

Research Article

Inclusion Properties of Certain Subclasses of p -Valent Functions Associated with the Integral Operator

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Received 2 May 2014; Accepted 18 May 2014; Published 29 May 2014

Academic Editor: Ming-Sheng Liu

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The purpose of the present paper is to introduce two subclasses of p -valent functions by using the integral operator and to investigate various properties for these subclasses.

1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions of the following form:

$$f(z) = z^p + \sum_{j=1}^{\infty} a_{p+j} z^{p+j}, \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}), \quad (1)$$

which are analytic and p -valent in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{P}_k(p, \gamma)$ be the class of functions g analytic in \mathbb{U} satisfying $g(0) = p$ and

$$\int_0^{2\pi} \left| \frac{\Re \{g(z)\} - \gamma}{p - \gamma} \right| d\theta \leq k\pi, \quad (z = re^{i\theta}; k \geq 2; 0 \leq \gamma < p). \quad (2)$$

The class $\mathcal{P}_k(p, \gamma)$ was introduced by Aouf [1] and we note the following:

- (i) the class $\mathcal{P}_k(1, \gamma) = \mathcal{P}_k(\gamma)$ was introduced by Padmanabhan and Parvatham [2];
- (ii) the class $\mathcal{P}_k(1, 0) = \mathcal{P}_k$ was introduced by Pinchuk [3];
- (iii) $\mathcal{P}_2(p, \gamma) = \mathcal{P}(p, \gamma)$ is the class of functions with positive real part greater than γ ($0 \leq \gamma < p$);
- (iv) $\mathcal{P}_2(1, \gamma) = \mathcal{P}(\gamma)$ is the class of functions with positive real part greater than γ ($0 \leq \gamma < 1$);
- (v) $\mathcal{P}_2(1, 0) = \mathcal{P}$ is the class of functions with positive real part.

From (1), we have $g \in \mathcal{P}_k(p, \gamma)$ if and only if there exists $g_1, g_2 \in \mathcal{P}(p, \gamma)$ such that

$$g(z) = \left(\frac{k}{4} + \frac{1}{2}\right) g_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) g_2(z), \quad (z \in \mathbb{U}). \quad (3)$$

It is known that [4] the class $\mathcal{P}_k(\gamma)$ is a convex set.

Motivated essentially by Jung et al. [5], Liu and Owa [6] introduced the integral operator $Q_{\beta, p}^{\alpha} : \mathcal{A}(p) \rightarrow \mathcal{A}(p)$ ($\alpha \geq 0; \beta > -p; p \in \mathbb{N}$) as follows:

$$Q_{\beta, p}^{\alpha} f(z) = \begin{cases} \left(\frac{p+\alpha+\beta-1}{p+\beta-1}\right) \frac{\alpha}{z^{\beta}} \int_0^z \left(1-\frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t) dt & (\alpha > 0), \\ f(z) & (\alpha = 0). \end{cases} \quad (4)$$

For $f \in \mathcal{A}(p)$ given by (1) and then from (4), we deduce that

$$Q_{\beta, p}^{\alpha} f(z) = z^p + \frac{\Gamma(\alpha+\beta+p)}{\Gamma(\beta+p)} \sum_{j=1}^{\infty} \frac{\Gamma(\beta+p+j)}{\Gamma(\alpha+\beta+p+j)} a_{p+j} z^{p+j} \quad (\alpha \geq 0; \beta > -p). \quad (5)$$

It is easily verified from (5) that (see [6])

$$z(Q_{\beta, p}^{\alpha+1} f(z))' = (\alpha + \beta + p) Q_{\beta, p}^{\alpha} f(z) - (\alpha + \beta) Q_{\beta, p}^{\alpha} f(z). \quad (6)$$

We note that (i) the one-parameter family of integral operator $Q_{\beta,1}^\alpha = Q_\beta^\alpha$ was defined by Jung et al. [5] and studied by Aouf [7] and Gao et al. [8].

(ii) Consider

$$Q_{c,p}^1 f(z) = F_{c,p}(f)(z) = \frac{c+p}{z^c} \int t^{c-1} f(z) dt, \quad (c > -p), \tag{7}$$

where the operator $F_{c,p}$ is the generalized Bernardi-Libera-Livingston integral operator (see [9]).

We have the following known subclasses $\mathcal{S}_k(p, \gamma)$ and $\mathcal{E}_k(p, \gamma)$ of the class $\mathcal{A}(p)$ for $0 \leq \gamma, \eta < p$, and $k \geq 2$ which are defined by

$$\mathcal{S}_k(p, \gamma) = \left\{ f \in \mathcal{A}(p) : \frac{zf'(z)}{f(z)} \in \mathcal{P}_k(p, \gamma), z \in \mathbb{U} \right\},$$

$$\mathcal{E}_k(p, \gamma) = \left\{ f \in \mathcal{A}(p) : \left(\frac{zf'(z)}{f(z)} \right)' \in \mathcal{P}_k(p, \gamma), z \in \mathbb{U} \right\}. \tag{8}$$

Next, by using the integral operator $Q_{\beta,p}^\alpha$, we introduce the following classes of analytic functions for $0 \leq \gamma < p$ and $k \geq 2$:

$$\mathcal{S}_k(p, \alpha; \gamma) = \left\{ f \in \mathcal{A}(p) : Q_{\beta,p}^\alpha f(z) \in \mathcal{S}_k(p, \gamma) \right\},$$

$$\mathcal{E}_k(p, \alpha; \gamma) = \left\{ f \in \mathcal{A}(p) : Q_{\beta,p}^\alpha f(z) \in \mathcal{E}_k(p, \gamma) \right\}. \tag{9}$$

We also note that

$$f \in \mathcal{E}_k(p, \alpha; \gamma) \iff \frac{zf'}{p} \in \mathcal{S}_k(p, \alpha; \gamma). \tag{10}$$

In particular, we set $\mathcal{S}_k(1, \alpha; \gamma) = \mathcal{S}_k(\alpha; \gamma)$ and $\mathcal{E}_k(1, \alpha; \gamma) = \mathcal{E}_k(\alpha; \gamma)$.

The following lemma will be required in our investigation.

Lemma 1 (see [10]). *Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ and let $\Psi(u, v)$ be a complex-valued function satisfying the following conditions:*

- (i) $\Psi(u, v)$ is continuous in a domain $D \in \mathbb{C}^2$;
- (ii) $(0, 1) \in D$ and $\Psi(1, 0) > 0$;
- (iii) $\Re\{\Psi(iu_2, v_1)\} > 0$ whenever $(iu_2, v_1) \in D$ and $v_1 \leq -(1/2)(1 + u_2^2)$.

If $h(z) = 1 + c_1z + c_2z^2 + \dots$ is analytic in \mathbb{U} such that $(h(z), zh'(z)) \in D$ and $\Re\{\Psi(h(z), zh'(z))\} > 0$ for $z \in \mathbb{U}$, then $\Re\{\Psi(h(z), zh'(z))\} > 0$ in \mathbb{U} .

Lemma 2 (see [11]). *Let $p(z)$ be analytic in \mathbb{U} with $p(0) = a$ and $\Re\{p(z)\} > 0, z \in \mathbb{U}$. Then, for $s > 0$ and $\mu \in \mathbb{C} \setminus \{-1\}$,*

$$\Re \left\{ p(z) + \frac{szp'(z)}{p(z) + \mu} \right\} > 0, \quad (|z| < r_0), \tag{11}$$

where r_0 is given by

$$r_0 = \frac{|\mu + 1|}{\sqrt{A + (A^2 - |\mu^2 - 1|)^{1/2}}}, \quad A = 2(s + 1)^2 + |\mu|^2 - 1, \tag{12}$$

and this radius is the best possible.

Lemma 3 (see [12]). *Let ψ be convex and let g be starlike in \mathbb{U} . Then, for F analytic in \mathbb{U} with $F(0) = 1, ((\psi * Fg)/(\psi * g))$ is contained in the convex hull of $F(\mathbb{U})$.*

In this paper, we obtain several inclusion properties of the classes $\mathcal{S}_k(p, \alpha; \gamma)$ and $\mathcal{E}_k(p, \alpha; \gamma)$ associated with the operator $Q_{\beta,p}^\alpha$.

2. Main Results

Unless otherwise mentioned, we assume throughout this paper that $k \geq 2, \alpha \geq 0, \beta > 0, 0 \leq \gamma < p$, and $p \in \mathbb{N}$.

Theorem 4. *One has*

$$\mathcal{S}_k(p, \alpha + 1; \gamma) \subset \mathcal{S}_k(p, \alpha; \gamma). \tag{13}$$

Proof. We begin by setting

$$\frac{z(Q_{\beta,p}^{\alpha+1} f(z))'}{Q_{\beta,p}^{\alpha+1} f(z)} = (p - \gamma)h(z) + \gamma$$

$$= \left(\frac{k}{4} + \frac{1}{2} \right) \{ (p - \gamma)h_1(z) + \gamma \}$$

$$- \left(\frac{k}{4} - \frac{1}{2} \right) \{ (p - \gamma)h_2(z) + \gamma \}, \tag{14}$$

where h_i is analytic in \mathbb{U} with $h_i(0) = 1, i = 1, 2$. Using the identity (6) in (14) and differentiating the resulting equation with respect to z , we obtain

$$\frac{z(Q_{\beta,p}^\alpha f(z))'}{Q_{\beta,p}^\alpha f(z)} = \left\{ \gamma + (p - \gamma)h(z) + \frac{(p - \gamma)zh'(z)}{(p - \gamma)h(z) + \gamma + \alpha + \beta} \right\} \in \mathcal{P}_k(p, \gamma). \tag{15}$$

This implies that

$$h_i(z) + \frac{zh'_i(z)}{(p - \gamma)h_i(z) + \gamma + \alpha + \beta} \in \mathcal{P}, \quad (z \in \mathbb{U}; i = 1, 2). \tag{16}$$

We form the functional $\Psi(u, v)$ by choosing $u = h_i(z)$ and $v = zh'_i(z)$:

$$\Psi(u, v) = u + \frac{v}{(p - \gamma)u + \gamma + \alpha + \beta}. \tag{17}$$

Clearly, the first two conditions of Lemma 1 are satisfied. Now, we verify condition (iii) as follows:

$$\Re \{ \Psi(iu_2, v_1) \} = \Re \left\{ \frac{v_1}{(p-\gamma)iu_2 + \gamma + \alpha + \beta} \right\} \leq -\frac{(\gamma + \alpha + \beta)(1 + u_2^2)}{2[(p-\gamma)^2u_2^2 + (\gamma + \alpha + \beta)^2]} < 0. \tag{18}$$

Therefore applying Lemma 1, $h_i \in \mathcal{P}$ ($i = 1, 2$) and consequently $h \in \mathcal{P}_k$ for $z \in \mathbb{U}$. This completes the proof of Theorem 4. \square

Theorem 5. *One has*

$$\mathcal{C}_k(p, \alpha + 1; \gamma) \subset \mathcal{C}_k(p, \alpha; \gamma). \tag{19}$$

Proof. Applying (10) and Theorem 4, we observe that

$$f \in \mathcal{C}_k(p, \alpha + 1; \gamma) \iff \frac{zf'}{p} \in \mathcal{S}_k(p, \alpha + 1; \gamma) \implies \frac{zf'}{p} \in \mathcal{S}_k(p, \alpha; \gamma) \iff f \in \mathcal{C}_k(p, \alpha; \gamma),$$

which evidently proves Theorem 5. \square

Theorem 6. *If $f \in \mathcal{S}_k(p, \alpha; \gamma)$, then $F_{c,p}(f) \in \mathcal{S}_k(p, \alpha; \gamma)$ ($c \geq 0$), where the generalized Libera integral operator $F_{c,p}$ is defined by (7).*

Proof. Let $f \in \mathcal{S}_k(p, \alpha; \gamma)$ and set

$$\begin{aligned} \frac{z(Q_{\beta,p}^\alpha F_{c,p}(f)(z))'}{Q_{\beta,p}^\alpha F_{c,p}(f)(z)} &= (p-\gamma)h(z) + \gamma \\ &= \left(\frac{k}{4} + \frac{1}{2}\right)\{(p-\gamma)h_1(z) + \gamma\} \\ &\quad - \left(\frac{k}{4} - \frac{1}{2}\right)\{(p-\gamma)h_2(z) + \gamma\}, \end{aligned} \tag{21}$$

where h is analytic in \mathbb{U} with $h(0) = 1$. From (21), we have

$$z(Q_{\beta,p}^\alpha F_{c,p}(f)(z))' = (c+p)Q_{\beta,p}^\alpha f(z) - cQ_{\beta,p}^\alpha F_{c,p}(f)(z). \tag{22}$$

Then, by using (21) and (22), we obtain

$$(c+p) \frac{Q_{\beta,p}^\alpha f(z)}{Q_{\beta,p}^\alpha F_{c,p}(f)(z)} = (p-\gamma)h(z) + \gamma + c. \tag{23}$$

Taking the logarithmic differentiation on both sides of (23) with respect to z and multiplying by z , we have

$$\begin{aligned} \frac{1}{p-\gamma} \left(\frac{z(Q_{\beta,p}^\alpha f(z))'}{Q_{\beta,p}^\alpha f(z)} - \gamma \right) & \\ = h(z) + \frac{zh'(z)}{(p-\gamma)h(z) + \gamma + c} &\in \mathcal{P}_k. \end{aligned} \tag{24}$$

This implies that

$$\left\{ h_i(z) + \frac{zh'_i(z)}{(p-\gamma)h_i(z) + \gamma + c} \right\} \in \mathcal{P}, \quad (z \in \mathbb{U}; i = 1, 2). \tag{25}$$

We form the functional $\Psi(u, v)$ by choosing $u = h_i(z)$ and $v = zh'_i(z)$:

$$\Psi(u, v) = u + \frac{v}{(p-\gamma)u + \gamma + c}. \tag{26}$$

Then clearly $\Psi(u, v)$ satisfies all the properties of Lemma 1. Hence, $h_i \in \mathcal{P}$ ($i = 1, 2$) and consequently $h \in \mathcal{P}_k$ for $z \in \mathbb{U}$, which implies that $F_{c,p}(f) \in \mathcal{S}_k(p, \alpha; \gamma)$. \square

Next, we derive an inclusion property for the subclass $\mathcal{C}_k(\alpha; \gamma)$ involving $F_{c,p}(f)$, which is given by the following theorem.

Theorem 7. *If $f \in \mathcal{C}_k(p, \alpha; \gamma)$, then $F_{c,p}(f) \in \mathcal{C}_k(p, \alpha; \gamma)$ ($c \geq 0$), where $F_{c,p}$ is defined by (7).*

Proof. By applying Theorem 6, it follows that

$$\begin{aligned} f \in \mathcal{C}_k(p, \alpha; \gamma) &\iff \frac{zf'}{p} \in \mathcal{S}_k(p, \alpha; \gamma) \\ &\implies F_{c,p} \left(\frac{zf'}{p} \right) \in \mathcal{S}_k(p, \alpha; \gamma) \\ &\quad \text{(by Theorem 5)} \\ &\iff \frac{z(F_{c,p}(f))'}{p} \in \mathcal{S}_k(p, \alpha; \gamma) \\ &\iff F_{c,p}(f) \in \mathcal{C}_k(p, \alpha; \gamma), \end{aligned} \tag{27}$$

which proves Theorem 7. \square

Theorem 8. *If $f \in \mathcal{C}_k(p, \alpha + 1; \gamma)$, for $z \in \mathbb{U}$, then $f \in \mathcal{C}_k(p, \alpha; \gamma)$ for*

$$|z| < r_0 = \frac{|\mu + 1|}{\sqrt{A + (A^2 - |\mu^2 - 1|)^{1/2}}}, \tag{28}$$

where $A = 2(s+1)^2 + |\mu|^2 - 1$, with $\mu = ((\gamma + \alpha + \beta)/(p - \gamma)) \neq -1$ and $s = (1/(p - \gamma))$. This radius is the best possible.

Proof. Let $f \in \mathcal{C}_k(p, \alpha + 1; \gamma)$ for $z \in \mathbb{U}$ and let

$$\begin{aligned} \frac{z(Q_{\beta,p}^{\alpha+1} f(z))'}{Q_{\beta,p}^{\alpha+1} f(z)} &= (p-\gamma)h(z) + \gamma \\ &= \left(\frac{k}{4} + \frac{1}{2}\right)\{(p-\gamma)h_1(z) + \gamma\} \\ &\quad - \left(\frac{k}{4} - \frac{1}{2}\right)\{(p-\gamma)h_2(z) + \gamma\}, \end{aligned} \tag{29}$$

where h_i is analytic in \mathbb{U} with $h_i(0) = 1$ and $\Re\{h_i(z)\} > 0$ for $i = 1, 2$. Using the identity (6) in (29) and differentiating the resulting equation with respect to z , we obtain

$$\begin{aligned} & \frac{1}{p-\gamma} \left\{ \frac{z(Q_{\beta,p}^\alpha f(z))'}{Q_{\beta,p}^\alpha f(z)} - \gamma \right\} \\ &= h(z) + \frac{(1/(p-\gamma))zh'(z)}{h(z) + ((\gamma + \alpha + \beta)/(p-\gamma))} \\ &= \left(\frac{k}{4} + \frac{1}{2}\right) \left\{ h_1(z) + \frac{(1/(p-\gamma))zh_1'(z)}{h_1(z) + ((\gamma + \alpha + \beta)/(p-\gamma))} \right\} \\ &\quad - \left(\frac{k}{4} - \frac{1}{2}\right) \left\{ h_2(z) \right. \\ &\quad \left. + \frac{(1/(p-\gamma))zh_2'(z)}{h_2(z) + ((\gamma + \alpha + \beta)/(p-\gamma))} \right\}, \end{aligned} \tag{30}$$

where $\Re\{h_i(z)\} > 0$ for $i = 1, 2$. Applying Lemma 2 with $s = ((\gamma + \alpha + \beta)/(p-\gamma))$ and $\mu = ((\gamma + \alpha + \beta)/(p-\gamma)) \neq -1$, we get

$$\Re \left\{ h_i(z) + \frac{(1/(p-\gamma))zh_i'(z)}{h_i(z) + ((\gamma + \alpha + \beta)/(p-\gamma))} \right\} > 0 \tag{31}$$

for $|z| < r_0$,

where r_0 is given by (28). This completes the proof of Theorem 8. \square

Theorem 9. Let ϕ be a convex function and $f \in \mathcal{S}_2(\alpha; \gamma)$. Then $G \in \mathcal{S}_2(\alpha; \gamma)$, where $G = \phi * f$.

Proof. Let $G = \phi * f$. Then

$$Q_{\beta,p}^\alpha G(z) = Q_{\beta,p}^\alpha (\phi * f)(z) = \phi(z) * Q_{\beta,p}^\alpha f(z). \tag{32}$$

Also, $f \in \mathcal{S}_2(\alpha; \gamma)$. Therefore, $Q_{\beta,p}^\alpha f \in \mathcal{S}_2(\gamma)$. By logarithmic differentiation of (32) and after some simplification, we obtain

$$\frac{z(Q_{\beta,p}^\alpha G(z))'}{pQ_{\beta,p}^\alpha G(z)} = \frac{\phi(z) * F(z)Q_{\beta,p}^\alpha f(z)}{\phi(z) * Q_{\beta,p}^\alpha f(z)}, \tag{33}$$

where $F = z(Q_{\beta,p}^\alpha f(z))'/pQ_{\beta,p}^\alpha f(z)$ is analytic in \mathbb{U} and $F(0) = 1$. From Lemma 3, we can see that $z(Q_{\beta,p}^\alpha G(z))'/pQ_{\beta,p}^\alpha G(z)$ is contained in the convex hull of $F(\mathbb{U})$. Since $z(Q_{\beta,p}^\alpha G(z))'/pQ_{\beta,p}^\alpha G(z)$ is analytic in \mathbb{U} and

$$F(\mathbb{U}) = \Omega = \left\{ w : \frac{z(Q_{\beta,p}^\alpha w(z))'}{pQ_{\beta,p}^\alpha w(z)} \in \mathcal{P}(\gamma) \right\}, \tag{34}$$

then $z(Q_{\beta,p}^\alpha G(z))'/pQ_{\beta,p}^\alpha G(z)$ lies in Ω ; this implies that $G = \phi * f \in \mathcal{S}_2(\alpha; \gamma)$. \square

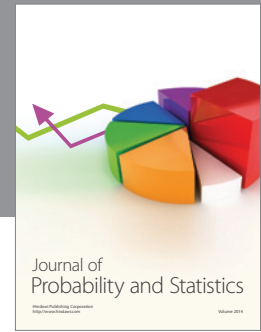
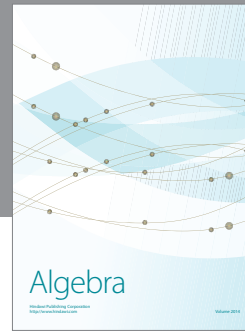
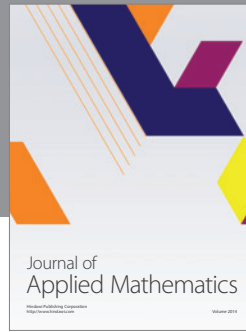
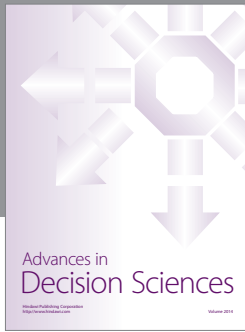
Remark 10. Putting $p = 1$ in the above results, we obtain corresponding results for the operator Q_β^α .

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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