Research Article

Periodic Solutions for a Class of Singular Hamiltonian Systems on Time Scales

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We are concerned with a class of singular Hamiltonian systems on time scales. Some results on the existence of periodic solutions are obtained for the system under consideration by means of the variational methods and the critical point theory.

1. Introduction

In recent years, dynamic equations on time scales have been studied intensively in the literature [1–7]. Some ideas and methods have been developed to study the existence and multiplicity of solutions for dynamic equations on time scales, for example, the fixed point theory, the method of the upper and lower solutions, the coincidence degree theory, and so on.

However, not much work has been seen on the existence of solutions to dynamic equations on time scales through the variational method and the critical point theory; for details see [4–10] and the references therein. For example, authors of [11] give some results on the existence and multiplicity of periodic solutions which are obtained for the Hamiltonian system by means of the saddle point theorem, the least action principle, and the three-critical-point theorem. To the best of our knowledge, it is still worth making an attempt to extend variational methods to study the existence of periodic solutions for various Hamiltonian systems. Naturally, it is interesting and necessary to study the existence of periodic solutions for Hamiltonian systems on time scales.

Besides, in [12], using Lyusternik-Schnirelmann theory with classical $(PS)^+$ compact condition, Ambrosetti-Coti Zelati studied the periodic solutions of a fixed energy $h \in \mathbb{R}$ for Hamiltonian systems with singular potential $V \in C^2(\mathbb{R}^n \setminus \{0, \mathbb{R}\})$:

$$\ddot{q} + V'(q) = 0,$$

$$\frac{1}{2} |\dot{q}|^2 + V(q) = h. \quad (1)$$

After Ambrosetti-Coti Zelati, a lot of mathematicians studied singular Hamiltonian systems (see, e.g., [13–15]).

Motivated by the above, in this paper, we consider the following second order Hamiltonian system with a fixed energy h on time scale $\mathbb{T}$:

$$q^{\Delta^2}(t) + \nabla V(\sigma(t), q^\sigma(t)) = 0, \quad (2)$$

$$\frac{1}{2} |q^\Delta(t)|^2 + V(\sigma(t), q^\sigma(t)) = h, \quad (3)$$

where $q^\Delta(t)$ denotes the delta (or Hilger) derivative of q at t , $q^{\Delta^2}(t) = (q^\Delta)^\Delta(t)$, σ is the forward jump operator, $h \in \mathbb{R}$, and $V: \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies the following assumption:

(H_0) $V(t, x)$ is measurable in t for every $x \in \mathbb{R}^n$ and continuously differentiable in x for $t \in \mathbb{T}$.

The paper is organized as follows. In Section 2, we introduce some definitions and make some preparations for later sections. We summarize our main results on the existence of periodic solutions of the second order Hamiltonian system on time scales in Section 3.

2. Preliminaries

In this section, we will first recall some fundamental definitions and lemmas which are used in what follows.

Definition 1 (see [3]). A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real set $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$. The forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ is defined by $\sigma(t) = \inf\{s \in \mathbb{T}, s > t\}$ for all $t \in \mathbb{T}$, while the backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$ is defined by $\rho(t) = \sup\{s \in \mathbb{T}, s < t\}$ for all $t \in \mathbb{T}$. Finally, the graininess function $\mu : \mathbb{T} \rightarrow [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$.

Definition 2 (see [3]). Assume that $f : \mathbb{T} \rightarrow \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\begin{aligned} & \left| [f(\sigma(t)) - f(s)] - f^\Delta(t) [\sigma(t) - s] \right| \\ & \leq \epsilon |\sigma(t) - s| \quad \forall s \in U. \end{aligned} \quad (4)$$

We call $f^\Delta(t)$ the delta (or Hilger) derivative of f at t . The function f is delta (or Hilger) differentiable on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The function $f^\Delta : \mathbb{T}^\kappa \rightarrow \mathbb{R}$ is then called the delta derivative of f on $\mathbb{T}^\kappa$. Then we define the function $f^\sigma : \mathbb{T}^\kappa \rightarrow \mathbb{R}$ by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}^\kappa$.

Definition 3 (see [3]). For a function $f : \mathbb{T} \rightarrow \mathbb{R}$ we will talk about the second derivative f^{Δ^2} provided f^Δ is differentiable on $\mathbb{T}^{\kappa^2} = (\mathbb{T}^\kappa)^\kappa$ with derivative $f^{\Delta^2} = (f^\Delta)^\Delta : \mathbb{T}^{\kappa^2} \rightarrow \mathbb{R}$.

Definition 4 (see [3]). A function $f : \mathbb{T} \rightarrow \mathbb{R}^N$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.

Lemma 5 (see [3]). A function $f : [a, b]_\mathbb{T} \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]_\mathbb{T}$ if and only if f is delta differentiable Δ -a.e. on $[a, b]_\mathbb{T}$ and

$$f(t) = f(a) + \int_{[a,b]_\mathbb{T}} f^\Delta(s) \Delta s, \quad \forall t \in [a, b]_\mathbb{T}. \quad (5)$$

Lemma 6 (see [3]). Assume the functions $f, g : [a, b]_\mathbb{T} \rightarrow \mathbb{R}$ are absolutely continuous on $[a, b]_\mathbb{T}$; then fg is absolutely continuous on $[a, b]_\mathbb{T}$ and the following equality is valid:

$$\begin{aligned} \int_{[a,b]_\mathbb{T}} (f^\Delta g + f^\sigma g^\Delta)(t) \Delta t &= f(b)g(b) - f(a)g(a) \\ &= \int_{[a,b]_\mathbb{T}} (fg^\Delta + f^\Delta g^\sigma)(t) \Delta t. \end{aligned} \quad (6)$$

In the following, we adopt the notations used in [4].

Lemma 7 (see [4]). There exists $K > 0$ such that the inequality

$$\|u\|_\infty \leq K \|u\|_{W_{\Delta,T}^{1,p}} \quad (7)$$

holds for all $u \in W_{\Delta,T}^{1,p}([0, T]_\mathbb{T}, \mathbb{R}^N)$, where $\|u\|_\infty = \max_{t \in [0, T]_\mathbb{T}} |u(t)|$.

Moreover, if $\int_{[0,T]_\mathbb{T}} u(t) \Delta t = 0$, then

$$\|u\|_\infty \leq K \|u^\Delta\|_{L_\Delta^p}. \quad (8)$$

Lemma 8 (see [4]). If the sequence $\{u_k\}_{k \in \mathbb{N}} \subset W_{\Delta,T}^{1,p}([0, T]_\mathbb{T}, \mathbb{R}^N)$ converges weakly to u in $W_{\Delta,T}^{1,p}([0, T]_\mathbb{T}, \mathbb{R}^N)$, then $\{u_k\}_{k \in \mathbb{N}}$ converges strongly in $C([0, T]_\mathbb{T}, \mathbb{R}^N)$ to u .

Lemma 9 (see [4]). Assume that $p \geq 1$ ($p \in \overline{\mathbb{R}}$). Then, for every $q \in [1, +\infty)$ the immersion $W_\Delta^{1,p}([a, b]_\mathbb{T}, \mathbb{R}^n) \hookrightarrow L_\Delta^q([a, b]_\mathbb{T}, \mathbb{R}^n)$ is compact.

Lemma 10 (see [11]). If f is weakly lower semicontinuous on a reflexive Banach space X and has a bounded minimizing sequence, then f has a minimum on X .

3. Existence of Periodic Solutions

3.1. Variational Structure. In this section, we present a recent approach via variational methods and critical point theory to obtain the existence of periodic solutions for the second order Hamiltonian systems on time scale $\mathbb{T}$.

By making a variational structure on $H_{\Delta,T}^1$, we can reduce the problem of finding solutions of (2) to the one of seeking the critical points of a corresponding functional.

Let

$$\begin{aligned} H_{\Delta,T}^1 &= W_{\Delta,T}^{1,2}(\mathbb{T}, \mathbb{R}^n) \\ &= \left\{ u : \mathbb{T} \rightarrow \mathbb{R}^n \mid u \in L_\Delta^2(\mathbb{T}, \mathbb{R}^n), \right. \\ &\quad \left. u^\Delta \in L_\Delta^2(\mathbb{T}, \mathbb{R}^n), u(t+T) = u(t), \forall t \in \mathbb{T} \right\}. \end{aligned} \quad (9)$$

It is easy to see that $H_{\Delta,T}^1$ is a Hilbert space with the norm defined by

$$\begin{aligned} \|u\| &= \|u\|_{H_{\Delta,T}^1} \\ &= \left(\int_{[0,T]_\mathbb{T}} |u(t)|^2 \Delta t + \int_{[0,T]_\mathbb{T}} |u^\Delta(t)|^2 \Delta t \right)^{1/2}. \end{aligned} \quad (10)$$

In addition, let $L_\Delta^2(\mathbb{T}, \mathbb{R}^n)$ denote the Hilbert space of T -periodic functions on time scale $\mathbb{T}$ with values in $\mathbb{R}^n$, and the norm is defined by

$$\|u\|_{L_\Delta^2} = \left(\int_{[0,T]_\mathbb{T}} |u(t)|^2 \Delta t \right)^{1/2}. \quad (11)$$

Let

$$W = \{u \in H_{\Delta,T}^1 \mid u(t) \neq 0, \forall t \in \mathbb{T}\},$$

$$W_0 = \left\{ u \in H_{\Delta,T}^1 = W_{\Delta,T}^{1,2}(\mathbb{T}, \mathbb{R}^n), \right. \quad (12)$$

$$\left. u\left(t + \frac{T}{2}\right) = -u(t), u(t) \neq 0, \forall t \in \mathbb{T} \right\}.$$

We define the equivalent norm in $E = \{u \in H_{\Delta,T}^1 = W_{\Delta,T}^{1,2}(\mathbb{T}, \mathbb{R}^n), u(t + (T/2)) = -u(t), \forall t \in \mathbb{T}\}$ as follows:

$$\|u\| = \|u\|_E = \left(\int_{[0,T]_{\mathbb{T}}} |u^\Delta(t)|^2 \Delta t \right)^{1/2}. \quad (13)$$

Let

$$\begin{aligned} \tilde{u} &= u - \bar{u}, \\ \tilde{H}_{\Delta,T}^1 &= \{u \in H_{\Delta,T}^1 : \bar{u} = 0\}, \end{aligned} \quad (14)$$

where $\bar{u} = (1/T) \int_{[0,T]_{\mathbb{T}}} u(t) \Delta t$. It is easy to see that $H_{\Delta,T}^1 = \mathbb{R}^n \oplus \tilde{H}_{\Delta,T}^1$.

Lemma 11 (see [16]). *The system (2) will be said to satisfy the strong force (SF) condition if and only if there exists a neighborhood N of $\{0\}$ and a C^2 function U on $N - \{0\}$ such that*

- (i) $U(x) \rightarrow -\infty$ as $x \rightarrow 0$;
- (ii) $-V(x) \geq |\nabla U(x)|^2$ for all x in $N - \{0\}$.

Let

∂W

$$= \{u \in H_{\Delta,T}^1 = W_{\Delta,T}^{1,2}(\mathbb{T}, \mathbb{R}^n), \exists t_0 \in \mathbb{T}, u(t_0) = 0\}. \quad (15)$$

Then we have

$$\int_{[0,T]_{\mathbb{T}}} V(\sigma(t), u_n^\sigma(t)) \Delta t \rightarrow -\infty, \quad \forall u_n \rightarrow u \in \partial W. \quad (16)$$

Let

$$\begin{aligned} \partial W_0 &= \left\{ u \in H_{\Delta,T}^1 = W_{\Delta,T}^{1,2}(\mathbb{T}, \mathbb{R}^n), u\left(t + \frac{T}{2}\right) = -u(t), \right. \\ &\quad \left. \exists t_0 \in \mathbb{T}, u(t_0) = 0 \right\}. \end{aligned} \quad (17)$$

Then we have

$$\begin{aligned} \int_{[0,T]_{\mathbb{T}}} V(\sigma(t), u_n^\sigma(t)) \Delta t &\rightarrow -\infty, \\ \forall u_n &\rightarrow u \in \partial W_0. \end{aligned} \quad (18)$$

Lemma 12. *The existence of a bounded minimizing sequence is insured when f is coercive in space $\tilde{H}_{\Delta,T}^1$.*

Proof. Since f is coercive, $\lim_{\|u\| \rightarrow \infty} f(u) = +\infty$. So we can choose a $u_0 \in \tilde{H}_{\Delta,T}^1$ and a positive number $\delta > 0$ such that $f(u) \geq 2f(u_0) > f(u_0)$ as $\|u\| > \delta$.

Let $B = \{u \in \tilde{H}_{\Delta,T}^1 \mid \|u\| < \delta\}$; it is easy to see that the set B is a weakly closed subset; there exists $u_1 \in B$ such that

$$f(u_1) = \inf_{u \in B} f(u) \leq f(u_0); \quad (19)$$

we obtain $f(u_1) = \inf_{u \in \tilde{H}_{\Delta,T}^1} f(u)$. The proof is complete. $\square$

Lemma 13. $\|u\| \rightarrow \infty$ if and only if $(|\bar{u}|^2 + \int_{[0,T]_{\mathbb{T}}} |u^\Delta(t)|^2 \Delta t)^{1/2} \rightarrow \infty$.

Proof. On the one hand, by the definition of $\|u\|$, we have

$$\begin{aligned} \|u\| &= \|u\|_{H_{\Delta,T}^1} \\ &= \left(\int_{[0,T]_{\mathbb{T}}} |u(t)|^2 \Delta t + \int_{[0,T]_{\mathbb{T}}} |u^\Delta(t)|^2 \Delta t \right)^{1/2} \\ &= \left(\int_{[0,T]_{\mathbb{T}}} |\bar{u} + \tilde{u}(t)|^2 \Delta t + \int_{[0,T]_{\mathbb{T}}} |u^\Delta(t)|^2 \Delta t \right)^{1/2} \\ &\leq \left(\int_{[0,T]_{\mathbb{T}}} |\bar{u}| + \|\tilde{u}\|_\infty^2 \Delta t + \int_{[0,T]_{\mathbb{T}}} |u^\Delta(t)|^2 \Delta t \right)^{1/2} \\ &\leq \left(\int_{[0,T]_{\mathbb{T}}} 4 \max\{|\bar{u}|, \|\tilde{u}\|_\infty\}^2 \Delta t + \int_{[0,T]_{\mathbb{T}}} |u^\Delta(t)|^2 \Delta t \right)^{1/2} \\ &\leq \left(4T \max\{|\bar{u}|, \|\tilde{u}\|_\infty\}^2 + \int_{[0,T]_{\mathbb{T}}} |u^\Delta(t)|^2 \Delta t \right)^{1/2} \\ &\leq \left(4T |\bar{u}|^2 + (1 + TC) \int_{[0,T]_{\mathbb{T}}} |u^\Delta(t)|^2 \Delta t \right)^{1/2} \\ &\leq M \left(|\bar{u}|^2 + \int_{[0,T]_{\mathbb{T}}} |u^\Delta(t)|^2 \Delta t \right)^{1/2}; \end{aligned} \quad (20)$$

thus if $\|u\| \rightarrow \infty$, then $(|\bar{u}|^2 + \int_{[0,T]_{\mathbb{T}}} |u^\Delta(t)|^2 \Delta t)^{1/2} \rightarrow \infty$.

On the other hand, if

$$\left(|\bar{u}|^2 + \int_{[0,T]_{\mathbb{T}}} |u^\Delta(t)|^2 \Delta t \right)^{1/2} \rightarrow \infty, \quad (21)$$

then $\int_{[0,T]_{\mathbb{T}}} |u^\Delta(t)|^2 \Delta t \rightarrow \infty$ or $|\bar{u}|^2 \rightarrow \infty$. If

$$\int_{[0,T]_{\mathbb{T}}} |u^\Delta(t)|^2 \Delta t \rightarrow \infty, \quad (22)$$

then

$$\begin{aligned} \|u\| &= \left(\int_{[0,T]_{\mathbb{T}}} |u(t)|^2 \Delta t + \int_{[0,T]_{\mathbb{T}}} |u^\Delta(t)|^2 \Delta t \right)^{1/2} \\ &\rightarrow \infty. \end{aligned} \quad (23)$$

If $|\bar{u}|^2 \rightarrow \infty$, then $(1/T) \int_{[0,T]_{\mathbb{T}}} |u(t)| \Delta t \rightarrow \infty$. Since

$$\int_{[0,T]_{\mathbb{T}}} |u(t)| \Delta t \leq \sqrt{T} \left(\int_{[0,T]_{\mathbb{T}}} |u(t)|^2 \Delta t \right)^{1/2}, \quad (24)$$

so $\|u\| \rightarrow \infty$. The proof is complete. $\square$

Lemma 14 (see [13]). *Let X be a Banach space, and let $E \subset X$ be a weakly closed subset. Suppose that $g(u)$ is defined on an open subset $W \subset X$ and $g(u) \neq -\infty$ for any $u \in W$. Let $g(u) = +\infty$ for $u \in \partial W$. Assume that $g(u) \neq +\infty$ and is*

weakly lower semicontinuous on $\overline{W} \cap E$ and that it is coercive on $W \cap E$:

$$\begin{aligned} g(u) &\longrightarrow +\infty, \\ \|u\| &\longrightarrow +\infty, \\ g(u_n) &\longrightarrow +\infty, \quad u_n \rightharpoonup u \in \partial W. \end{aligned} \quad (25)$$

Then g attains its infimum in $W \cap E$.

Lemma 15 (see [13]). *The functional $f(u)$ attains the infimum on W_0 ; furthermore, the minimizer is nonconstant.*

Consider the functional $f : H_{\Delta,T}^1 \rightarrow \mathbb{R}$ defined by

$$f(u) = \frac{1}{2} \int_{[0,T]_{\mathbb{T}}} |u^\Delta(t)|^2 (h - V(\sigma(t), u^\sigma(t))) \Delta t. \quad (26)$$

For any $v \in H_{\Delta,T}^1$ and $0 < |\lambda| < 1$, we have

$$\begin{aligned} &\frac{1}{\lambda} [f(u + \lambda v) - f(u)] \\ &= \frac{1}{2} \int_{[0,T]_{\mathbb{T}}} \frac{(u^\Delta + \lambda v^\Delta)^2 [h - V(\sigma(t), u^\sigma + \lambda v^\sigma)]}{\lambda} \Delta t \\ &\quad - \frac{1}{2} \int_{[0,T]_{\mathbb{T}}} \frac{(u^\Delta)^2 [h - V(\sigma(t), u^\sigma)]}{\lambda} \Delta t \\ &= \frac{1}{2} \int_{[0,T]_{\mathbb{T}}} \frac{(u^\Delta)^2 h}{\lambda} \Delta t \\ &\quad - \frac{1}{2} \int_{[0,T]_{\mathbb{T}}} \frac{(u^\Delta)^2 V(\sigma(t), u^\sigma + \lambda v^\sigma)}{\lambda} \Delta t \\ &\quad + \int_{[0,T]_{\mathbb{T}}} (u^\Delta, v^\Delta) h \Delta t \\ &\quad - \int_{[0,T]_{\mathbb{T}}} (u^\Delta, v^\Delta) V(\sigma(t), u^\sigma + \lambda v^\sigma) \Delta t \\ &\quad + \frac{1}{2} \int_{[0,T]_{\mathbb{T}}} \lambda (v^\Delta)^2 h \Delta t \\ &\quad - \frac{1}{2} \int_{[0,T]_{\mathbb{T}}} \lambda (v^\Delta)^2 V(\sigma(t), u^\sigma + \lambda v^\sigma) \Delta t \\ &\quad - \frac{1}{2} \int_{[0,T]_{\mathbb{T}}} \frac{(u^\Delta)^2 h}{\lambda} \Delta t + \frac{1}{2} \int_{[0,T]_{\mathbb{T}}} \frac{(u^\Delta)^2 V(\sigma(t), u^\sigma)}{\lambda} \Delta t \\ &= -\frac{1}{2} \int_{[0,T]_{\mathbb{T}}} \frac{(u^\Delta)^2 [V(\sigma(t), u^\sigma + \lambda v^\sigma) - V(\sigma(t), u^\sigma)]}{\lambda} \Delta t \\ &\quad + \int_{[0,T]_{\mathbb{T}}} (u^\Delta, v^\Delta) [h - V(\sigma(t), u^\sigma + \lambda v^\sigma)] \Delta t \\ &\quad + \frac{1}{2} \int_{[0,T]_{\mathbb{T}}} \lambda (v^\Delta)^2 [h - V(\sigma(t), u^\sigma + \lambda v^\sigma)] \Delta t. \end{aligned} \quad (27)$$

It follows from the dominated convergence theorem on time scales that

$$\begin{aligned} &(f'(u), v) \\ &= \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} [f(u + \lambda v) - f(u)] \\ &= -\frac{1}{2} \int_{[0,T]_{\mathbb{T}}} \lim_{\lambda \rightarrow 0} \frac{(u^\Delta)^2 [V(\sigma(t), u^\sigma + \lambda v^\sigma) - V(\sigma(t), u^\sigma)]}{\lambda} \Delta t \\ &\quad + \int_{[0,T]_{\mathbb{T}}} \lim_{\lambda \rightarrow 0} (u^\Delta, v^\Delta) [h - V(\sigma(t), u^\sigma + \lambda v^\sigma)] \Delta t \\ &\quad + \frac{1}{2} \int_{[0,T]_{\mathbb{T}}} \lim_{\lambda \rightarrow 0} \lambda (v^\Delta)^2 [h - V(\sigma(t), u^\sigma + \lambda v^\sigma)] \Delta t \\ &= -\frac{1}{2} \int_{[0,T]_{\mathbb{T}}} \lim_{\lambda \rightarrow 0} \frac{(u^\Delta)^2 [V(\sigma(t), u^\sigma + \lambda v^\sigma) - V(\sigma(t), u^\sigma)] v^\sigma}{\lambda v^\sigma} \Delta t \\ &\quad + \int_{[0,T]_{\mathbb{T}}} \lim_{\lambda \rightarrow 0} (u^\Delta, v^\Delta) [h - V(\sigma(t), u^\sigma + \lambda v^\sigma)] \Delta t \\ &\quad + \frac{1}{2} \int_{[0,T]_{\mathbb{T}}} \lim_{\lambda \rightarrow 0} \lambda (v^\Delta)^2 [h - V(\sigma(t), u^\sigma + \lambda v^\sigma)] \Delta t \\ &= -\frac{1}{2} \int_{[0,T]_{\mathbb{T}}} (u^\Delta)^2 (\nabla V(\sigma(t), u^\sigma), v^\sigma) \Delta t \\ &\quad + \int_{[0,T]_{\mathbb{T}}} (u^\Delta, v^\Delta) [h - V(\sigma(t), u^\sigma)] \Delta t. \end{aligned} \quad (28)$$

From the preceding discussions, we know that the critical points of functional f are classical periodic solutions of systems (2) and (3). It is obvious that the functional f is continuously differentiable and weakly lower semicontinuous on $H_{\Delta,T}^1$.

3.2. Results on the Existence. In this subsection, we present two results on the existence of periodic solutions for the Hamiltonian system on time scales.

Throughout this subsection, we assume that

(H₁) there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1_{\Delta}(\mathbb{T}, \mathbb{R}^+)$ such that

$$\begin{aligned} |V(t, x)| &\leq a(|x|) b(t), \\ |\nabla V(t, x)| &\leq a(|x|) b(t) \end{aligned} \quad (29)$$

for all $x \in \mathbb{R}^n$ and Δ -a.e. $t \in \mathbb{T}$, where $\nabla V(t, x)$ denotes the gradient of $V(t, x)$ in x .

Theorem 16. *Assume that conditions (H₀) and (H₁) hold, and the following two conditions are true:*

(H₂) $\int_{[0,T]_{\mathbb{T}}} V(\sigma(t), x) \Delta t \rightarrow +\infty$ as $|x| \rightarrow +\infty$;

(H₃) *there exists $g \in L^1([0, T], \mathbb{R}^+)$ such that $g^\sigma \in L^1_{\Delta}([0, T]_{\mathbb{T}}, \mathbb{R}^+)$ and*

$$|\nabla V(\sigma(t), x)| \leq g^\sigma(t). \quad (30)$$

Then, systems (2)-(3) have at least one periodic solution which minimizes the function f .

Proof. By Lemma 7, there exists $C_1 > 0$ such that

$$\|\tilde{u}\|_\infty^2 \leq C_1 \int_{[0,T]_\mathbb{T}} |u^\Delta(t)|^2 \Delta t. \quad (31)$$

It follows from (H_3) , Lemma 7, and (31) that

$$\begin{aligned} & \left| \int_{[0,T]_\mathbb{T}} (V(\sigma(t), u^\sigma(t)) - V(\sigma(t), \bar{u})) \Delta t \right| \\ & \leq \left| \int_{[0,T]_\mathbb{T}} \left(\int_0^1 (\nabla V(\sigma(t), \bar{u} + s\tilde{u}^\sigma(t)), \tilde{u}^\sigma(t)) ds \right) \Delta t \right| \\ & \leq \int_{[0,T]_\mathbb{T}} \left(\int_0^1 g^\sigma(t) |\tilde{u}^\sigma(t)| ds \right) \Delta t \\ & \leq \|\tilde{u}\|_\infty \int_{[0,T]_\mathbb{T}} g^\sigma(t) \Delta t \\ & \leq C_2 \left(\int_{[0,T]_\mathbb{T}} |u^\Delta(t)|^2 \Delta t \right)^{1/2} \end{aligned} \quad (32)$$

for all $u \in H_{\Delta,T}^1$, where $C_2 = (C_1)^{1/2} \int_{[0,T]_\mathbb{T}} g^\sigma(t) \Delta t$. Therefore, we have

$$\begin{aligned} f(u) &= \frac{1}{2} \int_{[0,T]_\mathbb{T}} |u^\Delta(t)|^2 (h - V(\sigma(t), u^\sigma(t))) \Delta t \\ &= \frac{h}{2} \int_{[0,T]_\mathbb{T}} |u^\Delta(t)|^2 \Delta t \\ &\quad - \frac{1}{2} \int_{[0,T]_\mathbb{T}} |u^\Delta(t)|^2 V(\sigma(t), u^\sigma(t)) \Delta t \\ &= \frac{h}{2} \int_{[0,T]_\mathbb{T}} |u^\Delta(t)|^2 \Delta t \\ &\quad - \frac{1}{2} \int_{[0,T]_\mathbb{T}} |u^\Delta(t)|^2 ((V(\sigma(t), u^\sigma(t)) - V(\sigma(t), \bar{u})) \\ &\quad \quad + V(\sigma(t), \bar{u})) \Delta t \\ &\geq \frac{h}{2} \int_{[0,T]_\mathbb{T}} |u^\Delta(t)|^2 \Delta t + \frac{C_2}{2} \left(\int_{[0,T]_\mathbb{T}} |u^\Delta(t)|^2 \Delta t \right)^{3/2} \\ &\quad - \frac{1}{2} \int_{[0,T]_\mathbb{T}} |u^\Delta(t)|^2 \times V(\sigma(t), \bar{u}) \Delta t \end{aligned} \quad (33)$$

for all $u \in H_{\Delta,T}^1$. It follows from Lemmas 11 and 13 and (H_2) that $f(u) \rightarrow +\infty$ as $\|u\| \rightarrow \infty$. Therefore, systems (2)-(3) have at least one periodic solution. $\square$

Example 17. Let $\mathbb{T} = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ and $T = 1$. Consider the following second order Hamiltonian system on time scale $\mathbb{T}$ of the form

$$\begin{aligned} q^{\Delta^2}(t) + \nabla V\left(\frac{t}{1-t}, q\left(\frac{t}{1-t}\right)\right) &= 0, \\ \frac{1}{2} |q^\Delta(t)|^2 + V\left(\frac{t}{1-t}, q\left(\frac{t}{1-t}\right)\right) &= h, \end{aligned} \quad (34)$$

where $V(\sigma(t), x) = 2t|x|^2$. It is easy to verify that (H_0) and all conditions of Theorem 16 are satisfied. By Theorem 16 we see that system (34) has at least one solution.

Theorem 18. Assume that $V \in C^1(\mathbb{T} \times \mathbb{R}^n \setminus \{0\}, \mathbb{R})$ satisfies condition (H_0) and the following two conditions:

(H_4) there exist constants $C_3 > 0$, $\alpha > 2$ such that for $t \in [0, T]_\mathbb{T}$ the following inequality uniformly holds:

$$\liminf_{|x| \rightarrow \infty} \frac{V(\sigma(t), x)}{|x|^\alpha} < -C_3; \quad (35)$$

(H_5) for $t \in [0, T]_\mathbb{T}$ the following equality uniformly holds:

$$V(\sigma(t), -x) = V(\sigma(t), x), \quad x \neq 0. \quad (36)$$

Then for any $h > 0$, systems (2)-(3) have at least one nonconstant periodic solution with the given energy h .

3.3. Proof of Theorem 18

Lemma 19. Assume (H_4) holds; then, for any weakly convergent sequence $u_n \rightharpoonup u \in \partial W_0$, one has

$$f(u_n) \longrightarrow +\infty. \quad (37)$$

Proof. Notice that (H_4) implies Gordon's strong force condition. It follows from (H_4) and there exist constants $0 < \beta < C_3$ and $M > 0$ such that

$$V(\sigma(t), u) < -C_3 |u|^\alpha + \beta |u|^\alpha \quad (38)$$

for all $u \in \mathbb{R}^n$ with $|u| > M$ and $t \in [0, T]_\mathbb{T}$. By $u_n \rightharpoonup u \in \partial W$ and V satisfying Gordon's strong force condition, we have

$$\begin{aligned} \int_{[0,T]_\mathbb{T}} -V(\sigma(t), u_n^\sigma(t)) \Delta t &\longrightarrow +\infty, \\ \forall u_n \rightharpoonup u \in \partial W. \end{aligned} \quad (39)$$

By $u_n \rightharpoonup u$ in the Hilbert space $H_{\Delta,T}^1$, we know that $\|u_n\|$ is bounded.

(1) If $u \equiv 0$, then by Sobolev's embedding theorem, we have the uniform convergence property:

$$|u_n|_\infty \longrightarrow 0, \quad n \longrightarrow +\infty. \quad (40)$$

By the symmetry of $u(t + (T/2)) = -u(t)$, we have $\int_{[0,T]_\mathbb{T}} u(t) \Delta t = 0$. So, by Sobolev's inequality

$$\int_{[0,T]_\mathbb{T}} |u^\Delta|^2 \Delta t \geq \frac{12}{T} |u|_\infty^2, \quad (41)$$

we have

$$\begin{aligned}
 f(u_n) &= \frac{1}{2} \int_{[0,T]_{\mathbb{T}}} |u_n^\Delta(t)|^2 (h - V(\sigma(t), u_n^\sigma(t))) \Delta t \\
 &\geq \frac{6(C_3 - \beta)}{T} |u_n|_\infty^{2+\alpha} \longrightarrow +\infty, \quad n \longrightarrow +\infty.
 \end{aligned} \tag{42}$$

(2) If $u \neq 0$, then, by the weakly lower semicontinuity for the norm, we have

$$\lim_{n \rightarrow +\infty} \inf \|u_n\| \geq \|u\| > 0. \tag{43}$$

So, by Gordon's lemma, we have

$$\begin{aligned}
 &\lim_{n \rightarrow +\infty} \inf f(u_n) \\
 &= \lim_{n \rightarrow +\infty} \inf \left(\frac{1}{2} \int_{[0,T]_{\mathbb{T}}} |u_n^\Delta(t)|^2 (h - V(\sigma(t), u_n^\sigma(t))) \Delta t \right) \\
 &= +\infty.
 \end{aligned} \tag{44}$$

The proof is complete. $\square$

Lemma 20. *The functional $f(u)$ is weakly lower semicontinuous on $\overline{W_0}$.*

Proof. Let

$$\begin{aligned}
 f_1(u) &= \frac{h}{2} \int_{[0,T]_{\mathbb{T}}} |u^\Delta(t)|^2 \Delta t, \\
 f_2(u) &= \frac{1}{2} \int_{[0,T]_{\mathbb{T}}} |u^\Delta(t)|^2 V(\sigma(t), u^\sigma(t)) \Delta t.
 \end{aligned} \tag{45}$$

Then, for given energy h , f_1 is continuous and convex. Hence, f_1 is weakly lower semicontinuous. On the other hand, let $\{u_n\}_{n \in \mathbb{N}} \subset \overline{W_0}$, $u_n \rightharpoonup u$ in $\overline{W_0}$. By Lemma 14, $\{u_k\}_{k \in \mathbb{N}}$ converges strongly in $C([0, T]_{\mathbb{T}}, \mathbb{R}^n)$ to u . By condition (H_1) , we have

$$\begin{aligned}
 &|f_2(u_n) - f_2(u)| \\
 &= \left| \frac{1}{2} \int_{[0,T]_{\mathbb{T}}} |u_n^\Delta(t)|^2 V(\sigma(t), u_n^\sigma(t)) \Delta t \right. \\
 &\quad \left. - \frac{1}{2} \int_{[0,T]_{\mathbb{T}}} |u^\Delta(t)|^2 V(\sigma(t), u^\sigma(t)) \Delta t \right| \\
 &\leq \frac{1}{2} \int_{[0,T]_{\mathbb{T}}} \left| |u_n^\Delta(t)|^2 V(\sigma(t), u_n^\sigma(t)) \right. \\
 &\quad \left. - |u^\Delta(t)|^2 V(\sigma(t), u^\sigma(t)) \right| \Delta t
 \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{1}{2} \int_{[0,T]_{\mathbb{T}}} \left| |u_n^\Delta(t)|^2 V(\sigma(t), u_n^\sigma(t)) \right. \\
 &\quad \left. - |u^\Delta(t)|^2 V(\sigma(t), u_n^\sigma(t)) \right| \Delta t \\
 &\quad + \frac{1}{2} \int_{[0,T]_{\mathbb{T}}} \left| |u^\Delta(t)|^2 V(\sigma(t), u_n^\sigma(t)) \right. \\
 &\quad \left. - |u^\Delta(t)|^2 V(\sigma(t), u^\sigma(t)) \right| \Delta t \\
 &\leq \frac{1}{2} \int_{[0,T]_{\mathbb{T}}} |V(\sigma(t), u_n^\sigma(t))| (|u_n^\Delta(t)|^2 - |u^\Delta(t)|^2) \Delta t \\
 &\quad + \frac{1}{2} \int_{[0,T]_{\mathbb{T}}} |u^\Delta(t)|^2 |V(\sigma(t), u_n^\sigma(t)) - V(\sigma(t), u^\sigma(t))| \Delta t \\
 &\longrightarrow 0.
 \end{aligned} \tag{46}$$

Thus, f_2 is weakly continuous. Consequently, $f = f_1 - f_2$ is weakly lower semicontinuous. The proof is complete. $\square$

Lemma 21. *$\overline{W_0}$ is a weakly closed subset of $H_{\Delta,T}^1$.*

Proof. From Lemma 9, $H_{\Delta,T}^1 = W_{\Delta,T}^{1,2}(\mathbb{T}, \mathbb{R}^n) \hookrightarrow \overline{W_0}$ is compact, so the proof is obvious. $\square$

Lemma 22. *The functional $f(u)$ is coercive on W_0 .*

Proof. By the definition of $f(u)$ and the assumption (H_4) , we have

$$\begin{aligned}
 f(u) &= \frac{1}{2} \int_{[0,T]_{\mathbb{T}}} |u^\Delta(t)|^2 (h - V(\sigma(t), u^\sigma(t))) \Delta t \\
 &\geq \frac{1}{2} \int_{[0,T]_{\mathbb{T}}} |u^\Delta(t)|^2 (h + (C_3 - \beta) |u|^\alpha) \Delta t \\
 &\geq \frac{h}{2} \|u^\Delta\|_{L_{\Delta,T}^2}^2, \quad \forall u \in W_0;
 \end{aligned} \tag{47}$$

that is, $f(u) \rightarrow +\infty$ as $\|u\| \rightarrow \infty$ for $u \in W_0$; then $f(u)$ is coercive in W_0 . The proof is complete. $\square$

Since f is lower semicontinuous and coercive, we obtain that f is bounded below and has a bounded minimizing sequence. By virtue of Lemmas 10 and 15, we know that f has at least one nonconstant critical point in W_0 . That is, systems (2)-(3) have at least one nonconstant periodic solution with the given energy h .

Example 23. Let $\mathbb{T} = \{n/2 : n \in \mathbb{N}_0\}$ and $T = 1$. Consider the second order Hamiltonian system on time scale $\mathbb{T}$:

$$\begin{aligned}
 q^{\Delta^2}(t) + \nabla V\left(t + \frac{1}{2}, q\left(t + \frac{1}{2}\right)\right) &= 0, \\
 \frac{1}{2} |q^\Delta(t)|^2 + V\left(t + \frac{1}{2}, q\left(t + \frac{1}{2}\right)\right) &= h,
 \end{aligned} \tag{48}$$

where $V(\sigma(t), x) = \lambda |x|^2$ ($0 < \lambda < C_3$). It is easy to verify that (H_0) and all conditions of Theorem 18 are satisfied. By Theorem 18 we see that system (34) has at least one nonconstant periodic solution with the given energy h .

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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