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Research Article

Harmonic Subtangent Structures

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The concept of harmonic subtangent structures on almost subtangent metric manifolds is introduced and a Bochner-type formula is proved for this case. Conditions for a subtangent harmonic structure to be preserved by harmonic maps are also given.

1. Introduction

Inspired by the paper of Jianming [1], we introduce the notion of *harmonic almost subtangent structure* and underline the connection between harmonic subtangent structures and harmonic maps. It is well known that harmonic maps play an important role in many areas of mathematics. They often appear in nonlinear theories because of the nonlinear nature of the corresponding partial differential equations. In theoretical physics, harmonic maps are also known as sigma models. Remark also that harmonic maps between manifolds endowed with different geometrical structures have been studied in many contexts: Ianus and Pastore treated the case of contact metric manifolds [2], Bejan and Benyounes the almost para-Hermitian manifolds [3], Sahin the locally conformal Kähler manifolds [4], Ianus et al. the quaternionic Kähler manifolds [5], Jaiswal the Sasakian manifolds [6], Fetcu the complex Sasakian manifolds [7], Li the Finsler manifolds [8], and so forth. Fotiadis studied the noncompact case, describing the problem of finding a harmonic map between noncompact manifolds [9].

Let M be a smooth, m-dimensional real manifold for which we denote by $C^\infty(M)$ the real algebra of smooth real functions on M, by $\Gamma(TM)$ the Lie algebra of vector fields on M, and by $T^r_s(M)$ the $C^\infty(M)$-module of tensor fields of (r,s)-type on M. An element of $T^1_1(M)$ is usually called *vector* 1-*form* or *affinor*.

Recall the concept of almost tangent geometry.

Definition 1 (see [10]). $J \in T^1_1(M)$ is called almost tangent structure on M if it has a constant rank and

$$\operatorname{Im} J = \ker J. \tag{1}$$

The pair (M, J) is called *almost tangent manifold*.

The name is motivated by the fact that (1) implies the nilpotence $J^2 = 0$ exactly as the natural tangent structure of tangent bundles. Denoting rank $J = n$ it results in $m = 2n$. If in addition, we assume that J is integrable, that is,

$$\begin{aligned} N_J(X,Y) &:= [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X,Y] \\ &= 0, \end{aligned} \tag{2}$$

then J is called *tangent structure* and (M, J) is called *tangent manifold*.

From [11] we deduce some aspects of tangent manifolds:

(i) the distribution $\operatorname{Im} J (= \ker J)$ defines a foliation;

(ii) there exist local coordinates $(x, y) = (x^i, y^i)_{1\le i\le n}$ on M such that $J = \partial/\partial y^i \otimes dx^i$; that is,

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \qquad J\left(\frac{\partial}{\partial y^i}\right) = 0. \tag{3}$$

We call (x, y) *canonical coordinates* and the change of canonical coordinates $(x, y) \to (\tilde{x}, \tilde{y})$ is given by

$$\tilde{x}^i = \tilde{x}^i(x),$$
$$\tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^a} y^a + B^i(x). \tag{4}$$

So another description can be obtained in terms of G-structures. Namely, a tangent structure is a G-structure with [12]

$$G = \left\{ C = \begin{pmatrix} A & O_n \\ B & A \end{pmatrix} \in GL(2n, \mathbb{R}); \right.$$
$$\left. A \in GL(n, \mathbb{R}), B \in gl(n, \mathbb{R}) \right\} \tag{5}$$

and G is the invariance group of matrix $J = \begin{pmatrix} O_n & O_n \\ I_n & O_n \end{pmatrix}$; that is, $C \in G$ if and only if $C \cdot J = J \cdot C$.

The natural almost tangent structure J of $M = TN$ is an example of tangent structure having exactly the expression (3) if (x^i) are the coordinates on N and (y^i) are the coordinates in the fibers of $TN \to N$. A class of examples is obtained by duality [12]: if J is an (integrable) endomorphism with $J^2 = 0$, then its dual $J^* : \Gamma(T^*M) \to \Gamma(T^*M)$, given by $J^*\alpha := \alpha \circ J$ for $\alpha \in \Gamma(T^*M)$, is (integrable) endomorphism with $(J^*)^2 = 0$.

If the condition in the Definition 1 is weakened, requiring that only J squares to 0, we call J *almost subtangent structure*. In this case, $\operatorname{Im} J \subset \ker J$.

2. Harmonic Subtangent Structures

Let (M, g, J) be an *almost subtangent metric manifold* of dimension $2n$, that is, a $2n$-dimensional smooth manifold M endowed with an almost subtangent structure J which is compatible with a pseudo-Riemannian metric g (i.e., $g(JX, Y) = g(X, JY)$, for any $X, Y \in \Gamma(TM)$) and let ∇ be the Levi-Civita connection associated with g. Consider the exterior differential and codifferential operators defined for any tangent bundle-valued p-form $T \in \Gamma(\Lambda^p T^*M \otimes TM)$ by

$$dT(X_0, \ldots, X_p) := \sum_{i=0}^{p} (-1)^i (\nabla_{X_i} T)(X_0, \ldots, \widehat{X_i}, \ldots, X_p),$$
$$\delta T(X_1, \ldots, X_{p-1}) := -\sum_{i=1}^{2n} (\nabla_{E_i} T)(E_i, X_1, \ldots, X_{p-1}), \tag{6}$$

for $\{E_i\}_{1 \le i \le 2n}$ an orthonormal frame field and the Hodge-Laplace operator on $\Gamma(\Lambda^p T^*M \otimes TM)$:

$$\Delta := d \circ \delta + \delta \circ d. \tag{7}$$

Jianming studied in [1] some properties of harmonic complex structures and we discussed in [13] the paracosymplectic case.

Definition 2. An almost subtangent structure J is called harmonic if $\Delta J = 0$.

If M is compact, from the definition it follows that J is harmonic if and only if $dJ = 0$ and $\delta J = 0$ which is equivalent to $(\nabla_X J)Y = (\nabla_Y J)X$, for any $X, Y \in \Gamma(TM)$ and $\operatorname{trace}(\nabla J) = 0$, ∇ being the Levi-Civita connection associated with the pseudo-Riemannian structure g.

Proposition 3. *On a compact almost subtangent manifold, any harmonic almost subtangent structure J is integrable (i.e., it is a subtangent structure).*

Proof. Let $X, Y \in \Gamma(TM)$. Then

$$(dJ)(X, Y) := (\nabla J)(X, Y) - (\nabla J)(Y, X)$$
$$= [X, JY] + \nabla_{JY} X - [Y, JX] - \nabla_{JX} Y - J[X, Y]. \tag{8}$$

As $\Delta J = 0$ implies $dJ = 0$, we get

$$0 = (dJ)(JX, Y) + (dJ)(X, JY)$$
$$= [JX, JY] + J^2[X, Y] - J[JX, Y] - J[X, JY], \tag{9}$$

which shows the integrability of J. □

Remark 4. As expected, the harmonicity of an almost subtangent structure is not always preserved under conformal transformations. Indeed, let J be a harmonic subtangent structure (with respect to Δ) and for a smooth positive function f on the $2n$-dimensional manifold M, let $\tilde{g} = f^2 g$. Then the Levi-Civita connection associated with $\tilde{g}$ is $\tilde{\nabla}_X Y = \nabla_X Y + (1/f)X(f) \cdot Y + (1/f)Y(f) \cdot X - (1/f)g(X, Y)\operatorname{grad}_g(f)$, for any $X, Y \in \Gamma(TM)$. The necessary and sufficient condition for J to be harmonic (with respect to $\tilde{\Delta}$) is

$$(\tilde{\nabla}_X J) Y = (\tilde{\nabla}_Y J) X, \quad \text{for any } X, Y \in \Gamma(TM),$$
$$\widetilde{\operatorname{trace}}(\tilde{\nabla} J) = 0, \tag{10}$$

but

$$(\tilde{\nabla}_X J) Y = (\nabla_X J) Y + \frac{1}{f} \left[(JY)(f) \cdot X - Y(f) \cdot JX \right.$$
$$- g(X, JY) \operatorname{grad}_g(f)$$
$$\left. + g(X, Y) J(\operatorname{grad}_g(f)) \right], \tag{11}$$

so the first relation is equivalent to

$$X(f) \cdot JY - (JX)(f) \cdot Y$$
$$= Y(f) \cdot JX - (JY)(f) \cdot X, \quad \text{for any } X, Y \in \Gamma(TM). \tag{12}$$

Taking $\{E_i\}_{1\le i\le 2n}$ an orthonormal frame field on (M,g) with $\nabla_{E_i}E_j=0$, $1\le i,j\le 2n$, and computing

$$\widetilde{\operatorname{trace}}\left(\widetilde{\nabla}J\right)=\operatorname{trace}(\nabla J)+\frac{1}{f}\sum_{i=1}^{2n}\Big[(JE_i)(f)\cdot E_i-E_i(f)\cdot JE_i+J\left(\operatorname{grad}_g(f)\right)-g(E_i,JE_i)\operatorname{grad}_g(f)\Big],\tag{13}$$

the second relation is equivalent to

$$\sum_{i=1}^{2n}\left[E_i(f)\cdot JE_i-(JE_i)(f)\cdot E_i\right]=\sum_{i=1}^{2n}\left[J\left(\operatorname{grad}_g(f)\right)-g(E_i,JE_i)\operatorname{grad}_g(f)\right].\tag{14}$$

In conclusion, J is also harmonic with respect to $\widetilde{\Delta}$ if and only if

$$df\otimes J-J\otimes df=(df\circ J)\otimes I-I\otimes(df\circ J)\tag{15}$$

$$\operatorname{trace}\left[df\otimes J-(df\circ J)\otimes I\right]=\frac{2n}{f}J\left(\operatorname{grad}_g(f)\right)-\operatorname{trace}\left[g\circ(I\times J)\right]\cdot\operatorname{grad}_g(f).\tag{16}$$

Now we want to see how a Bochner-type formula can be written on an almost subtangent metric manifold.

We know that for any tangent bundle-valued differential form, $T\in\Gamma(\Lambda^1T^*M\otimes TM)$, the following Weitzenböck formula holds [14]:

$$\Delta T=-\nabla^2T-S,\tag{17}$$

where $\nabla^2T:=\sum_{i=1}^{2n}\nabla_{E_i}\nabla_{E_i}T-\nabla_{\nabla_{E_i}E_i}T$ and $S(X):=\sum_{i=1}^{2n}(R_{E_iX}T)E_i$, $X\in\Gamma(TM)$, for $\{E_i\}_{1\le i\le 2n}$ an orthonormal frame field and $R_{XY}:=\nabla_X\nabla_Y-\nabla_Y\nabla_X-\nabla_{[X,Y]}$, $X,Y\in\Gamma(TM)$, the Riemann curvature tensor field. We will also use the notations $R_{XYZ}=:R_{XY}Z$ and $R_{XYZW}=:g(R_{XYZ},W)$, $X,Y,Z,W\in\Gamma(TM)$. Now, on the almost subtangent metric manifold (M,g,J), taking T equal to J, for any vector field X, we have

$$S(X):=\sum_{i=1}^{2n}\left(R_{E_iX}J\right)E_i=\sum_{i=1}^{2n}R_{E_iXJE_i}-\sum_{i=1}^{2n}J\left(R_{E_iXE_i}\right)=\sum_{i=1}^{2n}\left[R_{E_iXJE_i}-J\left(R_{E_iXE_i}\right)\right].\tag{18}$$

We can state the following theorem.

Theorem 5. *Let (M,g,J) be an almost subtangent metric manifold and assume that J is harmonic subtangent structure. Then a Bochner-type formula reduces to*

$$|\nabla J|^2=\sum_{1\le i,j\le 2n}R_{E_iE_jJE_iJE_j},\tag{19}$$

for $\{E_i\}_{1\le i\le 2n}$ an orthonormal frame field on (M,g) with $\nabla_{E_i}E_j=0$, $1\le i,j\le 2n$.

Proof. A similar computation like in [1] leads us to

$$\left\langle\nabla^2J,J\right\rangle=\sum_{i=1}^{2n}\langle\nabla_{E_i}\nabla_{E_i}J,J\rangle=-|\nabla J|^2,\qquad \langle S,J\rangle=\sum_{j=1}^{2n}\left\langle SE_j,JE_j\right\rangle=\sum_{1\le i,j\le 2n}g\left(R_{E_iE_jJE_i},JE_j\right).\tag{20}$$

Therefore, as J is harmonic if $\Delta J=0$, from (17), we obtain

$$0=\langle\Delta J,J\rangle=-\left\langle\nabla^2J,J\right\rangle-\langle S,J\rangle=|\nabla J|^2-\sum_{1\le i,j\le 2n}g\left(R_{E_iE_jJE_i},JE_j\right)=|\nabla J|^2-\sum_{1\le i,j\le 2n}R_{E_iE_jJE_iJE_j}.\tag{21}$$

□

Notice that if J is only almost subtangent structure, from the proof of the theorem, we deduce that

$$\langle\Delta J,J\rangle=|\nabla J|^2-\sum_{1\le i,j\le 2n}R_{E_iE_jJE_iJE_j}.\tag{22}$$

If M is compact, integrating this relation with respect to the canonical measure, we obtain the following characterization of a harmonic almost subtangent structure.

Corollary 6. *Let (M,g,J) be a compact almost subtangent metric manifold. Then the almost subtangent structure J is harmonic if and only if*

$$\int_M\left(|\nabla J|^2-\sum_{1\le i,j\le 2n}R_{E_iE_jJE_iJE_j}\right)d\mu_g=0.\tag{23}$$

Example 7. Concerning the existence of almost tangent structures of order 2 (i.e., those J with $J^2=0$) on the spheres, Rosendo and Gadea [15] proved that the only spheres that admit such structures are S^3 and S^{15}. Moreover, they proved that the only spheres that admit almost tangent structures (of different orders) are S^3 (of order 2), S^7 (of order 6), and S^{15} (of order 2 or 4). For these cases, let $JE_i=\sum_{1\le k\le 2n}J_i^kE_k$ and $\operatorname{trace}J=\sum_{1\le k\le 2n}J_k^k$. Computing $R_{E_iE_jJE_iJE_j}=J_i^iJ_j^j-J_i^jJ_j^i$ and taking into account that $\sum_{1\le k\le 2n}J_i^kJ_k^j=0$, for any $i,j\in\{1,\dots,2n\}$, from Corollary 6, we get

$$\frac{\int_{S^3}|\nabla J|^2d\mu_g}{(\operatorname{trace}J)^2}=\frac{4\pi}{3},\qquad\frac{\int_{S^{15}}|\nabla J|^2d\mu_g}{(\operatorname{trace}J)^2}=\frac{2\cdot(2\pi)^7}{15\,!!}.\tag{24}$$

3. Harmonic Maps and Harmonic Subtangent Structures

Let (M,g,J) and $(\widetilde{M},\widetilde{g},\widetilde{J})$ be two almost subtangent metric manifolds $2n$- and $2\widetilde{n}$-dimensional, respectively. Denote by

∇ and $\widetilde{\nabla}$, respectively, the Levi-Civita connections associated with g and $\widetilde{g}$, respectively.

Consider $\Phi : (M, g, J) \to (\widetilde{M}, \widetilde{g}, \widetilde{J})$ a smooth map and let

$$\tau(\Phi) := \sum_{i=1}^{2n} \left[\widetilde{\nabla}_{\Phi_* E_i} \Phi_* E_i - \Phi_* \left(\nabla_{E_i} E_i\right)\right] \tag{25}$$

be the tension field of Φ, where $\{E_i\}_{1\le i\le 2n}$ is an orthonormal frame field on (M, g).

Proposition 8. *Let $\Phi : (M, g, J) \to (\widetilde{M}, \widetilde{g}, \widetilde{J})$ be a smooth map between almost subtangent metric manifolds such that $\Phi_* \circ J = \widetilde{J} \circ \Phi_*$. Then*

$$\begin{aligned}\widetilde{J}(\tau(\Phi)) - \Phi_*(\operatorname{trace}(\nabla J)) + \operatorname{trace}\left(\left(\widetilde{\nabla}\widetilde{J}\right) \circ (\Phi_* \times \Phi_*)\right) \\ = \sum_{i=1}^{2n} \left[\widetilde{\nabla}_{\Phi_* E_i} \Phi_* (JE_i) - \Phi_* \left(\nabla_{E_i} JE_i\right)\right],\end{aligned} \tag{26}$$

for $\{E_i\}_{1\le i\le 2n}$ an orthonormal frame field on the $2n$-dimensional manifold (M, g).

Proof. Express $\operatorname{trace}(\nabla J) = \sum_{i=1}^{2n} (\nabla J)(E_i, E_i) = \sum_{i=1}^{2n} [\nabla_{E_i} JE_i - J(\nabla_{E_i} E_i)]$ and replace it in the left side of the relation. □

Proposition 9. *Let $\Phi : (M, g, J) \to (\widetilde{M}, \widetilde{g}, \widetilde{J})$ be a smooth map between almost subtangent metric manifolds such that $\Phi_* \circ J = \widetilde{J} \circ \Phi_*$. If for any $X \in \Gamma(TM)$, $\Phi_* \circ \nabla_X J = (\widetilde{\nabla}_{\Phi_* X} \widetilde{J}) \circ \Phi_*$, then*

$$\operatorname{trace}\left(\left(\widetilde{\nabla}\widetilde{J}\right) \circ (\Phi_* \times \Phi_*)\right) = \Phi_*(\operatorname{trace}(\nabla J)). \tag{27}$$

Proof. For any $X, Y \in \Gamma(TM)$, $\widetilde{\nabla}_{\Phi_* X} \widetilde{J}(\Phi_* Y) - \Phi_*(\nabla_X JY) = \widetilde{J}(\widetilde{\nabla}_{\Phi_* X} \Phi_* Y) - \Phi_*(J(\nabla_X Y))$ and for $X = Y = E_i$,

$$\begin{aligned}\widetilde{J}(\tau(\Phi)) - \Phi_*(\operatorname{trace}(\nabla J)) + \operatorname{trace}\left(\left(\widetilde{\nabla}\widetilde{J}\right) \circ (\Phi_* \times \Phi_*)\right) \\ = \widetilde{J}\left(\sum_{i=1}^{2n} \left[\widetilde{\nabla}_{\Phi_* E_i} \Phi_* E_i - \Phi_* \left(\nabla_{E_i} E_i\right)\right]\right) = \widetilde{J}(\tau(\Phi)).\end{aligned} \tag{28}$$

□

Definition 10. A smooth map $\Phi : (M, g, J) \to (\widetilde{M}, \widetilde{g}, \widetilde{J})$ is said to be harmonic if its tension field $\tau(\Phi)$ vanishes.

Proposition 11. *Let $\Phi : (M, g, J) \to (\widetilde{M}, \widetilde{g}, \widetilde{J})$ be a smooth map between almost subtangent metric manifolds such that $\Phi_* \circ J = \widetilde{J} \circ \Phi_*$. If Φ is harmonic map, then*

$$\begin{aligned}\Phi_*(\operatorname{trace}(\nabla J)) = \operatorname{trace}\left(\left(\widetilde{\nabla}\widetilde{J}\right) \circ (\Phi_* \times \Phi_*)\right) \\ - \sum_{i=1}^{2n} \left[\widetilde{\nabla}_{\Phi_* E_i} \Phi_* (JE_i) - \Phi_* \left(\nabla_{E_i} JE_i\right)\right],\end{aligned} \tag{29}$$

for $\{E_i\}_{1\le i\le 2n}$ an orthonormal frame field on the $2n$-dimensional manifold (M, g).

Moreover, if for any $X \in \Gamma(TM)$, $\Phi_* \circ \nabla_X J = (\widetilde{\nabla}_{\Phi_* X} \widetilde{J}) \circ \Phi_*$, then

$$\sum_{i=1}^{2n} \left[\widetilde{\nabla}_{\Phi_* E_i} \Phi_* (JE_i) - \Phi_* \left(\nabla_{E_i} JE_i\right)\right] = 0, \tag{30}$$

for $\{E_i\}_{1\le i\le 2n}$ an orthonormal frame field on the $2n$-dimensional manifold (M, g).

Corollary 12. *Let $\Phi : (M, g, J) \to (\widetilde{M}, \widetilde{g}, \widetilde{J})$ be a smooth map between almost subtangent metric manifolds such that $\Phi_* \circ J = \widetilde{J} \circ \Phi_*$ and J is harmonic subtangent structure.*

(1) *If for any $X \in \Gamma(TM)$, $\Phi_* \circ \nabla_X J = (\widetilde{\nabla}_{\Phi_* X} \widetilde{J}) \circ \Phi_*$, then*

$$\operatorname{trace}\left(\left(\widetilde{\nabla}\widetilde{J}\right) \circ (\Phi_* \times \Phi_*)\right) = 0. \tag{31}$$

Moreover, if Φ is surjective submersion, then $\widetilde{J}$ is harmonic subtangent structure, too.

(2) *If Φ is harmonic map, then*

$$\begin{aligned}\operatorname{trace}\left(\left(\widetilde{\nabla}\widetilde{J}\right) \circ (\Phi_* \times \Phi_*)\right) \\ = \sum_{i=1}^{2n} \left[\widetilde{\nabla}_{\Phi_* E_i} \Phi_* (JE_i) - \Phi_* \left(\nabla_{E_i} JE_i\right)\right],\end{aligned} \tag{32}$$

for $\{E_i\}_{1\le i\le 2n}$ an orthonormal frame field on the $2n$-dimensional manifold (M, g).

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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