Research Article

A Special Class of Infinite Dimensional Dirac Operators on the Abstract Boson-Fermion Fock Space

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Spectral properties of a special class of infinite dimensional Dirac operators $Q(\alpha)$ on the abstract boson-fermion Fock space $\mathcal{F}(\mathcal{H}, \mathcal{K})$ associated with the pair $(\mathcal{H}, \mathcal{K})$ of complex Hilbert spaces are investigated, where $\alpha \in \mathbb{C}$ is a perturbation parameter (a coupling constant in the context of physics) and the unperturbed operator $Q(0)$ is taken to be a free infinite dimensional Dirac operator. A variety of the kernel of $Q(\alpha)$ is shown. It is proved that there are cases where, for all sufficiently large $|\alpha|$ with $\alpha < 0$, $Q(\alpha)$ has infinitely many nonzero eigenvalues even if $Q(0)$ has no nonzero eigenvalues. Also Fredholm property of $Q(\alpha)$ restricted to a subspace of $\mathcal{F}(\mathcal{H}, \mathcal{K})$ is discussed.

1. Introduction

In a previous paper [1] (cf. [2]), the author introduced a general class of infinite dimensional Dirac type operators Q_A on the abstract boson-fermion Fock space $\mathcal{F}(\mathcal{H}, \mathcal{K})$ associated with the pair $(\mathcal{H}, \mathcal{K})$ of complex Hilbert spaces (for the definition of $\mathcal{F}(\mathcal{H}, \mathcal{K})$ and Q_A , see Section 2), where A is a densely defined closed linear operator from $\mathcal{H}$ to $\mathcal{K}$. The operator Q_A gives an infinite dimensional and abstract version of finite dimensional Dirac type operators. In applications to physics, Q_A unifies self-adjoint supercharges (generators of supersymmetry) of some supersymmetric quantum field models (e.g., [3–6]).

In the paper [1], basic properties of Q_A are discussed. As for the spectral properties of Q_A , only the zero-eigenvalue of Q_A is considered. Moreover, a class of perturbations V for Q_A is introduced and, under a suitable condition, a path (functional) integral representation for the index of the perturbed operator $Q_A(V) := Q_A + V$ restricted to a subspace of $\mathcal{F}(\mathcal{H}, \mathcal{K})$ is established. The (essential) self-adjointness of $Q_A(V)$ is partially discussed in [7]. Analysis of other properties of $Q_A(V)$ including spectral ones except for the zero-eigenvalue has been left open.

In this paper, we undertake a comprehensive operator-theoretical analysis for $Q_A(V)$. But, as a preliminary, we consider, in the present paper, only the case where V is a

simple form and see what kind of phenomena occurs under the perturbation V .

The outline of the present paper is as follows. In Section 2 we review briefly some contents in [1]. In Section 3 we identify the spectra of Q_A . This is done via spectral analysis of a second quantization operator on the boson-fermion Fock space $\mathcal{F}(\mathcal{H}, \mathcal{K})$. In Section 4, we introduce a perturbation $V = V_{g,v}(\alpha)$ with $g \in \mathcal{H} \setminus \{0\}$ and $v \in \mathcal{K} \setminus \{0\}$, where $\alpha \in \mathbb{C}$ is a perturbation parameter (a coupling constant in the context of physics). The perturbed operator $Q(\alpha) := Q_A + V_{g,v}(\alpha)$ is the main object of our analysis in the present paper. We prove the self-adjointness of $Q(\alpha)$ and the essential self-adjointness of it on a suitable dense subspace of $\mathcal{F}(\mathcal{H}, \mathcal{K})$ with some other properties (Theorem 14). Also the spectra of $Q(\alpha)$ are identified (Theorem 16). Moreover, it is shown that the domain of $Q(\alpha)$ is equal to that of $Q(0) = Q_A$ for all sufficiently small $|\alpha|$ (Theorem 17). Section 5 is devoted to analysis of the kernel of $Q(\alpha)$. We see that the kernel property of $Q(\alpha)$ may be sensitive to conditions for (A, A^*, α) . In Section 6 we consider Fredholm property of $Q(\alpha)_+$, the restriction of $Q(\alpha)$ to a subspace of $\mathcal{F}(\mathcal{H}, \mathcal{K})$, in comparison with that of $(Q_A)_+ = Q(0)_+$. We obtain some classification on Fredholm property of $Q(\alpha)_+$ (Theorem 26). An interesting phenomenon occurs in the following sense: for a constant $\alpha_0 \neq 0$ (resp., $\beta_0 \neq 0$), $Q(\alpha_0)_+$ is not semi-Fredholm even if $(Q_A)_+$ is Fredholm (resp., semi-Fredholm). In the last

section, we consider nonzero eigenvalues of $Q(\alpha)$. We show that, under a suitable condition, for all sufficiently large $|\alpha|$ with $\alpha < 0$, $Q(\alpha)$ has infinitely many nonzero eigenvalues even if Q_A has no nonzero eigenvalues (Theorem 27). This structure also is interesting, which may be regarded as a strong coupling effect.

2. Brief Review on the Theory of Dirac Operators on the Abstract Boson-Fermion Fock Space

2.1. Definitions. For a linear operator L on a Hilbert space, we use the following notation:

- (i) $D(L)$: the domain of L ,
- (ii) $R(L)$: the range of L ,
- (iii) $\ker L := \{\psi \in D(L) \mid L\psi = 0\}$: the kernel of L ,
- (iv) $\rho(L) \subset \mathbb{C}$: the resolvent set of L ($\lambda \in \rho(L) \Leftrightarrow L$ is injective with $R(L)$ being dense and the inverse operator $(L - \lambda)^{-1}$ is bounded),
- (v) $\sigma(L) := \mathbb{C} \setminus \rho(L)$: the spectrum of L ,
- (vi) $\sigma_p(L)$: the point spectrum of L (the set of eigenvalues of L).

For a complex Hilbert space $\mathcal{H}$ and a nonnegative integer n , we denote by $\otimes_s^n \mathcal{H}$ (resp., $\wedge^n(\mathcal{H})$) the n -fold symmetric (resp., antisymmetric) tensor product Hilbert space of $\mathcal{H}$ with convention $\otimes_s^0 \mathcal{H} := \mathbb{C}$ (the complex number field) and $\wedge^0(\mathcal{H}) := \mathbb{C}$. In terms of these Hilbert spaces, one can construct two types of larger Hilbert spaces. The one is the *boson (symmetric) Fock space over $\mathcal{H}$* , denoted by $\mathcal{F}_b(\mathcal{H})$, which is defined to be the complete infinite direct sum of $\otimes_s^n \mathcal{H}$, $n \geq 0$:

$$\begin{aligned} \mathcal{F}_b(\mathcal{H}) &:= \oplus_{n=0}^{\infty} \otimes_s^n \mathcal{H} \\ &= \left\{ \psi = \{\psi^{(n)}\}_{n=0}^{\infty} \mid \psi^{(n)} \in \otimes_s^n \mathcal{H}, n \geq 0, \sum_{n=0}^{\infty} \|\psi^{(n)}\|^2 < \infty \right\}, \end{aligned} \quad (1)$$

where, for a complex Hilbert space $\mathcal{H}$, we denote its inner product (resp., norm) by $\langle \cdot, \cdot \rangle$ (linear in the second variable) (resp., $\|\cdot\|$). The other is the *fermion (antisymmetric) Fock space over $\mathcal{H}$* , denoted by $\mathcal{F}_f(\mathcal{H})$, which is defined to be the complete infinite direct sum of $\wedge^n(\mathcal{H})$, $n \geq 0$:

$$\begin{aligned} \mathcal{F}_f(\mathcal{H}) &:= \oplus_{n=0}^{\infty} \wedge^n(\mathcal{H}) \\ &= \left\{ \psi = \{\psi^{(n)}\}_{n=0}^{\infty} \mid \psi^{(n)} \in \wedge^n(\mathcal{H}), n \geq 0, \sum_{n=0}^{\infty} \|\psi^{(n)}\|^2 < \infty \right\}. \end{aligned} \quad (2)$$

For general theories of boson and fermion Fock spaces, see, for example, [8–10].

Let $\mathcal{H}$ be a complex Hilbert space. Then the *abstract boson-fermion Fock space* associated with the pair $(\mathcal{H}, \mathcal{H})$ is defined as the tensor product Hilbert space

$$\mathcal{F}(\mathcal{H}, \mathcal{H}) := \mathcal{F}_b(\mathcal{H}) \otimes \mathcal{F}_f(\mathcal{H}) \quad (3)$$

of $\mathcal{F}_b(\mathcal{H})$ and $\mathcal{F}_f(\mathcal{H})$.

For each $f \in \mathcal{H}$, we denote by $a(f)$ the *bosonic annihilation operator* on $\mathcal{F}_b(\mathcal{H})$ with test vector f ; that is, $a(f)$ is the densely defined closed linear operator acting in $\mathcal{F}_b(\mathcal{H})$ such that its adjoint $a(f)^*$ is given by the following form:

$$\begin{aligned} (a(f)^* \psi)^{(0)} &= 0, \quad (a(f)^* \psi)^{(n)} = \sqrt{n} S_n(f \otimes \psi^{(n-1)}), \\ n &\geq 1, \quad \psi \in D(a(f)^*), \end{aligned} \quad (4)$$

where S_n is the symmetrization operator acting on the n -fold tensor product Hilbert space $\otimes^n \mathcal{H}$.

Let

$$\Omega_b := \{1, 0, 0, \dots\} \in \mathcal{F}_b(\mathcal{H}), \quad (5)$$

called the *bosonic Fock vacuum*. Then, for all $f \in \mathcal{H}$,

$$a(f) \Omega_b = 0. \quad (6)$$

For each $u \in \mathcal{H}$, there exists a unique linear operator $b(u)$ on $\mathcal{F}_f(\mathcal{H})$, called the *fermionic annihilation operator* with test vector u , which has the following properties: (i) $D(b(u)) = \mathcal{F}_f(\mathcal{H})$ and $b(u)$ is bounded with operator norm $\|b(u)\| = \|u\|$; (ii) its adjoint $b(u)^*$ is of the following form:

$$\begin{aligned} (b(u)^* \psi)^{(0)} &= 0, \quad (b(u)^* \psi)^{(n)} = \sqrt{n} A_n(u \otimes \psi^{(n-1)}), \\ n &\geq 1, \quad \psi \in \mathcal{F}_f(\mathcal{H}), \end{aligned} \quad (7)$$

where A_n is the antisymmetrization operator acting on the n -fold tensor product Hilbert space $\otimes^n \mathcal{H}$. The following canonical anticommutation relations (CAR) hold:

$$\{b(u), b(v)^*\} = \langle u, v \rangle, \quad \{b(u), b(v)\} = 0, \quad u, v \in \mathcal{H}, \quad (8)$$

where $\{X, Y\} := XY + YX$. In particular, one has

$$b(u)^2 = 0, \quad u \in \mathcal{H}. \quad (9)$$

The vector

$$\Omega_f := \{1, 0, 0, \dots\} \in \mathcal{F}_f(\mathcal{H}) \quad (10)$$

is called the *fermionic Fock vacuum* and obeys

$$b(u) \Omega_f = 0, \quad u \in \mathcal{H}. \quad (11)$$

2.2. Infinite Dimensional Exterior Differential Operators. For a subset $\mathcal{D}$ of a Hilbert space, we denote by $\mathcal{L}(\mathcal{D})$ the subspace algebraically spanned by all the vectors of $\mathcal{D}$.

Let A be a densely defined closed linear operator from $\mathcal{H}$ to $\mathcal{H}$ and define

$$\begin{aligned} \mathcal{D}_A^\infty := \mathcal{L} \left(\{ a(f_1)^* \cdots a(f_n)^* \Omega_b \otimes b(u_1)^* \right. \\ \left. \cdots b(u_p)^* \Omega_f \mid n, p \geq 0, f_j \in C^\infty(A^*A), \right. \\ \left. j = 1, \dots, n, u_k \in C^\infty(AA^*), k = 1, \dots, p \} \right), \end{aligned} \quad (12)$$

where $C^\infty(L) := \cap_{n=1}^\infty D(L^n)$ for a linear operator L on a Hilbert space (the symbol $X_1 \cdots X_n \Omega$ with $n = 0$ should read Ω). It follows that $\mathcal{D}_A^\infty$ is dense in $\mathcal{F}(\mathcal{H}, \mathcal{H})$.

The following proposition is proved in [1].

Proposition 1. *There exists a unique densely defined closed linear operator d_A on $\mathcal{F}(\mathcal{H}, \mathcal{H})$ such that the following (i) and (ii) hold:*

- (i) $\mathcal{D}_A^\infty \subset D(d_A)$ and $\mathcal{D}_A^\infty$ is a core of d_A .
- (ii) For each vector $\Psi \in \mathcal{D}_A^\infty$ of the form

$$\Psi = a(f_1)^* \cdots a(f_n)^* \Omega_b \otimes b(u_1)^* \cdots b(u_p)^* \Omega_f, \quad (13)$$

d_A acts as

$$\begin{aligned} d_A \Psi &= 0 \quad \text{for } n = 0 \\ d_A \Psi &= \sum_{j=1}^n a(f_1)^* \cdots \widehat{a(f_j)^*} \cdots a(f_n)^* \Omega_b \\ &\quad \otimes b(A f_j)^* b(u_1)^* \cdots b(u_p)^* \Omega_f \quad \text{for } n \geq 1, \end{aligned} \quad (14)$$

where $\widehat{a(f_j)^*}$ indicates the omission of $a(f_j)^*$. In particular, d_A leaves $\mathcal{D}_A^\infty$ invariant.

Moreover, the following (iii)–(v) hold:

- (iii) $\mathcal{D}_A^\infty \subset D(d_A^*)$ and

$$d_A^* \Psi = \sum_{k=1}^p (-1)^{k-1} a(A^* u_k)^* a(f_1)^* \cdots a(f_n)^* \Omega_b \quad (15)$$

$$\otimes b(u_1)^* \cdots \widehat{b(u_k)^*} \cdots b(u_p)^* \Omega_f$$

for all vectors Ψ of the form (13) with $p \geq 1$. In the case $p = 0$, we have $d_A^* \Psi = 0$. In particular, d_A^* leaves $\mathcal{D}_A^\infty$ invariant.

- (iv) $D(d_A^2) = D(d_A)$ and, for all $\Psi \in D(d_A)$, $d_A^2 \Psi = 0$.

- (v) Let B be a bounded linear operator from $\mathcal{H}$ to $\mathcal{H}$ with $D(B) = \mathcal{H}$. Then, for all $\Psi \in \mathcal{D}_A^\infty$ and $\alpha, \beta \in \mathbb{C}$,

$$\alpha d_A \Psi + \beta d_B \Psi = d_{\alpha A + \beta B} \Psi. \quad (16)$$

The operator d_A is an abstract version of finite dimensional exterior differential operators, including an infinite dimensional version of them if $\mathcal{H}$ and $\mathcal{H}$ are infinite dimensional.

2.3. Infinite Dimensional Dirac Operators. Using the operator d_A , we define a *free Dirac operator* on $\mathcal{F}(\mathcal{H}, \mathcal{H})$ by

$$Q_A := d_A + d_A^* \quad (17)$$

with $D(Q_A) = D(d_A) \cap D(d_A^*)$. We set

$$\Delta_A := d_A^* d_A + d_A d_A^*, \quad (18)$$

a Laplace-Beltrami operator on $\mathcal{F}(\mathcal{H}, \mathcal{H})$.

For a self-adjoint operator S on $\mathcal{H}$, we define an operator $d\Gamma_b^{(n)}(S)$ on $\otimes_s^n \mathcal{H}$ by

$$\begin{aligned} d\Gamma_b^{(0)}(S) &:= 0, \\ d\Gamma_b^{(n)}(S) &:= \overline{\sum_{j=1}^n I \otimes \cdots \otimes I \otimes \overset{j\text{th}}{S} \otimes I \otimes \cdots \otimes I}, \quad n \geq 1, \end{aligned} \quad (19)$$

where I denotes identity and, for a closable linear operator C , $\overline{C}$ denotes the closure of C . For each $n \geq 0$, $d\Gamma_b^{(n)}(S)$ is self-adjoint. The second quantization $d\Gamma_b(S)$ of S on $\mathcal{F}_b(\mathcal{H})$ is defined by

$$d\Gamma_b(S) := \oplus_{n=0}^\infty d\Gamma_b^{(n)}(S), \quad (20)$$

which is self-adjoint (see, e.g., [9, Section 5.2], [11, p.302, Example 2]). It follows that

$$0 \in \sigma_p(d\Gamma_b(S)), \quad \Omega_b \in \ker(d\Gamma_b(S)). \quad (21)$$

Similarly, for a self-adjoint operator T on $\mathcal{H}$, we can define the second quantization $d\Gamma_f(T)$ of T on $\mathcal{F}_f(\mathcal{H})$ by

$$d\Gamma_f(T) := \oplus_{p=0}^\infty d\Gamma_f^{(p)}(T), \quad (22)$$

where

$$\begin{aligned} d\Gamma_f^{(0)}(T) &:= 0, \\ d\Gamma_f^{(p)}(T) &:= \overline{\sum_{j=1}^p I \otimes \cdots \otimes I \otimes \overset{j\text{th}}{T} \otimes I \otimes \cdots \otimes I}, \quad p \geq 1, \end{aligned} \quad (23)$$

acting in $\wedge^p(\mathcal{H})$. Similarly to the case of $d\Gamma_b(S)$, one has

$$0 \in \sigma_p(d\Gamma_f(T)), \quad \Omega_f \in \ker(d\Gamma_f(T)). \quad (24)$$

The number operator on $\mathcal{F}_f(\mathcal{H})$ is defined by

$$N_f := d\Gamma_f(I). \quad (25)$$

Then the operator

$$\Gamma := I \otimes (-1)^{N_f} \quad (26)$$

on $\mathcal{F}(\mathcal{H}, \mathcal{H})$ is self-adjoint with $\Gamma^2 = I$. The spectrum of Γ is $\{\pm 1\}$ with eigenspaces

$$\begin{aligned} \mathcal{F}_+(\mathcal{H}, \mathcal{H}) &:= \oplus_{p=0}^{\infty} \mathcal{F}_b(\mathcal{H}) \otimes \wedge^{2p}(\mathcal{H}), \\ \mathcal{F}_-(\mathcal{H}, \mathcal{H}) &:= \oplus_{p=0}^{\infty} \mathcal{F}_b(\mathcal{H}) \otimes \wedge^{2p+1}(\mathcal{H}), \end{aligned} \quad (27)$$

giving the orthogonal decomposition

$$\mathcal{F}(\mathcal{H}, \mathcal{H}) = \mathcal{F}_+(\mathcal{H}, \mathcal{H}) \oplus \mathcal{F}_-(\mathcal{H}, \mathcal{H}). \quad (28)$$

Let P_+ and P_- be the orthogonal projections onto $\mathcal{F}_+(\mathcal{H}, \mathcal{H})$ and $\mathcal{F}_-(\mathcal{H}, \mathcal{H})$, respectively. Then

$$\Gamma = P_+ - P_-. \quad (29)$$

For notational simplicity we set

$$H_b(S) := d\Gamma_b(S), \quad H_f(T) := d\Gamma_f(T). \quad (30)$$

We introduce an operator

$$H_{BF}(A) := H_b(A^*A) \otimes I + I \otimes H_f(AA^*) \quad (31)$$

acting in $\mathcal{F}(\mathcal{H}, \mathcal{H})$, which is nonnegative and self-adjoint. It follows from (21) and (24) that

$$0 \in \sigma_p(H_{BF}), \quad \Omega_b \otimes \Omega_f \in \ker(H_{BF}(A)). \quad (32)$$

Theorem 2 (see [1]). (i) The operator Q_A is self-adjoint and essentially self-adjoint on every core of $H_{BF}(A)$. In particular, Q_A is essentially self-adjoint on $\mathcal{D}_A^{\infty}$.

(ii) The operator Γ leaves $D(Q_A)$ invariant and

$$\Gamma Q_A + Q_A \Gamma = 0 \quad (33)$$

on $D(Q_A)$.

(iii) The following operator equalities hold:

$$\Delta_A = H_{BF}(A) = Q_A^2. \quad (34)$$

Remark 3. The quadruple $\{\mathcal{F}(\mathcal{H}, \mathcal{H}), Q_A, H_{BF}(A), \Gamma\}$ is an abstract form of models of free supersymmetric quantum field theory, where Q_A , $H_{BF}(A)$, and Γ are a self-adjoint supercharge, a supersymmetric Hamiltonian, and a sign operator of states (a “fermion number operator”), respectively [1].

For a linear operator L , we set

$$\text{nul } L := \dim \ker L, \quad (35)$$

the nullity of L .

Theorem 4 (see [1, Lemma 3.4]).

$$\ker Q_A = \ker H_{BF}(A) = \oplus_{n,p=0}^{\infty} [(\otimes_s^n \ker A) \otimes \wedge^p(\ker A^*)]. \quad (36)$$

Hence the following hold.

(i) Let $\text{nul } A = 0$ and $\text{nul } A^* < \infty$. Then $\text{nul } Q_A = 2^{\text{nul } A^*} < \infty$.

(ii) Let $\text{nul } A = 0$ and $\text{nul } A^* = \infty$. Then $\text{nul } Q_A = \infty$.

(iii) Let $\text{nul } A \geq 1$. Then $\text{nul } Q_A = \infty$.

2.4. Fredholm Property and Index. Let Q be a self-adjoint operator on $\mathcal{F}(\mathcal{H}, \mathcal{H})$ such that Γ leaves $D(Q)$ invariant satisfying $\Gamma Q + Q \Gamma = 0$ on $D(Q)$. This class of operators Q may be an abstract category of Dirac operators on the abstract boson-fermion Fock space $\mathcal{F}(\mathcal{H}, \mathcal{H})$.

Then it is easy to show that the operator Q_+ defined by

$$\begin{aligned} D(Q_+) &:= D(Q) \cap \mathcal{F}_+(\mathcal{H}, \mathcal{H}), \\ Q_+ \Psi &:= Q \Psi, \quad \Psi \in D(Q_+), \end{aligned} \quad (37)$$

is a densely defined closed linear operator from $\mathcal{F}_+(\mathcal{H}, \mathcal{H})$ to $\mathcal{F}_-(\mathcal{H}, \mathcal{H})$. We define an index $\text{ind}_\Gamma Q$ of Q by

$$\text{ind}_\Gamma Q := \text{nul } Q_+ - \text{nul } Q_+^*, \quad (38)$$

provided that at least one of $\text{nul } Q_+$ and $\text{nul } Q_+^*$ is finite. Note that

$$\text{ind}_\Gamma Q = \text{ind } Q_+, \quad (39)$$

the index of Q_+ (e.g., [12, Chapter IV, Section 5]).

As for the Fredholm property of Q_A , one has the following theorem.

Theorem 5 (see [1, Proposition 3.2]). (i) Let A be injective and Fredholm. Then $(Q_A)_+$ is Fredholm with

$$\text{ind}_\Gamma Q_A = \delta_{0, \text{nul } A^*}. \quad (40)$$

(ii) Let A be semi-Fredholm with $\text{nul } A \geq 1$ and let A^* be injective. Then $(Q_A)_+$ is semi-Fredholm with

$$\text{ind}_\Gamma Q_A = \infty. \quad (41)$$

(iii) Let A be semi-Fredholm with $\text{nul } A \geq 1$ and $\text{nul } A^* \geq 1$. Then $(Q_A)_+$ is not semi-Fredholm.

3. Spectra of $H_{BF}(A)$ and Q_A

Now we go into analyzing new aspects in the operator theory of Q_A and its perturbations. In the previous paper [1], the identification of the spectra of $H_{BF}(A)$ and Q_A was not discussed except for zero-eigenvalues. This section is devoted to identifying completely the spectra of $H_{BF}(A)$ and Q_A .

In what follows, we assume that $\mathcal{H}$ and $\mathcal{K}$ are separable.

We say that a subset J of $\mathbb{R}$ is symmetric with respect to the origin if $\lambda \in J$ implies that $-\lambda \in J$.

The main results in this section are as follows.

Theorem 6.

$$\sigma(H_{BF}(A)) = \{0\} \cup \overline{\bigcup_{n=1}^{\infty} \left\{ \sum_{j=1}^n \lambda_j \mid \lambda_j \in \sigma(A^*A) \setminus \{0\}, j = 1, \dots, n \right\}}, \quad (42)$$

$$\begin{aligned} & \sigma_p(H_{BF}(A)) \\ &= \{0\} \cup \left(\bigcup_{n=1}^{\infty} \left\{ \sum_{j=1}^n \lambda_j \mid \lambda_j \in \sigma_p(A^*A) \setminus \{0\}, \right. \right. \\ & \quad \left. \left. j = 1, \dots, n \right\} \right). \end{aligned} \quad (43)$$

Theorem 7. The spectrum $\sigma(Q_A)$ and the point spectrum $\sigma_p(Q_A)$ of Q_A are symmetric with respect to the origin and

$$\begin{aligned} \sigma(Q_A) &= \{0\} \cup \overline{\bigcup_{n=1}^{\infty} \left\{ \pm \sqrt{\sum_{j=1}^n \lambda_j} \mid \lambda_j \in \sigma(A^*A) \setminus \{0\}, j = 1, \dots, n \right\}}, \\ \sigma_p(Q_A) &= \{0\} \cup \left(\bigcup_{n=1}^{\infty} \left\{ \pm \sqrt{\sum_{j=1}^n \lambda_j} \mid \lambda_j \in \sigma_p(A^*A) \setminus \{0\}, j = 1, \dots, n \right\} \right) \end{aligned} \quad (44)$$

with

$$\text{nul}(Q_A - \lambda) = \text{nul}(Q_A + \lambda), \quad \lambda \in \sigma_p(Q_A). \quad (45)$$

To prove these theorems, we need a series of lemmas.

Lemma 8. Let S be a self-adjoint operator on a Hilbert space.

(i) Suppose that $\sigma(S)$ is symmetric with respect to the origin. Then

$$\sigma(S) = \{\pm \sqrt{\lambda} \mid \lambda \in \sigma(S^2)\}. \quad (46)$$

(ii) Suppose that $\sigma_p(S)$ is symmetric with respect to the origin. Then

$$\sigma_p(S) = \{\pm \sqrt{\lambda} \mid \lambda \in \sigma_p(S^2)\}. \quad (47)$$

Moreover, for each $f \in \ker(S^2 - \lambda)$ with $\lambda \in \sigma_p(S^2)$, $(S \pm \sqrt{\lambda})f \in \ker(S \mp \sqrt{\lambda})$.

Proof. (i) Let Σ_S be the set on the right hand side of (46). Then, by the spectral mapping theorem, we have $\sigma(S) \subset \Sigma_S$. Conversely, let $\mu \in \Sigma_S$, so that $\mu^2 \in \sigma(S^2)$. Hence there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset D(S^2)$ such that $\|f_n\| = 1$ and $(S^2 - \mu^2)f_n \rightarrow 0$ ($n \rightarrow \infty$). Suppose that $\mu \notin \sigma(S)$. Then $S - \mu$ is bijective with $(S - \mu)^{-1}$ bounded. Hence $(S - \mu)^{-1}(S^2 - \mu^2)f_n \rightarrow 0$ ($n \rightarrow \infty$), which means that $(S + \mu)f_n \rightarrow 0$ ($n \rightarrow \infty$). Hence $-\mu \in \sigma(S)$. Then, by the symmetry of $\sigma(S)$, μ is in $\sigma(S)$. But this is a contradiction. Hence $\mu \in \sigma(S)$. Thus $\Sigma_S \subset \sigma(S)$.

(ii) Let $\Sigma_{S,p}$ be the set on the right hand side of (47). Then it is obvious that $\sigma_p(S) \subset \Sigma_{S,p}$. Conversely, let $\mu \in \Sigma_{S,p}$, so that $\mu^2 \in \sigma_p(S^2)$. Hence there exists a vector $f \in D(S^2)$ such that $f \neq 0$ and $S^2 f = \mu^2 f$. The second equation is written as $(S - \mu)(S + \mu)f = 0$. If $\phi := (S + \mu)f = 0$, then $-\mu \in \sigma_p(S)$.

Hence, by the symmetry of $\sigma_p(S)$, μ is in $\sigma_p(S)$. If $\phi \neq 0$, then $(S - \mu)\phi = 0$. Hence $\mu \in \sigma_p(S)$. Thus $\Sigma_{S,p} \subset \sigma_p(S)$. These arguments also imply the last statement. $\square$

A sufficient condition for the spectra of a self-adjoint operator to be symmetric with respect to the origin is given in the following lemma.

Lemma 9. Let S be a self-adjoint operator on a Hilbert space $\mathcal{X}$. Suppose that there exists a bounded self-adjoint operator τ on $\mathcal{X}$ such that $\tau^2 = I$ and τ leaves $D(S)$ invariant satisfying

$$\tau S + S\tau = 0 \quad \text{on } D(S). \quad (48)$$

Then $\sigma(S)$ and $\sigma_p(S)$ are symmetric with respect to the origin and, for all $\lambda \in \sigma_p(S)$,

$$\text{nul}(S - \lambda) = \text{nul}(S + \lambda). \quad (49)$$

Proof. Note that τ is unitary with $\tau^{-1} = \tau$. Equation (48) implies the operator equality $S = \tau(-S)\tau^{-1}$. Hence S is unitarily equivalent to $-S$. Thus the desired results follow. $\square$

Lemma 10. Let $\mathcal{H}_n$ ($n = 0, 1, 2, \dots$) be a complex Hilbert space and A_n be a self-adjoint operator on $\mathcal{H}_n$. Let $\oplus_{n=0}^{\infty} A_n$ be the direct sum of A_n 's acting in the direct sum Hilbert space $\oplus_{n=0}^{\infty} \mathcal{H}_n$. Then, $\oplus_{n=0}^{\infty} A_n$ is self-adjoint and

$$\sigma(\oplus_{n=0}^{\infty} A_n) = \overline{\bigcup_{n=0}^{\infty} \sigma(A_n)}, \quad \sigma_p(\oplus_{n=0}^{\infty} A_n) = \bigcup_{n=0}^{\infty} \sigma_p(A_n). \quad (50)$$

Proof. See, for example, [8, Proposition 3.5-(ii), Theorem 3.7-(i)]. $\square$

Lemma 11. Let S be a self-adjoint operator on $\mathcal{H}$. Then

$$\begin{aligned} \sigma(H_b(S)) &= \{0\} \cup \left(\overline{\bigcup_{n=1}^{\infty} \left\{ \sum_{j=1}^n \lambda_j \mid \lambda_j \in \sigma(S), j = 1, \dots, n \right\}} \right), \end{aligned} \quad (51)$$

$$\begin{aligned} \sigma_p(H_b(S)) &= \{0\} \cup \left(\overline{\bigcup_{n=1}^{\infty} \left\{ \sum_{j=1}^n \lambda_j \mid \lambda_j \in \sigma_p(S), j = 1, \dots, n \right\}} \right). \end{aligned} \quad (52)$$

Proof. The following are well known (e.g., [8, Proposition 4.12]): $\sigma(d\Gamma_b^{(0)}(S)) = \{0\} = \sigma_p(d\Gamma_b^{(0)}(S))$ and, for all $n \geq 1$,

$$\begin{aligned} \sigma(d\Gamma_b^{(n)}(S)) &= \overline{\left\{ \sum_{j=1}^n \lambda_j \mid \lambda_j \in \sigma(S), j = 1, \dots, n \right\}}, \\ \sigma_p(d\Gamma_b^{(n)}(S)) &= \overline{\left\{ \sum_{j=1}^n \lambda_j \mid \lambda_j \in \sigma_p(S), j = 1, \dots, n \right\}}. \end{aligned} \quad (53)$$

By these facts and Lemma 10, we obtain (51) and (52). $\square$

For a self-adjoint operator T on a Hilbert space $\mathcal{H}$ and $p = 1, 2, \dots$, we define

$$\begin{aligned} \Sigma^{(p)}(T) &:= \left\{ \sum_{j=1}^p \mu_j \mid \mu_j \in \sigma(T) (j = 1, \dots, p), \right. \\ &\quad \left. \# \{k \mid \mu_j = \mu_k\} \leq \text{nul}(T - \mu_j) \text{ if } \mu_j \in \sigma_d(T) \right\}, \end{aligned} \quad (54)$$

where, for a finite set F , $\#F$ denotes the number of the elements of F and $\sigma_d(T)$ denotes the discrete spectrum of T (the set of isolated eigenvalues of T with finite multiplicity).

Lemma 12 (see [13, Corollary 3.2]).

$$\sigma(H_f(T)) = \{0\} \cup \left(\overline{\bigcup_{p=1}^{\infty} \Sigma^{(p)}(T)} \right), \quad (55)$$

$$\begin{aligned} \sigma_p(H_f(T)) &= \{0\} \cup \left(\overline{\bigcup_{p=1}^{\infty} \left\{ \sum_{j=1}^p \mu_j \mid \mu_j \in \sigma_p(T), \right. \right. \\ &\quad \left. \left. \# \{k \mid \mu_j = \mu_k\} \leq \text{nul}(T - \mu_j), \right. \right. \\ &\quad \left. \left. j = 1, \dots, p \right\}} \right). \end{aligned} \quad (56)$$

Proof of Theorem 6. By a theorem on the spectrum of the sum of tensor products of self-adjoint operators (e.g., [14, Corollary 7.25]),

$$\begin{aligned} \sigma(H_{BF}(A)) &= \left\{ \lambda + \mu \mid \lambda \in \sigma(H_b(A^*A)), \mu \in \sigma(H_f(AA^*)) \right\}, \end{aligned} \quad (57)$$

where we have used the fact that the set on the right hand side is closed (note that $\sigma(H_b(A^*A)), \sigma(H_f(AA^*)) \subset [0, \infty)$). By (51) and (55),

$$\sigma(H_{BF}(A)) \subset \{0\} \cup \bar{\Sigma}_1 \cup \bar{\Sigma}_2 \cup \bar{\Sigma}_3 \quad (58)$$

with

$$\begin{aligned} \Sigma_1 &:= \bigcup_{n=1}^{\infty} \left\{ \sum_{j=1}^n \lambda_j \mid \lambda_j \in \sigma(A^*A), j = 1, \dots, n \right\}, \\ \Sigma_2 &:= \bigcup_{p=1}^{\infty} \left\{ \sum_{k=1}^p \mu_k \mid \mu_k \in \sigma(AA^*) (k = 1, \dots, p), \right. \\ &\quad \left. \# \{\ell \mid \mu_k = \mu_\ell\} \leq \text{nul}(AA^* - \mu_k) \right. \\ &\quad \left. \text{if } \mu_k \in \sigma_d(AA^*) \right\}, \\ \Sigma_3 &:= \bigcup_{n,p=1}^{\infty} \left\{ \sum_{j=1}^n \lambda_j + \sum_{k=1}^p \mu_k \mid \lambda_j \in \sigma(A^*A) (j = 1, \dots, n), \right. \\ &\quad \left. \mu_k \in \sigma(AA^*) (k = 1, \dots, p), \right. \\ &\quad \left. \# \{\ell \mid \mu_k = \mu_\ell\} \leq \text{nul}(AA^* - \mu_k) \right. \\ &\quad \left. \text{if } \mu_k \in \sigma_d(AA^*) \right\}. \end{aligned} \quad (59)$$

By Deift's theorem [15], we have

$$\sigma(A^*A) \setminus \{0\} = \sigma(AA^*) \setminus \{0\}, \quad (60)$$

Hence $\Sigma_2 \subset \{0\} \cup \Sigma_1$ and $\Sigma_3 \subset \Sigma_1$. Therefore it follows that $\sigma(H_{BF}(A)) \subset \{0\} \cup \bar{\Sigma}_1$. The converse inclusion relation

$\{0\} \cup \bar{\Sigma}_1 \subset \sigma(H_{BF}(A))$ is obvious. Thus $\sigma(H_{BF}(A)) = \{0\} \cup \bar{\Sigma}_1$. The set on the right hand side is equal to the set on the right hand side of (42). Thus (42) holds.

The proof of (43) is similar to that of (42), where we use another version of Deift's theorem [15]:

$$\sigma_p(A^*A) \setminus \{0\} = \sigma_p(AA^*) \setminus \{0\} \quad (61)$$

with $\text{nul}(A^*A - \lambda) = \text{nul}(AA^* - \lambda)$ for each positive eigenvalue $\lambda > 0$ of A^*A (AA^*). $\square$

Proof of Theorem 7. By Theorem 2(ii) and Lemma 9, $\sigma(Q_A)$ and $\sigma_p(Q_A)$ are symmetric with respect to the origin and (45) holds. By Lemma 8 and Theorem 2(iii), we have

$$\begin{aligned} \sigma(Q_A) &= \{\pm\sqrt{\lambda} \mid \lambda \in \sigma(H_{BF}(A))\}, \\ \sigma_p(Q_A) &= \{\pm\sqrt{\lambda} \mid \lambda \in \sigma_p(H_{BF}(A))\}. \end{aligned} \quad (62)$$

By these facts and Theorem 6, we obtain (44). $\square$

4. A Class of Perturbed Infinite Dimensional Dirac Operators

In this section we consider a class of perturbations of Q_A . It would be natural to perturb Q_A through a perturbation of d_A . Let $g \in D(A) \setminus \{0\}$, $v \in D(A^*) \setminus \{0\}$ and define

$$d(\alpha) := d_A + \alpha a(g) \otimes b(v)^*, \quad (63)$$

where $\alpha \in \mathbb{C}$ is a perturbation parameter (a coupling constant in the context of physics). We have by direct computation

$$d(\alpha)^2 = 0 \quad \text{on } \mathcal{D}_A^\infty. \quad (64)$$

The adjoint $d(\alpha)^*$ is densely defined with $\mathcal{D}_A^\infty \subset D(d(\alpha)^*)$ and

$$d(\alpha)^* = d_A^* + \alpha^* a(g)^* \otimes b(v) \quad \text{on } \mathcal{D}_A^\infty. \quad (65)$$

Hence $d(\alpha)$ is closable. We denote the closure of $d(\alpha) \upharpoonright \mathcal{D}_A^\infty$ by $\bar{d}(\alpha)$. Then, for all $\Psi \in D(\bar{d}(\alpha))$, $\bar{d}(\alpha)\Psi$ is in $D(\bar{d}(\alpha))$ and

$$\bar{d}(\alpha)^2 \Psi = 0. \quad (66)$$

The operator $\bar{d}(\alpha)$ yields a perturbed Dirac operator:

$$Q(\alpha) := \bar{d}(\alpha) + \bar{d}(\alpha)^*. \quad (67)$$

By an application of [2, Lemma 4.1], one sees that $Q(\alpha)$ is a closed symmetric operator. We have

$$Q(\alpha) = Q_A + V_{g,v}(\alpha) \quad \text{on } \mathcal{D}_A^\infty \quad (68)$$

with

$$V_{g,v}(\alpha) := \alpha a(g) \otimes b(v)^* + \alpha^* a(g)^* \otimes b(v). \quad (69)$$

In what follows, we investigate properties of $Q(\alpha)$.

4.1. Self-Adjointness of $Q(\alpha)$. We define a linear operator $T_{g,v}$ from $\mathcal{H}$ to $\mathcal{H}$ by

$$T_{g,v}f := \langle g, f \rangle v, \quad f \in \mathcal{H}. \quad (70)$$

It is obvious that $T_{g,v}$ is a bounded operator (a one-rank operator). Hence

$$A(\alpha) := A + \alpha T_{g,v} \quad (71)$$

is a densely defined closed linear operator with $D(A(\alpha)) = D(A)$.

The following lemma is a key fact.

Lemma 13. *For all $\alpha \in \mathbb{C}$, the following operator equality holds:*

$$\bar{d}(\alpha) = d_{A(\alpha)}. \quad (72)$$

Proof. Let $\Psi \in \mathcal{D}_A^\infty$ be a vector of the form (13). Then, using the canonical commutation relations (CCR)

$$[a(f), a(g)^*] = \langle f, g \rangle, \quad f, g \in \mathcal{H}, \quad (73)$$

on

$$\mathcal{F}_{b,\text{fin}}(\mathcal{H})$$

$$:= \mathcal{L} \{a(f_1)^* \cdots a(f_n)^* \Omega_b \mid n \geq 0, f_j \in \mathcal{H}, j = 1, \dots, n\} \quad (74)$$

and (6), we have

$$\begin{aligned} a(g) \otimes b(v)^* \Psi &= \sum_{j=1}^n \langle g, f_j \rangle a(f_1)^* \cdots \widehat{a(f_j)}^* \cdots a(f_n)^* \Omega_b \\ &\quad \otimes b(v)^* b(u_1)^* \cdots b(u_p)^* \Omega_f \\ &= \sum_{j=1}^n a(f_1)^* \cdots \widehat{a(f_j)}^* \cdots a(f_n)^* \Omega_b \\ &\quad \otimes b(T_{g,v} f_j)^* b(u_1)^* \cdots b(u_p)^* \Omega_f \\ &= d_{T_{g,v}} \Psi. \end{aligned} \quad (75)$$

Hence, by Proposition 1-(v)

$$\bar{d}(\alpha) \Psi = d_A \Psi + \alpha d_{T_{g,v}} \Psi = d_{A(\alpha)} \Psi. \quad (76)$$

By a limiting argument, we obtain the operator equality (72). $\square$

Theorem 14. (i) *For all $\alpha \in \mathbb{C}$, $Q(\alpha)$ is self-adjoint and*

$$Q(\alpha) = Q_{A(\alpha)}. \quad (77)$$

(ii) *For all $\alpha \in \mathbb{C}$, $Q(\alpha)$ is essentially self-adjoint on $\mathcal{D}_A^\infty$.*

(iii) *For all $\alpha \in \mathbb{C}$,*

$$Q(\alpha) = \overline{Q_A + V_{g,v}(\alpha)}. \quad (78)$$

(iv) The operator Γ leaves $D(Q(\alpha))$ invariant and

$$\Gamma Q(\alpha) + Q(\alpha) \Gamma = 0 \quad \text{on } D(Q(\alpha)). \quad (79)$$

(v) For all $\Psi \in \mathcal{D}_A^\infty$, the vector-valued function: $\alpha \mapsto Q(\alpha)\Psi$ is strongly continuous on $\mathbb{C}$. Moreover, for all $z \in \mathbb{C} \setminus \mathbb{R}$, $(Q(\alpha) - z)^{-1}$ is strongly continuous in $\alpha \in \mathbb{C}$.

Proof. (i) Operator equality (77) follows from (72) and (67). Then, by Theorem 2-(i) with A replaced by $A(\alpha)$, we conclude that $Q(\alpha)$ is self-adjoint.

(ii) We have the following operator equalities:

$$\begin{aligned} A(\alpha)^* A(\alpha) &= A^* A + X_{g,v}(\alpha), \\ A(\alpha) A(\alpha)^* &= A A^* + Y_{g,v}(\alpha), \end{aligned} \quad (80)$$

where $X_{g,v}(\alpha)$ and $Y_{g,v}(\alpha)$ are bounded linear operators on $\mathcal{H}$ and $\mathcal{K}$, respectively, given as follows:

$$\begin{aligned} X_{g,v}(\alpha) f &:= \alpha \langle g, f \rangle A^* v + \alpha^* \langle A^* v, f \rangle g + |\alpha|^2 \|v\|^2 \langle g, f \rangle g, \\ f &\in \mathcal{H}, \\ Y_{g,v}(\alpha) u &:= \alpha \langle A g, u \rangle v + \alpha^* \langle v, u \rangle A g + |\alpha|^2 \|g\|^2 \langle v, u \rangle v, \\ u &\in \mathcal{K}. \end{aligned} \quad (81)$$

Hence, by the Kato-Rellich theorem ([12, Chapter V, Theorems 4.3, 4.4], [10, Theorem X.12]), $A(\alpha)^* A(\alpha)$ (resp., $A(\alpha) A(\alpha)^*$) is essentially self-adjoint on each core of $A^* A$ (resp., $A A^*$). In particular, $C^\infty(A^* A)$ (resp., $C^\infty(A A^*)$) is a core of $A(\alpha)^* A(\alpha)$ (resp., $A(\alpha) A(\alpha)^*$). Hence, by a core

property of second quantization operators, $\mathcal{D}_A^\infty$ is a core of $H_{BF}(A(\alpha))$. By this fact and Theorem 2-(i), $\mathcal{D}_A^\infty$ is a core of $Q_{A(\alpha)} = Q(\alpha)$.

(iii) This follows from (ii) and (68).

(iv) This follows from (77) and Theorem 2-(ii) with A replaced by $A(\alpha)$.

(v) Let $\Psi \in \mathcal{D}_A^\infty$. Then, for all $\alpha, \beta \in \mathbb{C}$, we have

$$\begin{aligned} &\|Q(\beta)\Psi - Q(\alpha)\Psi\| \\ &\leq |\beta - \alpha| (\|a(g) \otimes b(v)^* \Psi\| + \|a(g)^* \otimes b(v) \Psi\|) \\ &\longrightarrow 0 \quad (\beta \longrightarrow \alpha). \end{aligned} \quad (82)$$

Since $\mathcal{D}_A^\infty$ is a common core for $\{Q(\alpha)\}_{\alpha \in \mathbb{C}}$ as proved in part (ii), a general fact [11, Theorem VIII.25] implies the strong continuity of $(Q(\alpha) - z)^{-1}$ ($z \in \mathbb{C} \setminus \mathbb{R}$) in $\alpha \in \mathbb{C}$. $\square$

Remark 15. Since the operators $X_{g,v}(\alpha)$ and $Y_{g,v}(\alpha)$ are finite rank, they are compact. Hence it follows from (80) and a general theorem (e.g., [16, p.113, Corollary 2]) that, for all $\alpha \in \mathbb{C}$,

$$\begin{aligned} \sigma_{\text{ess}}(A(\alpha)^* A(\alpha)) &= \sigma_{\text{ess}}(A^* A), \\ \sigma_{\text{ess}}(A(\alpha) A(\alpha)^*) &= \sigma_{\text{ess}}(A A^*), \end{aligned} \quad (83)$$

where, for a self-adjoint operator T , $\sigma_{\text{ess}}(T)$ denotes the essential spectrum of T . In particular, if the spectrum $\sigma(A^* A)$ is purely discrete (in this case, $\sigma(A A^*)$ also is purely discrete by (60)), then so are $\sigma(A(\alpha)^* A(\alpha))$ and $\sigma(A(\alpha) A(\alpha)^*)$.

4.2. Spectra of $Q(\alpha)$. We have (77). Hence, applying Theorem 7 with A replaced by $A(\alpha)$, we obtain the following result on the spectra of $Q(\alpha)$.

Theorem 16. For all $\alpha \in \mathbb{C}$, $\sigma(Q(\alpha))$ and $\sigma_p(Q(\alpha))$ are symmetric with respect to the origin and

$$\begin{aligned} \sigma(Q(\alpha)) &= \{0\} \cup \left(\bigcup_{n=1}^{\infty} \left\{ \pm \sqrt{\sum_{j=1}^n \lambda_j} \mid \lambda_j \in \sigma(A(\alpha)^* A(\alpha)) \setminus \{0\}, j = 1, \dots, n \right\} \right), \\ \sigma_p(Q(\alpha)) &= \{0\} \cup \left(\bigcup_{n=1}^{\infty} \left\{ \pm \sqrt{\sum_{j=1}^n \lambda_j} \mid \lambda_j \in \sigma_p(A(\alpha)^* A(\alpha)) \setminus \{0\}, j = 1, \dots, n \right\} \right) \end{aligned} \quad (84)$$

with

$$\text{nul}(Q(\alpha) - \lambda) = \text{nul}(Q(\alpha) + \lambda), \quad \lambda \in \sigma_p(Q(\alpha)). \quad (85)$$

This theorem shows that the spectrum and the point spectrum of $Q(\alpha)$ are completely determined from those of $A(\alpha)^* A(\alpha) \setminus \{0\}$.

4.3. Identification of the Domain of $Q(\alpha)$. The following theorem gives a sufficient condition for the domain of $Q(\alpha)$ to be equal to that of the unperturbed operator Q_A .

Theorem 17. Suppose that A is injective and $g \in D(A^{-1})$. Then, for all $|\alpha| < 1/\|v\| \|A^{-1}g\|$, $Q(\alpha)$ is self-adjoint with $D(Q(\alpha)) = D(Q_A)$ and

$$Q(\alpha) = Q_A + V_{g,v}(\alpha). \quad (86)$$

Moreover, $Q(\alpha)$ is essentially self-adjoint on any core for Q_A .

To prove this theorem, we recall well known facts (e.g., [17, Lemma 2.1], [8]).

Lemma 18. Let S be a nonnegative self-adjoint operator on $\mathcal{H}$ with $\ker S = \{0\}$. Then, for all $f \in D(S^{-1/2})$, $D(d\Gamma_b(S)^{1/2}) \subset D(a(f)) \cap D(a(f)^*)$ with

$$\|a(f)\psi\| \leq \|S^{-1/2}f\| \|d\Gamma_b(S)^{1/2}\psi\|, \quad (87)$$

$$\|a(f)^*\psi\|^2 \leq \|S^{-1/2}f\|^2 \|d\Gamma_b(S)^{1/2}\psi\|^2 + \|f\|^2 \|\psi\|^2, \quad (88)$$

$$\psi \in D(d\Gamma_b(S)^{1/2}).$$

Lemma 19. The operator $V_{g,v}(\alpha)$ is a closed symmetric operator with $D(V_{g,v}(\alpha)) = D(a(g) \otimes b(v)^*) \cap D(a(g)^* \otimes b(v))$.

Proof. We need only to prove the closedness of $V_{g,v}(\alpha)$. Using (9) and a limiting argument, we have, for all $\Psi, \Phi \in D(V_{g,v}(\alpha)) = D(a(f) \otimes b(v)^*) \cap D(a(f)^* \otimes b(v))$,

$$(a(g) \otimes b(v)^* \Psi, a(g)^* \otimes b(v) \Phi) = 0. \quad (89)$$

Hence, by a general fact [2, Lemma 4.1], $V_{g,v}(\alpha)$ is closed. $\square$

Lemma 20. Let A be injective and $g \in D(A^{-1})$. Then $D(Q_A) \subset D(V_{g,v}(\alpha))$.

Proof. Under the present assumption, A^*A is injective and $(A^*A)^{-1/2} = |A|^{-1}$. Let $\Psi \in \mathcal{D}_A^\infty$. Then, using (87) and the well known fact that

$$\|b(u)\| = \|b(u)^*\| = \|u\|, \quad u \in \mathcal{K}, \quad (90)$$

we can show that

$$\begin{aligned} \|a(g) \otimes b(v)^* \Psi\| &\leq \|v\| \|a(g) \otimes I \Psi\| \\ &\leq \|v\| \| |A|^{-1} g \| \|d\Gamma_b(A^*A)^{1/2} \otimes I \Psi\| \\ &\leq \|v\| \|A^{-1} g\| \|H_{BF}(A)^{1/2} \Psi\|. \end{aligned} \quad (91)$$

By this estimate and the fact that

$$\|H_{BF}(A)^{1/2} \Phi\| = \|Q_A \Phi\|, \quad \Phi \in D(Q_A), \quad (92)$$

we obtain

$$\|a(g) \otimes b(v)^* \Psi\| \leq \|v\| \|A^{-1} g\| \|Q_A \Psi\|. \quad (93)$$

Since $\mathcal{D}_A^\infty$ is a core of Q_A , this inequality extends to all $\Psi \in D(Q_A)$ showing that $D(Q_A) \subset D(a(g) \otimes b(v)^*)$. Similarly, using (88), we can show that $D(Q_A) \subset D(a(g)^* \otimes b(v))$ with

$$\begin{aligned} \|a(g)^* \otimes b(v) \Psi\|^2 &\leq \|v\|^2 (\|A^{-1} g\|^2 \|Q_A \Psi\|^2 + \|g\|^2 \|\Psi\|^2), \\ \Psi &\in D(Q_A). \end{aligned} \quad (94)$$

Thus $D(Q_A) \subset D(V_{g,v}(\alpha))$. $\square$

Proof of Theorem 17. By Lemma 20 and Theorem 14, the operator

$$T := Q_A + V_{g,v}(\alpha) \quad (95)$$

is essentially self-adjoint with $D(T) = D(Q_A)$ and $\bar{T} = Q(\alpha)$. By (89), the CCR (73), and the CAR (8), one can show that, for all $\Psi \in D(Q_A)$,

$$\begin{aligned} \|V_{g,v}(\alpha) \Psi\|^2 &= |\alpha|^2 (\|a(g) \otimes b(v)^* \Phi\|^2 + \|a(g)^* \otimes b(v) \Phi\|^2) \\ &= |\alpha|^2 (\|v\|^2 \|a(g) \otimes I \Psi\|^2 + \|g\|^2 \|I \otimes b(v) \Psi\|^2) \\ &\leq |\alpha|^2 \|v\|^2 (\|a(g) \otimes I \Psi\|^2 + \|g\|^2 \|\Psi\|^2). \end{aligned} \quad (96)$$

Hence, using (87) (cf. the proof of Lemma 20), we obtain

$$\begin{aligned} \|V_{g,v}(\alpha) \Psi\|^2 &\leq |\alpha|^2 \|v\|^2 (\|A^{-1} g\|^2 \|Q_A \Psi\|^2 + \|g\|^2 \|\Phi\|^2), \\ \Psi &\in D(Q_A). \end{aligned} \quad (97)$$

Therefore, by the Kato-Rellich theorem, for all $\alpha \in \mathbb{C}$ satisfying $|\alpha| \|v\| \|A^{-1} g\| < 1$, T is self-adjoint on $D(Q_A)$, implying that $T = Q(\alpha)$, and essentially self-adjoint on any core for Q_A . $\square$

5. Kernel of $Q(\alpha)$

In this section we investigate the kernel of $Q(\alpha)$. For this purpose, we need some conditions on $\{A, g, v\}$.

(C.1) A is injective, $v \in D(A^{-1})$, and $\langle g, A^{-1}v \rangle \neq 0$. In this case we introduce a constant

$$\alpha_0 := -\frac{1}{\langle g, A^{-1}v \rangle}. \quad (98)$$

(C.2) A^* is injective, $g \in D(A^{*-1})$, and $\langle v, A^{*-1}g \rangle \neq 0$. In this case we introduce a constant

$$\beta_0 := -\frac{1}{\langle A^{*-1}g, v \rangle}. \quad (99)$$

(C.3) (a) A is injective and $v \notin D(A^{-1})$ or (b) A is injective and $v \in D(A^{-1})$ with $\langle g, A^{-1}v \rangle = 0$.

(C.4) (a) A^* is injective and $g \notin D(A^{*-1})$ or (b) A^* is injective and $g \in D(A^{*-1})$ with $\langle v, A^{*-1}g \rangle = 0$.

Remark 21. (i) If (C.1) and (C.2) hold, then it follows that $\alpha_0 = \beta_0$.

(ii) If A is Fredholm, then A^* is Fredholm [12, Chapter IV, Corollary 5.14]. Hence, if (C.1) and (C.2) hold and A is Fredholm, then $\mathcal{H} = \overline{R(A^*)} = R(A^*)$, $\mathcal{H} = \overline{R(A)} = R(A)$. Hence A and A^* are bijective with $D(A^{-1}) = \mathcal{H}$ and $D(A^{*-1}) = \mathcal{H}$. Thus, in this case, conditions $v \in D(A^{-1})$ and $g \in D(A^{*-1})$ are trivially satisfied.

We first consider the kernel of $A(\alpha)$ and $A(\alpha)^*$.

Lemma 22. (i) Suppose that (C.1) holds. Then

$$\ker A(\alpha) = \{0\}, \quad \alpha \neq \alpha_0, \quad (100)$$

$$\ker A(\alpha_0) = \{cA^{-1}\nu \mid c \in \mathbb{C}\}. \quad (101)$$

(ii) Suppose that (C.2) holds. Then

$$\begin{aligned} \ker A(\alpha)^* &= \{0\}, \quad \alpha \neq \beta_0, \\ \ker A(\beta_0)^* &= \{cA^{*-1}g \mid c \in \mathbb{C}\}. \end{aligned} \quad (102)$$

(iii) Suppose that (C.3) holds. Then, for all $\alpha \in \mathbb{C}$,

$$\ker A(\alpha) = \{0\}. \quad (103)$$

(iv) Suppose that (C.4) holds. Then, for all $\alpha \in \mathbb{C}$,

$$\ker A(\alpha)^* = \{0\}. \quad (104)$$

Proof. (i) Let $f \in \ker A(\alpha)$. Then $f \in D(A)$ and

$$Af = -\alpha \langle g, f \rangle \nu. \quad (105)$$

Hence

$$f = -\alpha \langle g, f \rangle f_0 \quad (106)$$

with $f_0 := A^{-1}\nu$. This implies

$$\ker A(\alpha) \subset \{cf_0 \mid c \in \mathbb{C}\}. \quad (107)$$

Let $\alpha \neq \alpha_0$. Then, taking the inner product of (106) with g and using the fact that $\langle g, f_0 \rangle = -1/\alpha_0$, we have $\langle g, f \rangle = 0$. Hence, by (106) again, $f = 0$. Thus (100) follows.

Let $\alpha = \alpha_0$. Then

$$Af_0 = \nu = -\alpha_0 \langle g, f_0 \rangle \nu. \quad (108)$$

Hence $f_0 \in \ker A(\alpha_0)$. Thus, by (107), we obtain (101).

(ii) We have

$$A(\alpha)^* = A^* + \alpha^* \langle \nu, \cdot \rangle g. \quad (109)$$

Hence, by replacing $\{A, g, \nu, \alpha\}$ by $\{A^*, \nu, g, \alpha^*\}$ in the proof of (i), we can easily obtain (102).

(iii) For all $f \in \ker A(\alpha)$, we have (105). Let $\alpha \neq 0$. Then (105) means that $\langle g, f \rangle \nu \in D(A^{-1})$. Hence, if $\nu \notin (A^{-1})$, then $\langle g, f \rangle = 0$, which implies via (105) again that $Af = 0$; that is, $f = 0$. Thus (103) follows.

If $\nu \in D(A^{-1})$ with $\langle g, A^{-1}\nu \rangle = 0$, then $f = -\alpha \langle g, f \rangle A^{-1}\nu$ and hence $\langle g, f \rangle = 0$. Hence $f = 0$. Thus (103) follows.

(iv) It is similar to the proof of (iii). $\square$

Theorem 23. (i) Assume (C.1). Then

$$\begin{aligned} \ker Q(\alpha_0) &= \bigoplus_{n,p=0}^{\infty} \left[\left(\bigotimes^n \{zA^{-1}\nu \mid z \in \mathbb{C}\} \right) \otimes \wedge^p (\ker A(\alpha_0)^*) \right], \end{aligned} \quad (110)$$

and hence $\text{nul } Q(\alpha_0) = \infty$.

Moreover, for all $\alpha \neq \alpha_0$,

$$\ker Q(\alpha) = \bigoplus_{p=0}^{\infty} \mathbb{C} \otimes \wedge^p (\ker A(\alpha)^*). \quad (111)$$

(ii) Assume (C.2). Then

$$\begin{aligned} \ker Q(\beta_0) &= \bigoplus_{n=0}^{\infty} \left\{ \left[\bigotimes_s^n \ker(A(\beta_0)) \right] \otimes \left[\mathbb{C} \oplus \mathcal{L}(\{A^{*-1}g\}) \right] \right\}, \\ \ker Q(\alpha) &= \bigoplus_{n=0}^{\infty} \left[\bigotimes_s^n \ker A(\alpha) \otimes \mathbb{C} \right], \quad \alpha \neq \beta_0. \end{aligned} \quad (112)$$

(iii) Assume (C.3). Then, for all $\alpha \in \mathbb{C}$,

$$\ker Q(\alpha) = \bigoplus_{p=0}^{\infty} \left[\mathbb{C} \otimes \wedge^p (\ker A(\alpha)^*) \right]. \quad (113)$$

(iv) Assume (C.4). Then, for all $\alpha \in \mathbb{C}$,

$$\ker Q(\alpha) = \bigoplus_{n=0}^{\infty} \left[\bigotimes_s^n \ker A(\alpha) \otimes \mathbb{C} \right]. \quad (114)$$

Proof. Under condition (C.1), we have (100) and (101). Hence $\bigotimes_s^n \ker A(\alpha_0) = \bigotimes^n \{zA^{-1}\nu \mid z \in \mathbb{C}\}$. Therefore, applying (77) and Theorem 4, we obtain (110).

In the case $\alpha \neq \alpha_0$, (111) follows from (100), (77), and Theorem 4.

Using Lemma 22, one can prove (ii)–(iv) in the same manner as in the proof of (i). $\square$

Some compatible compositions of conditions (C.1)–(C.4) yield the following results.

Corollary 24. (i) Assume (C.1) and (C.2). Then

$$\begin{aligned} \ker Q(\alpha_0) &= \overline{\mathcal{L} \left(\left\{ a(A^{-1}\nu)^{*n} \Omega_b \otimes b(A^{*-1}g)^{*j} \Omega_f \mid n \geq 0, j = 0, 1 \right\} \right)}, \\ \ker Q(\alpha) &= \{c\Omega_b \otimes \Omega_f \mid c \in \mathbb{C}\}, \quad \alpha \neq \alpha_0. \end{aligned} \quad (115)$$

(ii) Assume (C.1) and (C.4). Then

$$\begin{aligned} \ker Q(\alpha_0) &= \overline{\mathcal{L} \left(\left\{ a(A^{-1}\nu)^{*n} \Omega_b \otimes \Omega_f \mid n \geq 0 \right\} \right)}, \\ \ker Q(\alpha) &= \{c\Omega_b \otimes \Omega_f \mid c \in \mathbb{C}\}, \quad \alpha \neq \alpha_0. \end{aligned} \quad (116)$$

(iii) Assume (C.2) and (C.3). Then

$$\ker Q(\beta_0) = \mathcal{L} \left(\left\{ \Omega_b \otimes b(A^{*-1}g)^{*j} \Omega_f \mid j = 0, 1 \right\} \right). \quad (117)$$

$$\ker Q(\alpha) = \{c\Omega_b \otimes \Omega_f \mid c \in \mathbb{C}\}, \quad \alpha \neq \beta_0.$$

(iv) Assume (C.3) and (C.4). Then, for all $\alpha \in \mathbb{C}$,

$$\ker Q(\alpha) = \{c\Omega_b \otimes \Omega_f \mid c \in \mathbb{C}\}. \quad (118)$$

6. Fredholm Property and the Index $\text{ind}_\Gamma Q(\alpha)$

Theorem 23 has also implications for Fredholm property of $Q(\alpha)$. Since we want to see how Fredholm property of Q_A may change under the perturbation $V_{g,v}(\alpha)$, we take our consideration under the same conditions as in Theorem 5.

We first note the following fact.

Lemma 25. *Let A be semi-Fredholm. Then, for all $\alpha \in \mathbb{C}$, $A(\alpha)$ is semi-Fredholm and*

$$\text{ind } A = \text{ind } A(\alpha). \quad (119)$$

Proof. We have (71) and $T_{g,v}$ is a one-rank operator, which is compact. Thus, by a general theorem [12, Chapter IV, Theorem 5.26], the lemma holds. $\square$

Theorem 26. (i) *Suppose that (C.1) holds and A is Fredholm. Then $Q(\alpha_0)_+$ is not semi-Fredholm and, for all $\alpha \in \mathbb{C} \setminus \{\alpha_0\}$, $Q(\alpha)_+$ is Fredholm with*

$$\text{ind}_\Gamma Q(\alpha) = \delta_{0, \text{ind } A^*} = \text{ind}_\Gamma Q_A. \quad (120)$$

(ii) *Suppose that (C.3) holds and A is Fredholm. Then, for all $\alpha \in \mathbb{C}$, $Q(\alpha)$ is Fredholm and (120) holds.*

(iii) *Suppose that (C.2) holds and A is semi-Fredholm. Then $Q(\beta_0)_+$ is not semi-Fredholm and, for all $\alpha \in \mathbb{C} \setminus \{\beta_0\}$, the following hold.*

(a) *If A is injective, then $Q(\alpha)_+$ is Fredholm with $\text{ind}_\Gamma Q(\alpha) = 1$.*

(b) *If A is not injective, then $Q(\alpha)_+$ is semi-Fredholm with $\text{ind}_\Gamma Q(\alpha) = \infty$.*

(iv) *Suppose that (C.4) holds and that A is semi-Fredholm.*

(a) *If A is injective, then, for all $\alpha \in \mathbb{C}$, $Q(\alpha)_+$ is Fredholm and $\text{ind}_\Gamma Q(\alpha) = 1$.*

(b) *If A is not injective, then, for all $\alpha \in \mathbb{C}$, $Q(\alpha)_+$ is semi-Fredholm and $\text{ind}_\Gamma Q(\alpha) = \infty$.*

(v) *Let A be semi-Fredholm with $\text{nul } A \geq 1$ and $\text{nul } A^* \geq 1$. Suppose that $g \in (\ker A)^\perp$ and $v \in (\ker A^*)^\perp$. Then $Q(\alpha)_+$ is not semi-Fredholm.*

Proof. In the proofs below, (77) is used without mentioning explicitly.

(i) By Lemma 22-(i), $\text{nul } A(\alpha_0) = 1$. By this fact and Lemma 25, we have

$$\text{nul } A(\alpha_0)^* = 1 + \text{nul } A^* \geq 1. \quad (121)$$

Hence, applying Theorem 5-(iii) with A replaced by $A(\alpha_0)$, one concludes that $Q(\alpha_0)_+$ is not semi-Fredholm.

Let $\alpha \neq \alpha_0$. Then, by Lemma 25, we have $\text{nul } A^* = \text{nul } A(\alpha)^*$. Hence, by Theorem 5-(i), $Q(\alpha)_+$ is Fredholm and $\text{ind}_\Gamma Q(\alpha) = \delta_{0, \text{nul } A(\alpha)^*} = \delta_{0, \text{nul } A^*}$. Thus (120) holds.

(ii) It is similar to (i).

(iii) In the present case, we have $\text{nul } A^* = 0$ and $\text{nul } A(\beta_0)^* = 1$. Hence, by Lemma 25, $\text{nul } A(\beta_0) = \text{nul } A + 1 \geq 1$. Hence, by applying Theorem 5-(iii) with A replaced by $A(\beta_0)$, we conclude that $Q(\beta_0)_+$ is not semi-Fredholm.

Let $\alpha \neq \beta_0$. Then, by Lemma 22-(ii), $A(\alpha)^*$ is injective. By Lemma 25, $\text{nul } A = \text{nul } A(\alpha)$.

(a) Let A be injective. Then $A(\alpha)$ is injective. Hence, we can apply Theorem 5-(i) with A replaced by $A(\alpha)$ to conclude that $Q(\alpha)_+$ is Fredholm with $\text{ind}_\Gamma Q(\alpha) = 1$.

(b) Let A be not injective. Then $\text{nul } A(\alpha) \geq 1$. Hence, we can apply Theorem 5-(ii) with A replaced by $A(\alpha)$ to conclude that $Q(\alpha)_+$ is semi-Fredholm with $\text{ind}_\Gamma Q(\alpha) = \infty$.

(iv) In the present case, we have $\text{nul } A(\alpha)^* = 0$ for all $\alpha \in \mathbb{C}$. Hence, by Lemma 25, $\text{nul } A = \text{nul } A(\alpha)$ for all $\alpha \in \mathbb{C}$.

(a) In this case, $\text{nul } A(\alpha) = 0$. Hence, by Theorem 5-(i) with A replaced by $A(\alpha)$, $Q(\alpha)$ is Fredholm and $\text{ind}_\Gamma Q(\alpha)_+ = 1$.

(b) In this case, we can apply Theorem 5-(ii) with A replaced by $A(\alpha)$ to conclude that $Q(\alpha)_+$ is semi-Fredholm and $\text{ind}_\Gamma Q(\alpha)_+ = \infty$.

(v) In the present case, we have $\ker A \subset \ker A(\alpha)$ and $\ker A^* \subset \ker A(\alpha)^*$. Hence $\text{nul } A(\alpha) \geq 1$ and $\text{nul } A(\alpha)^* \geq 1$. Therefore, we can apply Theorem 5-(iii) with A replaced by $A(\alpha)$ to conclude that $Q(\alpha)_+$ is not semi-Fredholm. $\square$

7. Nonzero Eigenvalues of $Q(\alpha)$

Finally we consider existence of nonzero eigenvalues of $Q(\alpha)$. In this section we need the following condition.

Hypothesis (I). (i) $\mathcal{H} = \mathcal{K}$;

(ii) A is an injective and nonnegative self-adjoint operator;

(iii) $g = v \in D(A^{-1})$.

Under Hypothesis (I), the constant α_0 defined by (98) takes the form

$$\alpha_0 = -\frac{1}{\langle v, A^{-1}v \rangle} < 0. \quad (122)$$

Theorem 27. *Let Hypothesis (I) be satisfied and $\alpha < \alpha_0 (< 0)$. Then, there exists a unique constant $x_0(\alpha) < 0$ such that $\alpha \langle v, (x_0(\alpha) - A)^{-1}v \rangle = 1$ and, for all $n \in \{0\} \cup \mathbb{N}$,*

$$\pm \sqrt{n} x_0(\alpha) \in \sigma_p(Q(\alpha)). \quad (123)$$

with eigenvectors

$$\begin{aligned} & [Q(\alpha) \pm \sqrt{n} x_0(\alpha)] \{a(\phi_\alpha)^{*n-p} \Omega_b \otimes b(\phi_\alpha)^{*p} \Omega_f\} \\ & \in \ker(Q(\alpha) \mp \sqrt{n} x_0(\alpha)) \quad (n \geq p \geq 0), \end{aligned} \quad (124)$$

where

$$\phi_\alpha := (x_0(\alpha) - A)^{-1}v. \quad (125)$$

Moreover, $x_0(\alpha)$, as a function of $\alpha < \alpha_0$, is strictly monotone increasing on $(-\infty, \alpha_0)$ with $\lim_{\alpha \rightarrow -\infty} x_0(\alpha) = -\infty$ and $\lim_{\alpha \rightarrow \alpha_0} x_0(\alpha) = 0$.

Note that Theorem 27 holds even if Q_A has no nonzero eigenvalues. This is an interesting phenomenon. Since the

condition $\alpha < \alpha_0 < 0$ implies that $|\alpha| > |\alpha_0|$, the phenomenon may be regarded as a *strong coupling effect*.

To prove Theorem 27, we prepare two lemmas.

We consider a condition slightly more general than Hypothesis (I).

Hypothesis (II). In addition to (i) and (ii) in Hypothesis (I), the following holds: $v \in D(A^a) \cap D(A^{-1})$ for some $a \in \mathbb{R}$ and

$$g = A^a v. \quad (126)$$

Under Hypothesis (II), the constant α_0 defined by (98) takes the form

$$\alpha_0 = -\frac{1}{\langle A^a v, A^{-1} v \rangle} < 0. \quad (127)$$

Lemma 28. *Let Hypothesis (II) be satisfied.*

(i) *Let $\alpha_0 < \alpha$. Then there exist no solutions $x \leq 0$ to the equation*

$$\alpha \langle A^a v, (x - A)^{-1} v \rangle = 1. \quad (128)$$

(ii) *Let $\alpha \leq \alpha_0 (< 0)$. Then there exists a unique constant $x_0(\alpha) \leq 0$ such that*

$$\alpha \langle A^a v, (x_0(\alpha) - A)^{-1} v \rangle = 1. \quad (129)$$

The constant $x_0(\alpha)$ is, as a function of $\alpha \leq \alpha_0 (< 0)$, strictly monotone increasing and differentiable on $(-\infty, \alpha_0)$ with $x_0(\alpha_0) = 0$ and $\lim_{\alpha \rightarrow -\infty} x_0(\alpha) = -\infty$.

Proof. (i) Since A is injective, nonnegative, and self-adjoint, we can define a function F on $(-\infty, 0]$ by

$$F(x) := -\langle A^a v, (x - A)^{-1} v \rangle > 0, \quad x \in (-\infty, 0]. \quad (130)$$

This function is differentiable on $(-\infty, 0)$ with

$$\frac{dF(x)}{dx} = \langle A^a v, (x - A)^{-2} v \rangle > 0, \quad x \in (-\infty, 0). \quad (131)$$

Hence F is strictly monotone increasing. It is easy to see that $\lim_{x \rightarrow -\infty} F(x) = 0$ and $\lim_{x \rightarrow 0} F(x) = -1/\alpha_0 > 0$ (note that $\alpha_0 < 0$). Hence, if $\alpha > \alpha_0$, then $- \alpha F(x) = 1$ has no solutions x in $(-\infty, 0]$.

(ii) Let $\alpha \leq \alpha_0$. Then, by the proof of (i), there exists a unique constant $x_0(\alpha) \leq 0$ such that $- \alpha F(x_0(\alpha)) = 1$. The inverse function F^{-1} of F exists on $(0, -1/\alpha_0]$ and is differentiable on $(0, -1/\alpha_0)$. Hence $x_0(\alpha) = F^{-1}(-1/\alpha)$, $\alpha \leq \alpha_0$. This implies the assertion on $x_0(\alpha)$. $\square$

Lemma 29. *Let Hypothesis (II) be satisfied.*

(i) *Let $\alpha > \alpha_0$. Then*

$$\sigma_p(A(\alpha)) \cap (-\infty, 0] = \emptyset. \quad (132)$$

(ii) *Let $\alpha \leq \alpha_0$ and ϕ_α be defined by (125). Then*

$$\sigma_p(A(\alpha)) \cap (-\infty, 0] = \{x_0(\alpha)\}, \quad (133)$$

$$\ker(A(\alpha) - x_0(\alpha)) = \{c\phi_\alpha \mid c \in \mathbb{C}\}. \quad (134)$$

Proof. Let $x \in \sigma_p(A(\alpha)) \cap (-\infty, 0]$. Then there is a nonzero vector $\phi \in D(A)$ such that $A\phi + \alpha \langle A^a v, \phi \rangle v = x\phi$. It follows that

$$\phi = \alpha \langle A^a v, \phi \rangle (x - A)^{-1} v. \quad (135)$$

If $\langle A^a v, \phi \rangle = 0$, then $\phi = 0$, which is a contradiction. Hence $\langle A^a v, \phi \rangle \neq 0$. Taking the inner product of (135) with $A^a v$, we obtain

$$\alpha \langle A^a v, (x - A)^{-1} v \rangle = 1. \quad (136)$$

(i) If $\alpha > \alpha_0$, then, by Lemma 28-(i), (136) has no solutions x in $(-\infty, 0]$. Hence (132) holds.

(ii) Let $\alpha \leq \alpha_0$. Then, by Lemma 28-(ii), we have $x = x_0(\alpha)$. Thus $\sigma_p(A(\alpha)) \cap (-\infty, 0] \subset \{x_0(\alpha)\}$. It is easy to check that $A(\alpha)\phi_\alpha = x_0(\alpha)\phi_\alpha$. Thus (133) follows. By (135), every vector $\phi \in \ker(A(\alpha) - x_0(\alpha))$ satisfies $\phi = \alpha \langle A^a v, \phi \rangle \phi_\alpha$. Hence $\phi \in \{c\phi_\alpha \mid c \in \mathbb{C}\}$. Thus (134) follows. $\square$

Proof of Theorem 27. Under the present assumption, $A(\alpha)$ is self-adjoint. Hence, by Lemma 29, $x_0(\alpha)^2 \in \sigma_p(A(\alpha)^2)$. Hence Theorem 16 implies (123). We have $A(\alpha)^2 \phi_\alpha = x_0(\alpha)^2 \phi_\alpha$. Hence

$$\Phi^{(n,p)}(\alpha) := a(\phi_\alpha)^{*n-p} \Omega_b \otimes b(\phi_\alpha)^{*p} \Omega_f \quad (n \geq p \geq 0) \quad (137)$$

are eigenvectors of $Q_{A(\alpha)}^2 = Q(\alpha)^2$ with eigenvalue $nx_0(\alpha)^2$. Hence, applying Lemma 8 with $S = Q(\alpha)$, we see that $(Q(\alpha) \pm \sqrt{n}x_0(\alpha))\Phi^{(n,p)}(\alpha) \in \ker(Q(\alpha) \mp \sqrt{n}x_0(\alpha))$. $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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