# On the Range of the Radon Transform on $\mathbb{Z}^{n}$ and the Related Volberg's Uncertainty Principle 

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#### Abstract

We characterize the image of exponential type functions under the discrete Radon transform $R$ on the lattice $\mathbb{Z}^{n}$ of the Euclidean space $\mathbb{R}^{n}(n \geq 2)$. We also establish the generalization of Volberg's uncertainty principle on $\mathbb{Z}^{n}$, which is proved by means of this characterization. The techniques of which we make use essentially in this paper are those of the Diophantine integral geometry as well as the Fourier analysis.


## 1. Introduction

First of all, we recall briefly that the uncertainty principle states, roughly speaking, that a nonzero function and its Fourier transform cannot both be sharply localized, which can be interpreted topologically by the fact that they cannot have simultaneously their supports in a same too small compact (see the Heisenberg uncertainty principle in [1]). Considerable attention has been devoted to discovering different forms of the uncertainty principle on many settings such as certain types of Lie groups and homogeneous trees. Several versions of the uncertainty principle have been established by many authors in the last few decades. Among the contributions dealing with this important topic, let us quote principally $[1-4]$. On the other hand, we note that the uncertainty principle is one of the major themes of the classical Fourier analysis as well as its neighboring parts of the mathematical analysis.

We consider here the lattice $\mathbb{Z}^{n}$ of the Euclidean space $\mathbb{R}^{n}(n \geq 2)$. For $a=\left(a_{1}, \ldots, a_{n}\right) \in \mathbb{Z}^{n} \backslash\{0\}$ and $k \in \mathbb{Z}$, the linear Diophantine equation $a x=k$ has an infinity of solutions in $\mathbb{Z}^{n}$ if and only if $k$ is an integral multiple of the greatest common divisor $d(a)$ of the integers $a_{1}, \ldots, a_{n}$, where $a x$ denotes the usual inner product of $a$ and $x$ regarded as two
vectors of the Euclidean space $\mathbb{R}^{n}$ (see [5] for more details). Therefore, for $a \in \mathscr{P}=\left\{m \in \mathbb{Z}^{n} \backslash\{0\} \mid d(m)=1\right\}$, the set $H(a, k)=\left\{x \in \mathbb{Z}^{n} \mid a x=k\right\}$ of all its solutions in $\mathbb{Z}^{n}$ is infinite and forms a discrete hyperplane in $\mathbb{Z}^{n}$. Let $\mathbb{G}$ be the set consisting of all hyperplanes $H(a, k)$ in $\mathbb{Z}^{n}$, where $(a, k) \in \mathscr{P} \times \mathbb{Z}$. We note that $\mathbb{G}$ plays a role as discrete Grassmannian and can be parametrized as $\mathscr{P} \times \mathbb{Z} / \pm 1$ (see [5, Section 2]). Moreover, $\mathbb{G}$ can be written as the following disjoint union:

$$
\begin{equation*}
\mathbb{G}=\mathbb{G}^{(1)} \cup \mathbb{G}^{(2)}, \tag{1}
\end{equation*}
$$

where

$$
\begin{align*}
\mathbb{G}^{(1)} & =\left\{H(a, k)\left|(a, k) \in \mathscr{P} \times \mathbb{Z},\|a\|^{2}\right| k\right\} \\
& =\left\{H\left(a,\|a\|^{2} k\right) \mid(a, k) \in \mathscr{P} \times \mathbb{Z}\right\}  \tag{2}\\
\mathbb{G}^{(2)} & =\left\{H(a, k) \mid(a, k) \in \mathscr{P} \times \mathbb{Z},\|a\|^{2}+k\right\} \\
& =\left\{H(a, k) \mid(a, k) \in \mathscr{P} \times \mathbb{Z}, k \notin\|a\|^{2} \mathbb{Z}\right\}
\end{align*}
$$

with $\|a\|^{2}=\sum_{i=1}^{n} a_{i}^{2}$, for all $a=\left(a_{1}, \ldots, a_{n}\right) \in \mathscr{P}$.

As an analogue of the Euclidean case, the discrete Radon transform $R$ on $\mathbb{Z}^{n}$, which maps a function $f \in l^{1}\left(\mathbb{Z}^{n}\right)$ to a function $R f$ on $\mathbb{G}$, is defined by

$$
\begin{equation*}
R f(H(a, k))=\sum_{m \in H(a, k)} f(m), \tag{3}
\end{equation*}
$$

for all $(a, k) \in \mathscr{P} \times \mathbb{Z}$, where $l^{1}\left(\mathbb{Z}^{n}\right)$ is the space of all complexvalued functions $f$ defined on $\mathbb{Z}^{n}$ such that $\sum_{m \in \mathbb{Z}^{n}}|f(m)|<$ $+\infty$ (see [5] for more details on this Radon transform).

In this paper, we are interested in developing the study of the restriction of the Radon transform $R f$ of $f \in l^{1}\left(\mathbb{Z}^{n}\right)$ to $\mathbb{G}^{(1)}$. By means of Theorem 1 stated below, considered here as the first main result, we succeeded in proving the second main result concerning the generalization of Volberg's UP on $\mathbb{Z}^{n}$ (see Theorem 2). We precise that $\mathbb{G}^{(1)}$ is the most important subset of the discrete Grassmannian $\mathbb{G}$ for our study of both fundamental results (see Sections 3 and 4).

The purpose of this paper is to study the characterization of the image of exponential type functions under $R$, as well as the generalization of Volberg's UP on $\mathbb{Z}^{n}$.

Our work is motivated by the fact that the uncertainty principle for the discrete Radon transform $R$ on $\mathbb{Z}^{n}$ plays a fundamental role in the field of physics, especially in quantum mechanics.

Our paper is organized as follows.
In Section 2, we fix, once and for all, some notation and also give certain properties of the discrete Radon transform $R$ on $\mathbb{Z}^{n}$, which will be useful in the sequel of this paper. Moreover, we recall Volberg's theorem on $\mathbb{Z}$ in the same section.

Section 3 deals with the characterization of the image of exponential type functions under $R$, which is given by the following main theorem (see Theorem 4).

Theorem 1 (characterization of the image of exponential type functions under $R$ ). Let $f$ be a positive function of $l^{1}\left(\mathbb{Z}^{n}\right)$. Then
(i) the following two conditions are equivalent:
(1) $f(m)=O\left(e^{-\alpha\|m\|^{2}}\right)$,
(2) $R f\left(H\left(a,\|a\|^{2} k\right)\right)=O\left(e^{-\alpha k^{2}}\right), \forall a \in \mathscr{P}$,
where $\alpha>1$ is an absolute constant;
(ii) the following two equivalences hold:
(3) $\left[|R f(H(a, k))| \leq \exp \left(-\beta k^{2}\right), \forall(a, k) \in \mathscr{P} \times\right.$ $\mathbb{Z}] \Leftrightarrow\left[f(m)=0, \forall m \in \mathbb{Z}^{n} \backslash\{0\}\right]$,
where $\beta>0$ is an absolute constant,
(4) $\left[|R f(H(a, k))| \leq|k| \exp \left(-\gamma k^{2}\right), \forall(a, k) \in \mathscr{P} \times\right.$ $\mathbb{Z}] \Leftrightarrow f=0$,
where $\gamma>0$ is an absolute constant.

Section 4 is devoted to establishing the generalization of Volberg's UP on the lattice $\mathbb{Z}^{n}$ (see Theorem 2 below). We make here use of the discrete Fourier transform $\mathscr{F}$, which maps a function $f \in l^{1}\left(\mathbb{Z}^{n}\right)$ to a function $\mathscr{F} f$ on $\mathbb{T}^{n}$ defined by

$$
\begin{equation*}
\mathscr{F} f(\lambda)=\sum_{m \in \mathbb{Z}^{n}} f(m) \exp (-2 i \pi \lambda m), \quad \forall \lambda \in \mathbb{T}^{n} \tag{4}
\end{equation*}
$$

where $\mathbb{T}^{n}=\mathbb{R}^{n} / \mathbb{Z}^{n}$ is the $n$-dimensional torus.
For $a \in \mathscr{P}$ and $f \in l^{1}\left(\mathbb{Z}^{n}\right)$, we denote by $B(a)$ (resp., $f_{\mid B(a)}$ ) the set

$$
\begin{equation*}
B(a)=\left\{m \in \mathbb{Z}^{n} \left\lvert\, \frac{a m}{\|a\|^{2}} \in \mathbb{Z}\right.\right\} \tag{5}
\end{equation*}
$$

(resp., the restriction of $f$ to $B(a)$ ), where $a m=\sum_{i=1}^{n} a_{i} m_{i}$ by putting $a=\left(a_{1}, \ldots, a_{n}\right)$ and $m=\left(m_{1}, \ldots, m_{n}\right)$.

In this section, we consider the function $\psi$ defined on $\mathscr{P} \times$ Tby

$$
\begin{equation*}
\psi(a, s)=\frac{a s}{\|a\|^{2}}, \quad \forall(a, s) \in \mathscr{P} \times \mathbb{T}, \tag{6}
\end{equation*}
$$

and also $\mathscr{F}\left(f_{\mid B(\cdot)}\right)(\psi(\cdot, s))$ (where $s \quad \in \quad \mathbb{T}$ ) and $\left\|\mathscr{F}\left(f_{\mid B(\cdot)}\right)(\psi(\cdot, s))\right\|_{\infty}$, respectively, given by

$$
\begin{align*}
\mathscr{F}\left(f_{\mid B(\cdot)}\right)(\psi(\cdot, s))(a) & =\mathscr{F}\left(f_{\mid B(a)}\right)\left(\frac{a s}{\|a\|^{2}}\right) \\
& =\sum_{m \in B(a)} f(m) \exp \left(-2 i \pi \frac{a m s}{\|a\|^{2}}\right), \tag{7}
\end{align*}
$$

for all $(a, s) \in \mathscr{P} \times \mathbb{T}$, with $B(a)=\left\{m \in \mathbb{Z}^{n} \mid a m /\|a\|^{2} \in \mathbb{Z}\right\}$, respectively,

$$
\begin{equation*}
\left\|\mathscr{F}\left(f_{\mid B(\cdot)}\right)(\psi(\cdot, s))\right\|_{\infty}=\sup _{a \in \mathscr{P}}\left|\mathscr{F}\left(f_{\mid B(a)}\right)(\psi(a, s))\right|, \quad \forall s \in \mathbb{T}, \tag{8}
\end{equation*}
$$

$\mathbb{T}$ being the one-dimensional torus. Now, we state the following main theorem (see Theorem 10).

Theorem 2 (generalization of Volberg's UP on the lattice $\mathbb{Z}^{n}$ ). Let $f \in l^{1}\left(\mathbb{Z}^{n}\right)$ satisfying the following three conditions
(1) $|f(m)| \leq C_{0} \exp (-\alpha\|m\|)$, for all $m \in \mathbb{Z}^{n}$, where $C_{0}, \alpha$ are strictly positive absolute constants.
(2) The function $\mathbb{T} \ni s \mapsto\left\|\mathscr{F}\left(f_{\mid B(\cdot)}\right)(\psi(\cdot, s))\right\|_{\infty}$ belongs to $L^{1}(\mathbb{T})$.
(3) $\int_{\mathbb{T}} \log \left(\left\|\mathscr{F}\left(f_{\mid B(\cdot)}\right)(\psi(\cdot, s))\right\|_{\infty}\right) d s=-\infty$.

Then $f=0$.

## 2. Notations and Preliminaries

In this section, we fix some notation which will be useful in the sequel of this paper and recall certain properties of the discrete Radon transform on $\mathbb{Z}^{n}(n \geq 2)$. We also introduce various functional spaces. For $1 \leq p<+\infty$,
let $l^{p}\left(\mathbb{Z}^{n}\right)$ (resp., $l^{\infty}\left(\mathbb{Z}^{n}\right)$ ) be the space of all complex-valued functions $f$ defined on $\mathbb{Z}^{n}$ such that $\sum_{m \in \mathbb{Z}^{n}}|f(m)|^{p}<$ $+\infty$ (resp., $\left.\sup _{m \in \mathbb{Z}^{n}}|f(m)|<+\infty\right)$. Let us denote by $C_{0}\left(\mathbb{Z}^{n}\right)$ the space of all complex-valued functions $f$ defined on $\mathbb{Z}^{n}$ such that $f(m) \rightarrow 0$ as $\|m\| \rightarrow+\infty$, with $\|m\|^{2}=m_{1}^{2}+$ $\cdots+m_{n}^{2}$ for all $m=\left(m_{1}, \ldots, m_{n}\right) \in \mathbb{Z}^{n}$. It is clear that, for $1 \leq p<q<+\infty$, we have the following inclusions:

$$
\begin{align*}
l^{1}\left(\mathbb{Z}^{n}\right) & \subset l^{2}\left(\mathbb{Z}^{n}\right) \subset \cdots \subset l^{p}\left(\mathbb{Z}^{n}\right) \subset \cdots \subset l^{q}\left(\mathbb{Z}^{n}\right)  \tag{9}\\
& \subset C_{0}\left(\mathbb{Z}^{n}\right) \subset l^{\infty}\left(\mathbb{Z}^{n}\right)
\end{align*}
$$

For $1 \leq p<+\infty$, we denote by $\|\cdot\|_{p}$ the discrete norm on the space $l^{p}\left(\mathbb{Z}^{n}\right)$ defined by

$$
\begin{equation*}
\|f\|_{p}=\left(\sum_{m \in \mathbb{Z}^{n}}|f(m)|^{p}\right)^{1 / p}, \quad \forall f \in l^{p}\left(\mathbb{Z}^{n}\right) \tag{10}
\end{equation*}
$$

We define the discrete Radon transform $R$ on $\mathbb{Z}^{n}$ as follows:

$$
\begin{align*}
R f(H(a, k)) & =\sum_{m \in H(a, k)} f(m) \\
& =\sum_{m \in \mathbb{Z}^{n}, a m=k} f(m) \tag{11}
\end{align*}
$$

for all $(a, k) \in \mathscr{P} \times \mathbb{Z}$ and $f \in l^{1}\left(\mathbb{Z}^{n}\right)$, where $H(a, k)$ is the hyperplane in $\mathbb{Z}^{n}$ defined by

$$
\begin{equation*}
H(a, k)=\left\{m \in \mathbb{Z}^{n} \mid a m=k\right\} \tag{12}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathscr{P}=\left\{a=\left(a_{1}, \ldots, a_{n}\right) \in \mathbb{Z}^{n} \backslash\{0\} \mid d(a)=1\right\}, \tag{13}
\end{equation*}
$$

$d(a)$ being the greatest common divisor of the integers $a_{1}, \ldots, a_{n}$ (see [5] for more details), and ax denotes the usual inner product of $a$ and $x$ regarded as two vectors of the Euclidean space $\mathbb{R}^{n}$.

The set $B(a)=\left\{m \in \mathbb{Z}^{n} \mid a m /\|a\|^{2} \in \mathbb{Z}\right\}$ can be written as follows:

$$
\begin{equation*}
B(a)=\bigcup_{\alpha \in \mathbb{Z}} H\left(a, \alpha\|a\|^{2}\right) \quad \text { (disjoint union). } \tag{14}
\end{equation*}
$$

Because, for all $(\alpha, \beta) \in \mathbb{Z}^{2}$ such that $\alpha \neq \beta$, we have

$$
\begin{equation*}
H\left(a, \alpha\|a\|^{2}\right) \bigcap H\left(a, \beta\|a\|^{2}\right)=\varnothing . \tag{15}
\end{equation*}
$$

We note that

$$
\begin{equation*}
\bigcup_{a \in \mathscr{P}} B(a)=\mathbb{Z}^{n} \tag{16}
\end{equation*}
$$

Indeed, for $m \in \mathbb{Z}^{n} \backslash\{0\}$, we can take $a=m / d(m) \in \mathscr{P}$, then $a m /\|a\|^{2}=d(m) \in \mathbb{Z}$. Moreover, $0 \in B(a)$ for all $a \in \mathscr{P}$.

For a function $f \in l^{1}\left(\mathbb{Z}^{n}\right)$, we define its discrete Fourier transform $\mathscr{F} f$ on the $n$-dimensional torus $\mathbb{T}^{n}$ as follows:

$$
\begin{equation*}
\mathscr{F} f(\lambda)=\sum_{m \in \mathbb{Z}^{n}} f(m) \exp (-2 i \pi \lambda m), \quad \forall \lambda \in \mathbb{T}^{n} . \tag{17}
\end{equation*}
$$

We define the discrete one-dimensional Fourier transform $\mathscr{F}_{1}$ by

$$
\begin{equation*}
\mathscr{F}_{1} g(x)=\sum_{t \in \mathbb{Z}} g(t) \exp (-2 i \pi t x), \quad \forall x \in \mathbb{T} \tag{18}
\end{equation*}
$$

where $\mathbb{T}$ is the one-dimensional torus.
For $f \in L^{1}\left(\mathbb{T}^{n}\right)$, the Fourier coefficients of $f$ are denoted by $\widehat{f}(m)\left(m \in \mathbb{Z}^{n}\right)$ and defined by

$$
\begin{equation*}
\widehat{f}(m)=\int_{\mathbb{T}^{n}} f(x) \exp (2 i \pi m x) d x, \quad \forall m \in \mathbb{Z}^{n} \tag{19}
\end{equation*}
$$

Let $a_{j}=\left(1, j, j^{2}, \ldots, j^{n-1}\right) \in \mathscr{P}$, with $j \in \mathbb{N} \backslash\{0\}$. The inversion formula for the discrete Radon transform is given by

$$
\begin{equation*}
\lim _{j \rightarrow \infty} R f\left(H\left(a_{j}, a_{j} m\right)\right)=f(m) \tag{20}
\end{equation*}
$$

for all $m \in \mathbb{Z}^{n}$ and $f \in l^{1}\left(\mathbb{Z}^{n}\right)$ (see [5, Theorem 4.1] and also [6]).

At the end of this section, we recall Volberg's theorem on $\mathbb{Z}$.

Theorem 3 (Volberg's theorem, see [1]). Let $\alpha>0$ and suppose that $f$ is a nontrivial function on $\mathbb{Z}$ such that $f(t)=$ $O(\exp (-\alpha|t|))$ as $t \rightarrow-\infty$. Moreover, suppose that its Fourier transform $\mathscr{F}_{1} f$ is integrable on the one-dimensional torus $\mathbb{T}$. Then $\int_{\mathbb{T}} \log \left|\mathscr{F}_{1} f(x)\right| d x>-\infty$.

## 3. Characterization of the Image of Exponential Type Functions under $R$

In this section, we study the characterization of the image of exponential type functions under the discrete Radon transform $R$ on $\mathbb{Z}^{n}$. More precisely, we state the following main theorem which will be proved after introducing some intermediate lemmas.

Theorem 4 (characterization of the image of exponential type functions under $R$ ). Let $f$ be a positive function of $l^{1}\left(\mathbb{Z}^{n}\right)$. Then
(i) the following two conditions are equivalent:
(1) $f(m)=O\left(e^{-\alpha\|m\|^{2}}\right)$,
(2) $R f\left(H\left(a,\|a\|^{2} k\right)\right)=O\left(e^{-\alpha k^{2}}\right), \forall a \in \mathscr{P}$,
where $\alpha>1$ is an absolute constant;
(ii) the following two equivalences hold:
(3) $\left[|R f(H(a, k))| \leq \exp \left(-\beta k^{2}\right), \forall(a, k) \in \mathscr{P} \times\right.$ $\mathbb{Z}] \Leftrightarrow\left[f(m)=0, \forall m \in \mathbb{Z}^{n} \backslash\{0\}\right]$,
where $\beta>0$ is an absolute constant,
(4) $\left[|R f(H(a, k))| \leq|k| \exp \left(-\gamma k^{2}\right), \forall(a, k) \in \mathscr{P} \times\right.$ $\mathbb{Z}] \Leftrightarrow f=0$,
where $\gamma>0$ is an absolute constant.

In order to avoid any too long proof of Theorem 4 and then prove it clearly, we need the following useful lemmas.

Lemma 5. Let $\alpha>0$ and $f$ be the function defined on $\mathbb{Z}^{n}$ by: $f(m)=\exp \left(-\alpha\|m\|^{2}\right)$, for all $m \in \mathbb{Z}^{n}$. Then there exists a constant $C>0$ (which depends only on $\alpha$ and $n$ ) such that, for all $(a, k) \in \mathscr{P} \times \mathbb{Z}$, we have

$$
\begin{equation*}
0<R f\left(H\left(a, k\|a\|^{2}\right)\right) \leq C \exp \left(-\alpha k^{2}\right) \tag{21}
\end{equation*}
$$

Proof. Let $S(a, k)=\sum_{m \in \mathbb{Z}^{n}, a m=k\|a\|^{2}} \exp \left(\alpha\left(k^{2}-\|m\|^{2}\right)\right)$. The left inequality of (21) is trivial since $f>0$ implies clearly that $R f>0$. To show the right inequality of (21), it suffices to prove the inequality $S(a, k)<C$, for all $(a, k) \in \mathscr{P} \times \mathbb{Z}$. For this, we distinguish two cases.
(1) The Case When $\|a\|^{2} \geq 2$. We have $a m=k\|a\|^{2}$ and $a \neq$ 0 ; then, by the Cauchy-Schwarz inequality, $k\|a\| \leq\|m\|$. It follows that $k \leq\|a\|^{-1}\|m\|$; thus

$$
\begin{equation*}
S(a, k) \leq \sum_{m \in \mathbb{Z}^{n}, a m=k\|a\|^{2}} \exp \left(\alpha\|m\|^{2}\left(\|a\|^{-2}-1\right)\right) \tag{22}
\end{equation*}
$$

since $\|a\|^{2} \geq 2$ by hypothesis. The above inequality can be transformed as follows:

$$
\begin{equation*}
S(a, k) \leq \sum_{m \in \mathbb{Z}^{n}} \exp \left(\frac{-\alpha}{2}\|m\|^{2}\right)=C_{1} \tag{23}
\end{equation*}
$$

which implies (21).
(2) The Case When $\|a\|^{2}<2$, That $I s,\|a\|^{2}=1$. Since $a \in \mathscr{P} \subset$ $\mathbb{Z}^{n} \backslash\{0\}$, we can change the order of the coordinates of $a$, then we can take $a=(0, \ldots, 0,1)$, and thus $a m=m_{n}=k$, with $m=\left(m_{1}, m_{2}, \ldots, m_{n}\right)$. It follows that

$$
\begin{equation*}
S(a, k)=\sum_{m \in \mathbb{Z}^{n-1}} \exp \left(-\alpha\|m\|^{2}\right)=C_{2} . \tag{24}
\end{equation*}
$$

Let $C=\operatorname{Max}\left(C_{1}, C_{2}\right)$. From (23) and (24), we have $S(a, k) \leq C$ for all $(a, k) \in \mathscr{P} \times \mathbb{Z}$. Consequently,

$$
\begin{align*}
& \sum_{m \in \mathbb{Z}^{n}, a m=k\|a\|^{2}} \exp \left(\alpha\left(k^{2}-\|m\|^{2}\right)\right) \\
& =\left(\exp \left(\alpha k^{2}\right)\right)\left(\sum_{m \in H\left(a,\|a\|^{2} k\right)} f(m)\right) \leq C, \tag{25}
\end{align*}
$$

where $f(m)=\exp \left(-\alpha\|m\|^{2}\right)$. Then by the above inequality, we obtain

$$
\begin{align*}
& \sum_{m \in H\left(a,\|a\|^{2} k\right)} \exp \left(-\alpha\|m\|^{2}\right) \leq C \exp \left(-\alpha k^{2}\right)  \tag{26}\\
& \forall(a, k) \in \mathscr{P} \times \mathbb{Z}
\end{align*}
$$

And Lemma 5 is proved.

Lemma 6. Let $f \in l^{1}\left(\mathbb{Z}^{n}\right)$. We assume that there exists a function $\varphi: \mathbb{N} \rightarrow[0,+\infty[$ such that $\varphi(t) \rightarrow 0$ as $t \rightarrow+\infty$, satisfying the following condition:

$$
\begin{equation*}
|R f(H(a, k))| \leq \varphi(|k|), \quad \forall(a, k) \in \mathscr{P} \times \mathbb{Z} \tag{27}
\end{equation*}
$$

Then $f$ is supported at the origin; that is, $f(m)=0$ for all $m \in \mathbb{Z}^{n} \backslash\{0\}$.

Proof. Let $m=\left(m_{1}, m_{2}, \ldots, m_{n}\right) \in \mathbb{Z}^{n} \backslash\{0\}$. Permuting the coordinates of $m$, we can assume that $m_{n} \neq 0$. Applying hypothesis (27) to $a=a_{j}=\left(1, j, j^{2}, \ldots, j^{n-1}\right)$ and $k=a_{j} m=$ $m_{1}+m_{2} j+\cdots+m_{n} j^{n-1}$, where $j \in \mathbb{N} \backslash\{0\}$, we obtain

$$
\begin{equation*}
\left|R f\left(H\left(a_{j}, a_{j} m\right)\right)\right| \leq \varphi\left(\left|a_{j} m\right|\right), \quad \forall j \in \mathbb{N} \backslash\{0\} \tag{28}
\end{equation*}
$$

Since $m_{n} \neq 0$, this implies that $\left|a_{j} m\right| \rightarrow+\infty$ as $j \rightarrow+\infty$; therefore, the right hand side of inequality (28) tends to zero as $j \rightarrow+\infty$. By the inversion formula (see [5, Theorem 4.1]), the left hand side of the same inequality converges to $|f(m)|$ as $j \rightarrow+\infty$. Then $f(m)=0$. And Lemma 6 is proved.

We cannot hope to obtain more than this result: given a function $f$ on $\mathbb{Z}^{n}$ such that $f(m)=0$ for $m \neq 0$ and $f(0)=1$, we have $\operatorname{Rf}(H(a, k))=0$ if $k \neq 0$, and $\operatorname{Rf}(H(a, 0))=$ 1. Inequality (27) is verified for every function $\varphi$ such that $\varphi(0)=1$.

Lemma 7. Let $f \in l^{1}\left(\mathbb{Z}^{n}\right)$ verifying the following condition:

$$
\begin{equation*}
|R f(H(a, k))| \leq|k| e^{-\alpha k^{2}}, \quad \forall(a, k) \in \mathscr{P} \times \mathbb{Z} \tag{29}
\end{equation*}
$$

where $\alpha$ is a strictly positive absolute constant. Then $f=0$.
Proof. By applying Lemma 6, we obtain $f(m)=0$, for all $m \in$ $\mathbb{Z}^{n} \backslash\{0\}$. Now, we show that $f(0)=0$. Condition (29) implies that $\operatorname{Rf}(H(a, 0))=0$, for all $a \in \mathscr{P}$. Then, since we have

$$
\begin{equation*}
R f(H(a, 0))=\sum_{\substack{a m=0 \\ m \in \mathbb{Z}^{n}}} f(m)=f(0)+\sum_{\substack{a m=0 \\ m \in \mathbb{Z}^{n}\{\{0\}}} f(m), \tag{30}
\end{equation*}
$$

$\forall a \in \mathscr{P}$,
with $\sum_{a m=0, m \in \mathbb{Z}^{n} \backslash\{0\}} f(m)=0$, we infer that $f(0)=0$. Thus, $f(m)=0$, for all $m \in \mathbb{Z}^{n}$. And this proves Lemma 7.

We now return to the proof of Theorem 4.
Proof of Theorem 4. The implication (1) $\Rightarrow(2)$ of Theorem 4 follows from Lemma 5. On the other hand, we deduce equivalence (3) (resp., (4)) of Theorem 4 from Lemma 6 (resp., Lemma 7). Consequently, to complete the proof of Theorem 4, it remains to prove the implication (2) $\Rightarrow(1)$ of this theorem. For this, suppose that (2) is satisfied and prove (1). Under our assumption, we have

$$
\begin{equation*}
0<R f\left(H\left(a,\|a\|^{2} k\right)\right)<C \exp \left(-\alpha k^{2}\right), \quad \forall a \in \mathscr{P} \tag{31}
\end{equation*}
$$

where $C>0$ is an absolute constant. Multiplying the members of (31) by $k^{2 \beta} / \beta$ !, with $k \neq 0$ and $\beta \in \mathbb{N}$, (31) becomes

$$
\begin{equation*}
0<\frac{k^{2 \beta}}{\beta!} R f\left(H\left(a,\|a\|^{2} k\right)\right)<C \frac{k^{2 \beta}}{\beta!} \exp \left(-\alpha k^{2}\right), \quad \forall a \in \mathscr{P} \tag{32}
\end{equation*}
$$

but

$$
\begin{equation*}
\frac{k^{2 \beta}}{\beta!}=\frac{\left(k^{2}\right)^{\beta}}{\beta!} \leq e^{k^{2}}, \quad \forall \beta \in \mathbb{N} . \tag{33}
\end{equation*}
$$

Then (32) can be transformed as follows:

$$
\begin{equation*}
0<\frac{k^{2 \beta}}{\beta!} R f\left(H\left(a,\|a\|^{2} k\right)\right)<C e^{(1-\alpha) k^{2}}, \quad \forall(a, \beta) \in \mathscr{P} \times \mathbb{N} . \tag{34}
\end{equation*}
$$

Since $\alpha>1$, there exists a constant $C_{0}>0$ (which does not depend on $\beta$ ) such that

$$
\begin{equation*}
0<\sum_{k \in \mathbb{Z}} \frac{k^{2 \beta}}{\beta!} R f\left(H\left(a,\|a\|^{2} k\right)\right)<C_{0}, \quad \forall a \in \mathscr{P} \tag{35}
\end{equation*}
$$

but

$$
\begin{align*}
& \sum_{k \in \mathbb{Z}} \frac{k^{2 \beta}}{\beta!} R f\left(H\left(a,\|a\|^{2} k\right)\right) \\
& \quad=\sum_{k \in \mathbb{Z}} \frac{\left(k^{2}\right)^{\beta}}{\beta!} R f\left(H\left(a,\|a\|^{2} k\right)\right)  \tag{36}\\
& \quad=\sum_{k \in \mathbb{Z}} \frac{\left(k^{2}\right)^{\beta}}{\beta!}\left(\sum_{a m=\|a\|^{2} k} f(m)\right)<C_{0} .
\end{align*}
$$

For $\|a\|=1$, equality (36) becomes

$$
\begin{align*}
\sum_{k \in \mathbb{Z}} \frac{k^{2 \beta}}{\beta!} R f\left(H\left(e_{j}, k\right)\right) & =\sum_{k \in \mathbb{Z}} \frac{k^{2 \beta}}{\beta!}\left(\sum_{e_{j} m=k} f(m)\right) \\
& =\sum_{m \in \mathbb{Z}^{n}} \frac{m_{j}^{2 \beta}}{\beta!} f(m)  \tag{37}\\
& =\frac{1}{\beta!} \sum_{m \in \mathbb{Z}^{n}} m_{j}^{2 \beta} f(m)<C_{0}
\end{align*}
$$

where $\left(e_{j}\right)_{1 \leq j \leq n}$ is the canonical orthonormal basis of $\mathbb{R}^{n}$. Moreover, (37) implies that the series $\sum_{m \in \mathbb{Z}^{n}} m_{j}^{2 \beta} f(m)$ is convergent; then there exists a constant $C>0$ (which does not depend on $\beta$ ) such that

$$
\begin{equation*}
m_{j}^{2 \beta} f(m)<C, \tag{38}
\end{equation*}
$$

for all $j \in\{1,2, \ldots, n\}$ and $m \in \mathbb{Z}^{n}$ such that $\|m\|^{2} \rightarrow+\infty$. Therefore, for all $\beta \in \mathbb{N}$ and $j \in\{1,2, \ldots, n\}$

$$
\begin{equation*}
\frac{(\alpha n)^{\beta}}{\beta!} m_{j}^{2 \beta} f(m)<C \frac{(\alpha n)^{\beta}}{\beta!} \tag{39}
\end{equation*}
$$

and thus

$$
\begin{equation*}
\frac{\left(\alpha n m_{j}^{2}\right)^{\beta}}{\beta!} f(m)<C \frac{(\alpha n)^{\beta}}{\beta!}, \tag{40}
\end{equation*}
$$

which gives by summing with respect to $\beta$

$$
\begin{equation*}
\left(\sum_{\beta=0}^{+\infty} \frac{\left(n \alpha m_{j}^{2}\right)^{\beta}}{\beta!}\right) f(m)<C \sum_{\beta=0}^{+\infty} \frac{(\alpha n)^{\beta}}{\beta!} . \tag{41}
\end{equation*}
$$

Inequality (41) can be transformed as follows:

$$
\begin{equation*}
e^{n \alpha m_{j}^{2}} f(m)<C e^{n \alpha}, \quad \forall j \in\{1,2, \ldots, n\} . \tag{42}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
f(m)<C e^{n \alpha} e^{-n \alpha m_{j}^{2}}, \quad \forall j \in\{1,2, \ldots, n\} . \tag{43}
\end{equation*}
$$

Since $f(m) \geq 0$, inequality (43) implies

$$
\begin{align*}
f(m)^{2} & \leq C e^{n \alpha} e^{-n \alpha m_{j}^{2}} f(m)  \tag{44}\\
& \leq\left(C e^{n \alpha}\right)^{2} e^{-n \alpha m_{j}^{2}} e^{-n \alpha m_{i}^{2}},
\end{align*}
$$

for all $i, j \in\{1,2, \ldots, n\}$. It follows from inequality (44) that

$$
\begin{equation*}
f(m)^{n} \leq\left(C e^{n \alpha}\right)^{n} e^{-n \alpha\|m\|^{2}}, \tag{45}
\end{equation*}
$$

for all $m \in \mathbb{Z}^{n}$ such that $\|m\|^{2} \rightarrow+\infty$. Thus,

$$
\begin{equation*}
f(m) \leq C e^{n \alpha} e^{-\alpha\|m\|^{2}} \tag{46}
\end{equation*}
$$

for all $m \in \mathbb{Z}^{n}$ such that $\|m\|^{2} \rightarrow+\infty$, which proves the implication (2) $\Rightarrow$ (1). And this completes the proof of Theorem 4.

Remark 8. Let $f$ be a positive function of $L^{1}\left(\mathbb{T}^{n}\right)$ such that $\hat{f} \in l^{1}\left(\mathbb{Z}^{n}\right)$. Then

$$
\begin{align*}
& {\left[|R \widehat{f}(H(a, k))| \leq|k| \exp \left(-\gamma k^{2}\right), \forall(a, k) \in \mathscr{P} \times \mathbb{Z}\right]} \\
& \Longleftrightarrow f=0, \tag{47}
\end{align*}
$$

where $\gamma>0$ is an absolute constant. It suffices to apply Theorem 4 to $\widehat{f}$.

Now, denote by $A^{1}\left(\mathbb{T}^{n}\right)$ the subspace of $L^{1}\left(\mathbb{T}^{n}\right)$ consisting of all functions $G \in L^{1}\left(\mathbb{T}^{n}\right)$ such that $\sum_{m \in \mathbb{Z}^{n}}|\widehat{G}(m)|<+\infty$, and let $L_{*}^{1}(\mathbb{G})$ be the subspace of $L^{1}(\mathbb{G})$ consisting of all functions $F \in L^{1}(\mathbb{G})$ such that there exists $G \in A^{1}\left(\mathbb{T}^{n}\right)$ satisfying the condition

$$
\begin{equation*}
\mathscr{F}_{1} F(H(a, \cdot))(\theta)=G(\theta a), \quad \forall \theta \in \mathbb{T}, \tag{48}
\end{equation*}
$$

where $L^{1}(\mathbb{G})$ is the space of all complex-valued functions $F$ defined on $\mathbb{G}$ such that $\sum_{k \in \mathbb{Z}}|F(H(a, k))|$ is finite for all $a \in$ $\mathscr{P}$ (see [5]), and $\mathbb{T}$ is the one-dimensional torus. The authors
of [6] have proved that the discrete Radon transform $R$ is a continuous bijection of $l^{1}\left(\mathbb{Z}^{n}\right)$ onto $L_{*}^{1}(\mathbb{G})$ (see [6, Corollary 7]).

For $\alpha>0$, we put

$$
\begin{gather*}
\mathbb{A}_{\alpha}=\left\{f \in l^{1}\left(\mathbb{Z}^{n}\right) \mid f(m)=O\left(e^{-\alpha\|m\|^{2}}\right)\right\}, \\
\mathbb{B}_{\alpha}=\left\{R f: \mathbb{G} \longrightarrow \mathbb{C} \mid \operatorname{Rf}\left(H\left(a,\|a\|^{2} k\right)\right)=O\left(e^{-\alpha k^{2}}\right),\right. \\
\text { with } \left.f \in \mathbb{A}_{\alpha}\right\} . \tag{49}
\end{gather*}
$$

Since the map: $\left.l^{1}\left(\mathbb{Z}^{n}\right) \ni f \mapsto R f\right|_{\mathbb{G}^{(1)}}$ is not injective, we define here two equivalence relations $\mathscr{R}$ on $l^{1}\left(\mathbb{Z}^{n}\right)$ and $\mathscr{T}$ on $L_{*}^{1}(\mathbb{G})$ as follows:

$$
\begin{align*}
& {\left[f \mathscr{R} g \Longleftrightarrow|(f-g)(m)|=O\left(e^{-\alpha\|m\|^{2}}\right)\right],}  \tag{50}\\
& \forall f, g \in l^{1}\left(\mathbb{Z}^{n}\right), \\
& {\left[(R f) \mathscr{T}(R g) \Longleftrightarrow R(|f-g|)\left(H\left(a,\|a\|^{2} k\right)\right)=O\left(e^{-\alpha k^{2}}\right)\right],} \\
& \forall R f, R g \in L_{*}^{1}(\mathbb{G}), \tag{51}
\end{align*}
$$

in order to state the following interesting theorem which gives a one-to-one correspondence between the quotient sets $l^{1}\left(\mathbb{Z}^{n}\right) / \mathbb{A}_{\alpha}$ and $L_{*}^{1}(\mathbb{G}) / \mathbb{B}_{\alpha}$.

Theorem 9. The function $\Psi: l^{1}\left(\mathbb{Z}^{n}\right) / \mathbb{A}_{\alpha} \rightarrow L_{*}^{1}(\mathbb{G}) / \mathbb{B}_{\alpha}$, which maps each equivalence class $\dot{f}$ to the equivalence class of $R f$, is a bijection.

Proof. It suffices to prove the injectivity of the function $\Psi$. It is easy to see that

$$
\begin{equation*}
\Psi(\dot{f})=\Psi(\dot{g}) \Longleftrightarrow R(|f-g|)=O\left(e^{-\alpha k^{2}}\right) \tag{52}
\end{equation*}
$$

Then, from Theorem 4, we infer that

$$
\begin{equation*}
|(f-g)(m)|=O\left(e^{-\alpha\|m\|^{2}}\right) \tag{53}
\end{equation*}
$$

Therefore

$$
\begin{equation*}
\dot{f}=\dot{g} \tag{54}
\end{equation*}
$$

Hence, $\Psi$ is injective. And the theorem is proved.

## 4. Generalization of Volberg's Uncertainty Principle on the Lattice $\mathbb{Z}^{n}$

In this section which deals with the generalization of Volberg's UP on the lattice $\mathbb{Z}^{n}$, we state the following main theorem which will be proved after introducing some intermediate lemmas.

Theorem 10 (generalization of Volberg's UP on the lattice $\left.\mathbb{Z}^{n}\right)$. Let $f \in l^{1}\left(\mathbb{Z}^{n}\right)$ satisfying the following three conditions
(1) $|f(m)| \leq C_{0} \exp (-\alpha\|m\|)$, for all $m \in \mathbb{Z}^{n}$, where $C_{0}, \alpha$ are strictly positive absolute constants.
(2) The function $\mathbb{T} \ni s \mapsto\left\|\mathscr{F}\left(f_{\mid B(\cdot)}\right)(\psi(\cdot, s))\right\|_{\infty}$ belongs to $L^{1}(\mathbb{T})$.
(3) $\int_{\mathbb{T}} \log \left(\left\|\mathscr{F}\left(f_{\mid B(\cdot)}\right)(\psi(\cdot, s))\right\|_{\infty}\right) d s=-\infty$.

Then $f=0$.
Here, for the definition of the function $\psi: \mathscr{P} \times$ $\mathbb{T} \rightarrow \mathbb{C}$ and the expression of $\mathscr{F}\left(f_{\mid B(\cdot)}\right)(\psi(\cdot, s))$, as well as $\left\|\mathscr{F}\left(f_{\mid B(\cdot)}\right)(\psi(\cdot, s))\right\|_{\infty}($ where $s \in \mathbb{T})$, we just refer the reader to the Introduction (see (6), (7), and (8)).

In order to prove Theorem 10 clearly, we need the following useful lemmas.

Lemma 11. Let $\alpha>0$ and $f_{0}$ be the function defined on $\mathbb{Z}^{n}$ by: $f_{0}(m)=\exp (-\alpha\|m\|)$, for all $m \in \mathbb{Z}^{n}$. Then there exists a constant $C>0$ (which depends only on $\alpha$ and $n$ ) such that, for all $(a, k) \in \mathscr{P} \times \mathbb{Z}$, we have

$$
\begin{equation*}
0<R f_{0}\left(H\left(a,\|a\|^{2} k\right)\right) \leq C \exp \left(-\alpha \frac{|k|}{\sqrt{2}}\right) . \tag{55}
\end{equation*}
$$

Proof. Let $\widetilde{S}(a, k)=\sum_{m \in \mathbb{Z}^{n}, a m=k\|a\|^{2}} \exp (\alpha(|k| / \sqrt{2}-\|m\|))$. The left inequality of (55) is trivial since $f_{0}>0$ implies clearly that $R f_{0}>0$. To show the right inequality of (55), it suffices to prove the inequality $\widetilde{S}(a, k)<C$, for all $(a, k) \in \mathscr{P} \times \mathbb{Z}$. For this, we distinguish two possible cases.
(1) The Case When $\|a\|=1$. Since $a \in \mathbb{Z}^{n} \backslash\{0\}$, we can take $a=(0,0, \ldots, 0,1)$ by permuting the coordinates of $a$, which implies that $a m=m_{n}=k$. Consequently

$$
\begin{align*}
R f_{0}\left(H\left(a,\|a\|^{2} k\right)\right) & =\sum_{m \in \mathbb{Z}^{n}, a m=\|a\|^{2} k} f_{0}(m) \\
& =\sum_{\substack{m_{n}=k \\
\left(m_{1}, \ldots, m_{n-1}\right) \in \mathbb{Z}^{n-1}}} f_{0}(m) \\
& =\sum_{t=\left(m_{1}, \ldots, m_{n-1}\right) \in \mathbb{Z}^{n-1}} \exp \left(-\alpha \sqrt{k^{2}+\|t\|^{2}}\right) . \tag{56}
\end{align*}
$$

Since $\sqrt{\|t\|^{2}+k^{2}} \geq(\|t\|+|k|) / \sqrt{2}$, it is clear that $-\sqrt{\|t\|^{2}+k^{2}} \leq-|k| / \sqrt{2}-\|t\| / \sqrt{2}$. Thus, the left hand side of (56) can be majorized as follows:

$$
\begin{align*}
& R f_{0}\left(H\left(a,\|a\|^{2} k\right)\right) \\
& \leq \sum_{t=\left(m_{1}, \ldots, m_{n-1}\right) \in \mathbb{Z}^{n-1}} \exp \left(-\alpha \frac{\|t\|}{\sqrt{2}}\right) \cdot \exp \left(-\alpha \frac{|k|}{\sqrt{2}}\right) . \tag{57}
\end{align*}
$$

This inequality can be transformed as follows:

$$
\begin{equation*}
\tilde{S}(a, k)<C_{1}=\sum_{t=\left(m_{1}, \ldots, m_{n-1}\right) \in \mathbb{Z}^{n-1}} \exp \left(-\alpha \frac{\|t\|}{\sqrt{2}}\right) . \tag{58}
\end{equation*}
$$

And (55) holds in this first case.
(2) The Case When $\|a\|>1$, That $I s,\|a\| \geq \sqrt{2}$. In this case, since $a m=\|a\|^{2} k$ and $a \neq 0$, we obtain

$$
\begin{equation*}
|k|\|a\|^{2}=\|a\| \cdot\|m\| \cdot|\cos (a, m)| \leq\|a\|\|m\| \tag{59}
\end{equation*}
$$

by applying the Cauchy-Schwarz inequality. Therefore

$$
\begin{equation*}
|k| \leq\|a\|^{-1} \cdot\|m\| \tag{60}
\end{equation*}
$$

It follows that

$$
\begin{align*}
A(a, k) & =\sum_{m \in \mathbb{Z}^{n}, a m=\|a\|^{2} k} \exp (\alpha(|k|-\|m\|)) \\
& \leq \sum_{m \in \mathbb{Z}^{n}, a m=\|a\|^{2} k} \exp \left(\alpha\|m\|\left(\frac{1}{\|a\|}-1\right)\right) \tag{61}
\end{align*}
$$

Since $\|a\| \geq \sqrt{2}$, we have

$$
\begin{equation*}
\|a\|^{-1}-1 \leq \frac{\sqrt{2}}{2}-1 \approx-0,3 \leq-\frac{1}{5} \tag{62}
\end{equation*}
$$

Then

$$
\begin{equation*}
A(a, k) \leq \sum_{m \in \mathbb{Z}^{n}} \exp \left(-\frac{1}{5} \alpha\|m\|\right)=C_{2} \tag{63}
\end{equation*}
$$

but

$$
\begin{equation*}
\widetilde{S}(a, k) \leq A(a, k) \tag{64}
\end{equation*}
$$

Thus

$$
\begin{equation*}
\widetilde{S}(a, k) \leq \sum_{m \in \mathbb{Z}^{n}} \exp \left(-\frac{1}{5} \alpha\|m\|\right)=C_{2} \tag{65}
\end{equation*}
$$

Consequently

$$
\begin{align*}
\widetilde{S}(a, k) & =\sum_{m \in \mathbb{Z}^{n}, a m=\|a\|^{2} k} \exp \left(\alpha\left(\frac{|k|}{\sqrt{2}}-\|m\|\right)\right) \\
& =\exp \left(\alpha \frac{|k|}{\sqrt{2}}\right)\left(\sum_{m \in H\left(a,\|a\|^{2} k\right)} f_{0}(m)\right) \leq C_{2} \tag{66}
\end{align*}
$$

where $f_{0}(m)=\exp (-\alpha\|m\|)$; therefore

$$
\begin{equation*}
R f_{0}\left(H\left(a,\|a\|^{2} k\right)\right)=\sum_{m \in H\left(a,\|a\|^{2} k\right)} f_{0}(m) \leq C_{2} \exp \left(-\alpha \frac{|k|}{\sqrt{2}}\right) \tag{67}
\end{equation*}
$$

And (55) holds in this second case.
Now, by putting $C=\operatorname{Max}\left(C_{1}, C_{2}\right)$, it follows from inequalities (57) and (67) that

$$
\begin{equation*}
R f_{0}\left(H\left(a,\|a\|^{2} k\right)\right) \leq C \exp \left(-\alpha \frac{|k|}{\sqrt{2}}\right), \quad \forall(a, k) \in \mathscr{P} \times \mathbb{Z} \tag{68}
\end{equation*}
$$

This proves the lemma.

Lemma 12. Let $m_{0} \in \mathbb{Z}^{n}$ and $f \in l^{1}\left(\mathbb{Z}^{n}\right)$ satisfying conditions (1), (2), and (3) of Theorem 10. Then $e_{\alpha}\left(m_{0}\right)\left({ }_{m_{0}} f\right)$ verifies also these conditions, where ${ }_{m_{0}} f(m)=f\left(m+m_{0}\right)$ for all $m \in \mathbb{Z}^{n}$ and $e_{\alpha}\left(m_{0}\right)=\exp \left(-\alpha\left\|m_{0}\right\|\right)$.

Proof. Let $g_{m_{0}, \alpha}(m)=e_{\alpha}\left(m_{0}\right) f\left(m+m_{0}\right)$ for all $m \in \mathbb{Z}^{n}$. Since $f$ satisfies the condition $|f(m)| \leq C_{0} \exp (-\alpha\|m\|)$ for all $m \in$ $\mathbb{Z}^{n}$, then we have

$$
\begin{equation*}
\left|f\left(m+m_{0}\right)\right| \leq C_{0} \exp \left(-\alpha\left\|m+m_{0}\right\|\right), \quad \forall m \in \mathbb{Z}^{n} \tag{69}
\end{equation*}
$$

But $\left\|m+m_{0}\right\| \geq\|m\|-\left\|m_{0}\right\|$; then

$$
\begin{equation*}
\exp \left(-\alpha\left\|m+m_{0}\right\|\right) \leq \exp (-\alpha\|m\|) \cdot \exp \left(\alpha\left\|m_{0}\right\|\right) \tag{70}
\end{equation*}
$$

It follows from (69) that

$$
\begin{equation*}
\left|f\left(m+m_{0}\right)\right| \leq C_{0} \exp (-\alpha\|m\|) \cdot \exp \left(\alpha\left\|m_{0}\right\|\right) \tag{71}
\end{equation*}
$$

Therefore

$$
e_{\alpha}\left(m_{0}\right)\left|f\left(m+m_{0}\right)\right|=\left|g_{m_{0}, \alpha}(m)\right| \leq C_{0} \exp (-\alpha\|m\|)
$$

$\forall m \in \mathbb{Z}^{n}$.

Show that $g_{m_{0}, \alpha}$ verifies condition (2) of Theorem 10. Assume that $m_{0} \in B(a)$. Then we have

$$
\begin{align*}
& \mathscr{F}\left(g_{m_{0}, \alpha \mid B(a)}\right)\left(\frac{a s}{\|a\|^{2}}\right) \\
& =\sum_{m \in B(a)} g_{m_{0}, \alpha}(m) \exp \left(-2 i \pi \frac{a m s}{\|a\|^{2}}\right)  \tag{73}\\
& =\sum_{m \in B(a)} e_{\alpha}\left(m_{0}\right) f\left(m+m_{0}\right) \exp \left(-2 i \pi \frac{a m s}{\|a\|^{2}}\right) .
\end{align*}
$$

By putting $m+m_{0}=t$ for $m \in B(a)$, we have $t \in B(a)$, since $m, m_{0} \in B(a)$ implies that $a t /\|a\|^{2}=a m /\|a\|^{2}+a m_{0} /\|a\|^{2} \in$ $\mathbb{Z}$. Then

$$
\begin{align*}
& \mathscr{F}\left(g_{m_{0}, \alpha \mid B(a)}\right)\left(\frac{a s}{\|a\|^{2}}\right) \\
& =e_{\alpha}\left(m_{0}\right) \sum_{t \in B(a)} f(t) \exp \left(-2 i \pi \frac{a t s}{\|a\|^{2}}\right) \cdot \exp \left(2 i \pi \frac{m_{0} s}{\|a\|^{2}}\right) \\
& =e_{\alpha}\left(m_{0}\right) \exp \left(2 i \pi \frac{a m_{0} s}{\|a\|^{2}}\right) \mathscr{F}\left(f_{\mid B(a)}\right)\left(\frac{a s}{\|a\|^{2}}\right) \\
& =\exp \left(-\alpha\left\|m_{0}\right\|+2 i \pi \frac{a m_{0} s}{\|a\|^{2}}\right) \mathscr{F}\left(f_{\mid B(a)}\right)\left(\frac{a s}{\|a\|^{2}}\right) \tag{74}
\end{align*}
$$

We deduce from (74) that

$$
\begin{equation*}
\left\|\mathscr{F}\left(g_{m_{0}, \alpha \mid B(\cdot)}\right)(\psi(\cdot, s))\right\|_{\infty} \leq\left\|\mathscr{F}\left(f_{\mid B(\cdot)}\right)(\psi(\cdot, s))\right\|_{\infty} . \tag{75}
\end{equation*}
$$

Hence condition (2) of Theorem 10 is satisfied.

It remains to show condition (3) of Theorem 10. By using (75) and the fact that $f$ satisfies condition (3), we get

$$
\begin{align*}
& \int_{\mathbb{T}} \log \left(\left\|\mathscr{F}\left(g_{m_{0}, \alpha \mid B(\cdot)}\right)(\psi(\cdot, s))\right\|_{\infty}\right) d s  \tag{76}\\
& \leq \int_{\mathbb{T}} \log \left(\left\|\mathscr{F}\left(f_{\mid B(\cdot)}\right)(\psi(\cdot, s))\right\|_{\infty}\right) d s=-\infty .
\end{align*}
$$

It follows that

$$
\begin{equation*}
\int_{\mathbb{T}} \log \left(\left\|\mathscr{F}\left(g_{m_{0}, \alpha \mid B(\cdot)}\right)(\psi(\cdot, s))\right\|_{\infty}\right) d s=-\infty . \tag{77}
\end{equation*}
$$

This completes the proof of Lemma 12.
We now give the proof of Theorem 10.
Proof of Theorem 10. We prove this theorem by two steps as follows.
(1) First Step: Show That $f(0)=0$. Let $\varphi$ be an arbitrary function of $l^{1}(\mathscr{P})$, where $l^{1}(\mathscr{P})$ is the space of all complexvalued functions $\phi$ defined on $\mathbb{Z}^{n}$ such that $\sum_{a \in \mathscr{P}}|\phi(a)|<$ $+\infty$. Define a function $\kappa_{\varphi, f}$ on $\mathbb{Z}$ as follows:

$$
\begin{equation*}
\kappa_{\varphi, f}(h)=\sum_{a \in \mathscr{P}} \varphi(a) R f\left(H\left(a,\|a\|^{2} h\right)\right), \quad \forall h \in \mathbb{Z} . \tag{78}
\end{equation*}
$$

Prove that $\kappa_{\varphi, f}$ satisfies the conditions of Volberg's uncertainty principle on $\mathbb{Z}$. First, the function $\kappa_{\varphi, f}$ is clearly well defined on $\mathbb{Z}$, since $\varphi \in l^{1}(\mathscr{P})$ and

$$
\begin{align*}
\left|R f\left(H\left(a,\|a\|^{2} h\right)\right)\right| & =\left|\sum_{m \in \mathbb{Z}^{n}, a m=\|a\|^{2} h} f(m)\right|  \tag{79}\\
& \leq \sum_{m \in \mathbb{Z}^{n}}|f(m)|=\|f\|_{1} .
\end{align*}
$$

It is easy to check the following estimate of $\kappa_{\varphi, f}(h)$ :

$$
\begin{equation*}
\left|\kappa_{\varphi, f}(h)\right| \leq \sum_{a \in \mathscr{P}}|\varphi(a)|\left(\sum_{m \in H\left(a,\|a\|^{2} h\right)}|f(m)|\right) . \tag{80}
\end{equation*}
$$

But

$$
\begin{equation*}
|f(m)| \leq C_{0} \exp (-\alpha\|m\|), \quad \forall m \in \mathbb{Z}^{n} . \tag{81}
\end{equation*}
$$

It follows from (80) and Lemma 11 that

$$
\begin{equation*}
\left|\kappa_{\varphi, f}(h)\right| \leq C\|\varphi\|_{1} \exp \left(-\alpha \frac{|h|}{\sqrt{2}}\right), \quad \forall h \in \mathbb{Z}, \tag{82}
\end{equation*}
$$

where $\|\varphi\|_{1}=\sum_{a \in \mathscr{P}}|\varphi(a)|$ and $C>0$ is an absolute constant. In addition, by Fubini theorem, we obtain the following equalities for all $s \in \mathbb{T}$ :

$$
\begin{align*}
\mathscr{F} \kappa_{\varphi, f}(s) & =\sum_{h \in \mathbb{Z}} \kappa_{\varphi, f}(h) \exp (-2 i \pi h s) \\
& =\sum_{h \in \mathbb{Z}}\left(\sum_{a \in \mathscr{P}} \varphi(a) R f\left(H\left(a,\|a\|^{2} h\right)\right)\right) \exp (-2 i \pi h s) \\
& =\sum_{a \in \mathscr{P}} \varphi(a) \mathscr{F}\left(f_{\mid B(a)}\right)\left(\frac{a s}{\|a\|^{2}}\right) . \tag{83}
\end{align*}
$$

It remains to show that $\kappa_{\varphi, f}$ satisfies the following condition:

$$
\begin{equation*}
\int_{\mathbb{T}} \log \left|\mathscr{F}\left(\kappa_{\varphi, f}\right)(s)\right| d s=-\infty \tag{84}
\end{equation*}
$$

From (83), we have

$$
\begin{align*}
& \int_{\mathbb{T}} \log \left|\mathscr{F}\left(\kappa_{\varphi, f}\right)(s)\right| d s \\
& \quad=\int_{\mathbb{T}} \log \left|\sum_{a \in \mathscr{P}} \varphi(a) \mathscr{F}\left(f_{\mid B(a)}\right)\left(\frac{a s}{\|a\|^{2}}\right)\right| d s, \tag{85}
\end{align*}
$$

but

$$
\begin{align*}
& \int_{\mathbb{T}} \log \left|\sum_{a \in \mathscr{P}} \varphi(a) \mathscr{F}\left(f_{\mid B(a)}\right)\left(\frac{a s}{\|a\|^{2}}\right)\right| d s \\
& \leq \int_{\mathbb{T}} \log \left|\|\varphi\|_{1} \cdot \sup _{a \in \mathscr{P}}\right| \mathscr{F}\left(f_{\mid B(a)}\right)(\psi(a, s))| | d s  \tag{86}\\
& \leq \log \left(\|\varphi\|_{1}\right)+\int_{\mathbb{T}} \log \left(\left\|\mathscr{F}\left(f_{\mid B(\cdot)}\right)(\psi(\cdot, s))\right\|_{\infty}\right) d s \\
& =-\infty .
\end{align*}
$$

Then

$$
\begin{equation*}
\int_{\mathbb{U}} \log \left|\mathscr{F}\left(\kappa_{\varphi, f}\right)(s)\right| d s=-\infty . \tag{87}
\end{equation*}
$$

Now, since the function $\kappa_{\varphi, f}: \mathbb{Z} \rightarrow \mathbb{C}$ verifies Volberg's uncertainty principle on $\mathbb{Z}$, then we conclude from [1, Volberg's Theorem, page 222] that $\kappa_{\varphi, f}=0$ for all $\varphi \in l^{1}(\mathscr{P})$. It follows from (78) that $\operatorname{Rf}\left(H\left(a,\|a\|^{2} h\right)\right)=0$ for all $(a, h) \in$ $\mathscr{P} \times \mathbb{Z}$ (since $\varphi$ is arbitrary in $l^{1}(\mathscr{P})$ ), which implies that $R f(H(a, 0))=0$ for all $a \in \mathscr{P}$. In particular, $R f\left(H\left(a_{j}, 0\right)\right)=0$ for all $j \in \mathbb{N} \backslash\{0\}$, where $a_{j}=\left(1, j, j^{2}, \ldots, j^{n-1}\right)$. By the inversion formula (see [5, Theorem 4.1]), we infer that $f(0)=$ 0 .
(2) Second Step: Prove That $f=0$ on $\mathbb{Z}^{n}$. Let $m_{0} \in B(a)$. From Lemma 12 and the first step, we deduce that $g_{m_{0}, \alpha}(0)=0$; then $f\left(m_{0}\right)=0$. It follows that $f_{\mid B(a)}=0$ for all $a \in \mathscr{P}$. Since $\mathbb{Z}^{n}=\bigcup_{a \in \mathscr{P}} B(a)$ (see (16)), we infer that $f$ is identically zero. And Volberg's theorem on $\mathbb{Z}^{n}$ is proved.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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