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} x^\beta I_{0,x}^{\alpha,\beta,\eta} [\Phi_1 u(x)] \\ & \quad - I_{0,x}^{\alpha,\beta,\eta} [\Phi_1(x)] I_{0,x}^{\alpha,\beta,\eta} [u(x)] \end{aligned}$$

$$\begin{aligned} & + \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} x^\beta I_{0,x}^{\alpha,\beta,\eta} [\Phi_2 u(x)] \\ & \quad - I_{0,x}^{\alpha,\beta,\eta} [\Phi_2(x)] I_{0,x}^{\alpha,\beta,\eta} [u(x)] \\ & \quad + I_{0,x}^{\alpha,\beta,\eta} [\Phi_1(x)] I_{0,x}^{\alpha,\beta,\eta} [\Phi_2(x)] \\ & \quad - \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} x^\beta I_{0,x}^{\alpha,\beta,\eta} [\Phi_1 \Phi_2(x)] \\ & = T(u, \Phi_1, \Phi_2), \\ & \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} x^\beta I_{0,x}^{\alpha,\beta,\eta} [u^2(x)] - (I_{0,x}^{\alpha,\beta,\eta} [u(x)])^2 \\ & \leq (I_{0,x}^{\alpha,\beta,\eta} [\Psi_2(x)] - I_{0,x}^{\alpha,\beta,\eta} [u(x)]) \\ & \quad \cdot (I_{0,x}^{\alpha,\beta,\eta} [u(x)] - I_{0,x}^{\alpha,\beta,\eta} [\Psi_1(x)]) \\ & \quad + \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} x^\beta I_{0,x}^{\alpha,\beta,\eta} [\Psi_1 u(x)] \\ & \quad - I_{0,x}^{\alpha,\beta,\eta} [\Psi_1(x)] I_{0,x}^{\alpha,\beta,\eta} [u(x)] \\ & \quad + \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} x^\beta I_{0,x}^{\alpha,\beta,\eta} [\Psi_2 u(x)] \\ & \quad - I_{0,x}^{\alpha,\beta,\eta} [\Psi_2(x)] I_{0,x}^{\alpha,\beta,\eta} [u(x)] \\ & \quad + I_{0,x}^{\alpha,\beta,\eta} [\Psi_1(x)] I_{0,x}^{\alpha,\beta,\eta} [\Psi_2(x)] \\ & \quad - \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} x^\beta I_{0,x}^{\alpha,\beta,\eta} [\Psi_1 \Psi_2(x)] \\ & = T(u, \Psi_1, \Psi_2). \end{aligned} \quad (25)$$

Combining (23) and (25) we get the required inequality (17). $\square$

Theorem 8. Suppose that u is an integrable function defined on $[0, \infty[$ and condition (A_1) holds. Then for all $x > 0$, $\alpha > \max\{0, -\beta\}$, $\psi > \max\{0, -\phi\}$, $\beta < 1$, $\beta - 1 < \eta < 0$, $\phi < 1$, $\phi - 1 < \zeta < 0$,

$$\begin{aligned} & I_{0,x}^{\psi,\phi,\zeta} [\Phi_1(x)] I_{0,x}^{\alpha,\beta,\eta} [u(x)] \\ & \quad + I_{0,x}^{\alpha,\beta,\eta} [\Phi_2(x)] I_{0,x}^{\psi,\phi,\zeta} [u(x)] \\ & \geq I_{0,x}^{\alpha,\beta,\eta} [\Phi_2(x)] I_{0,x}^{\psi,\phi,\zeta} [\Phi_1(x)] \\ & \quad + I_{0,x}^{\alpha,\beta,\eta} [u(x)] I_{0,x}^{\psi,\phi,\zeta} [u(x)]. \end{aligned} \quad (26)$$

Proof. From the condition (A_1) , for all $\rho, \sigma \geq 0$, we have

$$(\Phi_2(\rho) - u(\rho))(u(\sigma) - \Phi_1(\sigma)) \geq 0,$$

$$\Phi_2(\rho) u(\sigma) - \Phi_2(\rho) \Phi_1(\sigma) - u(\rho) u(\sigma) + u(\rho) \Phi_1(\sigma) \geq 0, \quad (27)$$

which implies that

$$\begin{aligned} & \Phi_2(\rho) u(\sigma) + u(\rho) \Phi_1(\sigma) \\ & \geq \Phi_2(\rho) \Phi_1(\sigma) + u(\rho) u(\sigma). \end{aligned} \quad (28)$$

Multiplying both sides of (28) by $G(x, \rho)$ which is defined by (12) and remain positive because for all $\rho \in (0, x)$ ($x > 0$) since each term of (12) is positive. Then integrating resulting identity with respect to ρ from 0 to x , we get

$$\begin{aligned} & u(\sigma) \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \\ & \cdot \int_0^x (x-\rho)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\rho}{x}\right) \Phi_2(\rho) d\rho \\ & + \Phi_1(\sigma) \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \\ & \cdot \int_0^x (x-\rho)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\rho}{x}\right) u(\rho) d\rho \\ & \geq \Phi_1(\sigma) \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \\ & \cdot \int_0^x (x-\rho)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\rho}{x}\right) \Phi_2(\rho) d\rho \\ & \cdot u(\sigma) \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \\ & \cdot \int_0^x (x-\rho)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\rho}{x}\right) u(\rho) d\rho. \end{aligned} \quad (29)$$

Therefore,

$$\begin{aligned} & u(\sigma) I_{0,x}^{\alpha,\beta,\eta} [\Phi_2(x)] + \Phi_1(\sigma) I_{0,x}^{\alpha,\beta,\eta} [u(x)] \\ & \geq \Phi_1(\sigma) I_{0,x}^{\alpha,\beta,\eta} [\Phi_2(x)] + u(\sigma) I_{0,x}^{\alpha,\beta,\eta} [u(x)]. \end{aligned} \quad (30)$$

Multiplying (30) by $(x^{-\psi-\phi}/\Gamma(\psi))(x-\sigma)^{\psi-1} {}_2F_1(\psi+\phi, -\zeta; \psi; 1-\sigma/x)$ ($\sigma \in (0, x)$, $x > 0$), which remains positive. Then integrating the resulting identity with respect to σ from 0 to x , we have

$$\begin{aligned} & I_{0,x}^{\alpha,\beta,\eta} [\Phi_2(x)] \frac{x^{-\psi-\phi}}{\Gamma(\psi)} \\ & \cdot \int_0^x (x-\sigma)^{\alpha-1} {}_2F_1\left(\psi+\phi, -\zeta; \psi; 1-\frac{\sigma}{x}\right) u(\sigma) d\sigma \\ & \cdot I_{0,x}^{\alpha,\beta,\eta} [u(x)] \frac{x^{-\psi-\phi}}{\Gamma(\psi)} \\ & \cdot \int_0^x (x-\sigma)^{\alpha-1} {}_2F_1\left(\psi+\phi, -\zeta; \psi; 1-\frac{\sigma}{x}\right) \Phi_1(\sigma) d\sigma \end{aligned}$$

$$\begin{aligned} & \geq I_{0,x}^{\alpha,\beta,\eta} [\Phi_2(x)] \frac{x^{-\psi-\phi}}{\Gamma(\psi)} \\ & \cdot \int_0^x (x-\sigma)^{\alpha-1} {}_2F_1\left(\psi+\phi, -\zeta; \psi; 1-\frac{\sigma}{x}\right) \Phi_1(\sigma) d\sigma \\ & \cdot I_{0,x}^{\alpha,\beta,\eta} [u(x)] \frac{x^{-\psi-\phi}}{\Gamma(\psi)} \\ & \cdot \int_0^x (x-\sigma)^{\alpha-1} {}_2F_1\left(\psi+\phi, -\zeta; \psi; 1-\frac{\sigma}{x}\right) u(\sigma) d\sigma, \end{aligned} \quad (31)$$

which gives (26) and proves the theorem. $\square$

Remark 9. Suppose that u is an integrable function defined on $[0, \infty[$, such that $p \leq u(x) \leq P$, for all $x \in [0, \infty[$ and $p, P \in \mathbb{R}$. Then for all $x > 0$, $\alpha > \max\{0, -\beta\}$, $\psi > \max\{0, -\phi\}$, $\beta < 1$, $\beta-1 < \eta < 0$, $\phi < 1$, $\phi-1 < \zeta < 0$, we have

$$\begin{aligned} & p \frac{\Gamma(1-\phi+\zeta)}{\Gamma(1-\phi)\Gamma(1+\psi+\zeta)} x^\phi I_{0,x}^{\alpha,\beta,\eta} u(x) \\ & + P \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} x^\phi I_{0,x}^{\psi,\phi,\zeta} [u(x)] \\ & \geq Pp \frac{\Gamma(1-\beta+\eta)\Gamma(1-\phi+\zeta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)\Gamma(1-\phi)\Gamma(1+\psi+\zeta)} x^\beta x^\phi \\ & + I_{0,x}^{\alpha,\beta,\eta} [u(x)] I_{0,x}^{\psi,\phi,\zeta} [u(x)]. \end{aligned} \quad (32)$$

Theorem 10. Suppose that u and v are two integrable functions defined on $[0, \infty[$ and suppose that (A_1) and (A_2) hold. Then for all $x > 0$, $\alpha > \max\{0, -\beta\}$, $\psi > \max\{0, -\phi\}$, $\beta < 1$, $\beta-1 < \eta < 0$, $\phi < 1$, $\phi-1 < \zeta < 0$, the following inequalities are satisfied:

$$\begin{aligned} (a) \quad & I_{0,x}^{\psi,\phi,\zeta} [\Psi_1(x)] I_{0,x}^{\alpha,\beta,\eta} [u(x)] + I_{0,x}^{\alpha,\beta,\eta} [\Phi_2(x)] I_{0,x}^{\psi,\phi,\zeta} [v(x)] \\ & \geq I_{0,x}^{\psi,\phi,\zeta} [\Psi_1(x)] I_{0,x}^{\alpha,\beta,\eta} [\Phi_2(x)] \\ & + I_{0,x}^{\alpha,\beta,\eta} [u(x)] I_{0,x}^{\psi,\phi,\zeta} [v(x)]. \\ (b) \quad & I_{0,x}^{\psi,\phi,\zeta} [\Phi_1(x)] I_{0,x}^{\alpha,\beta,\eta} [v(x)] + I_{0,x}^{\alpha,\beta,\eta} [\Psi_2(x)] I_{0,x}^{\psi,\phi,\zeta} [u(x)] \\ & \geq I_{0,x}^{\psi,\phi,\zeta} [\Phi_1(x)] I_{0,x}^{\alpha,\beta,\eta} [\Psi_2(x)] \\ & + I_{0,x}^{\psi,\phi,\zeta} [u(x)] I_{0,x}^{\alpha,\beta,\eta} [v(x)]. \\ (c) \quad & I_{0,x}^{\alpha,\beta,\eta} [\Phi_2(x)] I_{0,x}^{\psi,\phi,\zeta} [\Psi_2(x)] + I_{0,x}^{\alpha,\beta,\eta} [u(x)] I_{0,x}^{\psi,\phi,\zeta} [v(x)] \\ & \geq I_{0,x}^{\alpha,\beta,\eta} [\Phi_2(x)] I_{0,x}^{\psi,\phi,\zeta} [v(x)] \\ & + I_{0,x}^{\psi,\phi,\zeta} [\Psi_2(x)] I_{0,x}^{\alpha,\beta,\eta} [u(x)]. \end{aligned}$$

$$\begin{aligned}
(d) \quad & I_{0,x}^{\alpha,\beta,\eta} [\Phi_1(x)] I_{0,x}^{\psi,\phi,\zeta} [\Psi_1(x)] + I_{0,x}^{\alpha,\beta,\eta} u(x) I_{0,x}^{\psi,\phi,\zeta} [v(x)] \\
& \geq I_{0,x}^{\alpha,\beta,\eta} [\Phi_1(x)] I_{0,x}^{\psi,\phi,\zeta} [v(x)] \\
& \quad + I_{0,x}^{\psi,\phi,\zeta} [\Psi_1(x)] I_{0,x}^{\alpha,\beta,\eta} [u(x)].
\end{aligned} \quad (33)$$

Proof. To show (a), we use conditions (A₁) and (A₂), for all $x \in [0, \infty[$; we have

$$(\Phi_2(\rho) - u(\rho))(v(\sigma) - \Psi_1(\sigma)) \geq 0, \quad (34)$$

which implies that

$$\begin{aligned}
& \Phi_2(\rho) v(\sigma) + u(\rho) \Psi_1(\sigma) \\
& \geq \Phi_2(\rho) \Psi_1(\sigma) + u(\rho) u(\sigma).
\end{aligned} \quad (35)$$

Multiplying both sides of (35) by $G(x, \rho)$, which is defined by (12) and remain positive because for all $\rho \in (0, x)$ ($x > 0$), since each term of (12) is positive. Then integrating the resulting identity with respect to ρ from 0 to x , we get

$$\begin{aligned}
& v(\sigma) \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \\
& \cdot \int_0^x (x-\rho)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\rho}{x}\right) \Phi_2(\rho) d\rho \\
& + \Psi_1(\sigma) \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \\
& \cdot \int_0^x (x-\rho)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\rho}{x}\right) u(\rho) d\rho \\
& \geq \Psi_1(\sigma) \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \\
& \cdot \int_0^x (x-\rho)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\rho}{x}\right) \Phi_2(\rho) d\rho \\
& \cdot v(\sigma) \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \\
& \cdot \int_0^x (x-\rho)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\rho}{x}\right) u(\rho) d\rho.
\end{aligned} \quad (36)$$

Therefore,

$$\begin{aligned}
& v(\sigma) I_{0,x}^{\alpha,\beta,\eta} \Phi_2(x) + \Psi_1(\sigma) I_{0,x}^{\alpha,\beta,\eta} u(x) \\
& \geq \Psi_1(\sigma) I_{0,x}^{\alpha,\beta,\eta} \Phi_2(x) + v(\sigma) I_{0,x}^{\alpha,\beta,\eta} u(x).
\end{aligned} \quad (37)$$

Multiplying (37) by $(x^{-\psi-\phi}/\Gamma(\psi))(x-\sigma)^{\psi-1} {}_2F_1(\psi+\phi, -\zeta; \psi; 1-\sigma/x)$ ($\sigma \in (0, x)$, $x > 0$), which remains positive.

Then by integrating the resulting identity with respect to σ from 0 to x , we have

$$\begin{aligned}
& I_{0,x}^{\alpha,\beta,\eta} [\Phi_2(x)] \frac{x^{-\psi-\phi}}{\Gamma(\psi)} \\
& \cdot \int_0^x (x-\sigma)^{\alpha-1} {}_2F_1\left(\psi+\phi, -\zeta; \psi; 1-\frac{\sigma}{x}\right) v(\sigma) d\sigma \\
& \cdot I_{0,x}^{\alpha,\beta,\eta} [u(x)] \frac{x^{-\psi-\phi}}{\Gamma(\psi)} \\
& \cdot \int_0^x (x-\sigma)^{\alpha-1} {}_2F_1\left(\psi+\phi, -\zeta; \psi; 1-\frac{\sigma}{x}\right) \Psi_1(\sigma) d\sigma \\
& \geq I_{0,x}^{\alpha,\beta,\eta} [\Phi_2(x)] \frac{x^{-\psi-\phi}}{\Gamma(\psi)} \\
& \cdot \int_0^x (x-\sigma)^{\alpha-1} {}_2F_1\left(\psi+\phi, -\zeta; \psi; 1-\frac{\sigma}{x}\right) \Psi_1(\sigma) d\sigma \\
& \cdot I_{0,x}^{\alpha,\beta,\eta} [u(x)] \frac{x^{-\psi-\phi}}{\Gamma(\psi)} \\
& \cdot \int_0^x (x-\sigma)^{\alpha-1} {}_2F_1\left(\psi+\phi, -\zeta; \psi; 1-\frac{\sigma}{x}\right) v(\sigma) d\sigma.
\end{aligned} \quad (38)$$

This gives the desired inequality (a). Using the following condition we prove (b)–(d):

$$\begin{aligned}
& (\Psi_2(\rho) - v(\rho))(u(\sigma) - \Phi_1(\sigma)) \geq 0, \\
& (\Phi_2(\rho) - u(\rho))(v(\sigma) - \Psi_2(\sigma)) \leq 0, \\
& (\Phi_2(\rho) - u(\rho))(v(\sigma) - \Psi_1(\sigma)) \leq 0.
\end{aligned} \quad (39)$$

□

Remark 11. Suppose that u and v are two integrable functions defined on $[0, \infty[$. Assume that there exist real constants P, p, Q , and q such that

$$\begin{aligned}
& p \leq u(x) \leq P, \\
& q \leq v(x) \leq Q, \\
& \forall x \in [0, \infty[.
\end{aligned} \quad (40)$$

Then for all $x > 0$, $\alpha > \max\{0, -\beta\}$, $\psi > \max\{0, -\phi\}$, $\beta < 1$, $\beta - 1 < \eta < 0$, $\phi < 1$, $\phi - 1 < \zeta < 0$, the following inequalities are satisfied:

$$\begin{aligned}
& (a_0) \quad q \frac{\Gamma(1-\phi+\zeta)}{\Gamma(1-\phi)\Gamma(1+\psi+\zeta)} x^\phi I_{0,x}^{\alpha,\beta,\eta} [u(x)] \\
& \quad + P I_{0,x}^{\alpha,\beta,\eta} I_{0,x}^{\psi,\phi,\zeta} [v(x)] \\
& \geq q P \frac{\Gamma(1-\beta+\eta)\Gamma(1-\phi+\zeta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)\Gamma(1-\phi)\Gamma(1+\psi+\zeta)} x^\beta x^\phi \\
& \quad + I_{0,x}^{\alpha,\beta,\eta} [u(x)] I_{0,x}^{\psi,\phi,\zeta} [v(x)].
\end{aligned}$$

$$\begin{aligned}
(b_0) \quad & P \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} I_{0,x}^{\alpha,\beta,\eta} [v(x)] \\
& + Q \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} I_{0,x}^{\psi,\phi,\zeta} [u(x)] \\
& \geq PQ \frac{\Gamma(1-\beta+\eta)\Gamma(1-\phi+\zeta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)\Gamma(1-\phi)\Gamma(1+\psi+\zeta)} x^\beta x^\phi \\
& + I_{0,x}^{\psi,\phi,\zeta} [u(x)] I_{0,x}^{\alpha,\beta,\eta} [v(x)]. \\
(c_0) \quad & PQ \frac{\Gamma(1-\beta+\eta)\Gamma(1-\phi+\zeta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)\Gamma(1-\phi)\Gamma(1+\psi+\zeta)} x^\beta x^\phi \\
& + I_{0,x}^{\alpha,\beta,\eta} u(x) I_{0,x}^{\psi,\phi,\zeta} [v(x)] \\
& \geq P \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} x^\beta I_{0,x}^{\psi,\phi,\zeta} [v(x)] \\
& + P \frac{\Gamma(1-\phi+\zeta)}{\Gamma(1-\phi)\Gamma(1+\psi+\zeta)} x^\phi I_{0,x}^{\alpha,\beta,\eta} [u(x)]. \\
(d_0) \quad & pq \frac{\Gamma(1-\phi+\zeta)}{\Gamma(1-\phi)\Gamma(1+\psi+\zeta)} x^\phi \\
& + I_{0,x}^{\alpha,\beta,\eta} [u(x)] I_{0,x}^{\psi,\phi,\zeta} [v(x)] \\
& \geq P \frac{\Gamma(1-\phi+\zeta)}{\Gamma(1-\phi)\Gamma(1+\psi+\zeta)} x^\phi I_{0,x}^{\psi,\phi,\zeta} [v(x)] \\
& + q \frac{\Gamma(1-\phi+\zeta)}{\Gamma(1-\phi)\Gamma(1+\psi+\zeta)} x^\phi I_{0,x}^{\alpha,\beta,\eta} [u(x)].
\end{aligned} \tag{41}$$

It is noted that the results give some contributions to the theory of integral inequalities and fractional calculus. Moreover, they are expected to lead to some applications for establishing uniqueness of solutions in fractional differential equations.

Conflict of Interests

The authors declare that they have no conflict of interests.

Authors' Contribution

All authors have equal contributions. All authors read and approved the final paper.

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