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Research Article

The Partition Function of the Dirichlet Operator $\mathscr{D}_{2s} = \sum_{i=1}^{d} (-\partial_i^2)^s$ on a d-Dimensional Rectangle Cavity

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We study the asymptotic behavior of the free partition function in the $t \to 0^+$ limit for a diffusion process which consists of d-independent, one-dimensional, symmetric, $2s$-stable processes in a hyperrectangular cavity $K \subset \mathbb{R}^d$ with an absorbing boundary. Each term of the partition function for this polyhedron in d-dimensions can be represented by a quermassintegral and the geometrical information conveyed by the eigenvalues of the fractional Dirichlet Laplacian for this solvable model is now transparent. We also utilize the intriguing method of images to solve the same problem, in one and two dimensions, and recover identical results to those derived in the previous analysis.

1. Introduction

Trace formulas for heat kernels of the fractional Laplacian $(-\Delta)^s$, $s \in (0,1)$ [1], and its Schrödinger perturbations in spectral theory [2] have attracted a lot of attention recently due to the numerous applications to the mathematical physics, mathematical biology, and finance. From a probabilistic point of view the fractional Laplacian on a domain $K \subset \mathbb{R}^d$ is a nonlocal operator which arises as the generator of a pure jump Lévy process killed upon exiting K.

In a recent paper [3] we investigated the distribution of eigenvalues of the Dirichlet pseudodifferential operator $\sum_{i=1}^{d}(-\partial_i^2)^s$, $s \in (0,1)$, on an open and bounded subdomain $K \subset \mathbb{R}^d$, which is defined by the principal value integral:

$$\begin{aligned}\left(-\partial_i^2\right)^s \psi_i(x_i) &= C_{1,s}\text{P.V.}\int_{\mathbb{R}} \frac{\psi_i(x_i)-\psi_i(y_i)}{|x_i-y_i|^{1+2s}}dx_i \\ &= C_{1,s}\lim_{\epsilon\to 0^+}\int_{|x_i-y_i|>\epsilon}\frac{\psi_i(x_i)-\psi_i(y_i)}{|x_i-y_i|^{1+2s}}dx_i, \\ C_{1,s} &= -2\cos(\pi s)\,\Gamma(-2s).\end{aligned} \tag{1}$$

The $\psi_i(x_i)$ are restrictions of functions that belong to the fractional Sobolev space [4], $H^s(\mathbb{R}) = W^{s,2}(\mathbb{R})$, $s \in (0,1)$, given by

$$H^s(\mathbb{R}) = \left\{\psi_i \in L^2(\mathbb{R}) : \frac{|\psi_i(x_i)-\psi_i(y_i)|}{|x_i-y_i|^{1/2+s}} \in L^2(\mathbb{R}\times\mathbb{R})\right\} \tag{2}$$

and satisfy $\psi(x) = \prod_{i=1}^{d}\psi_i(x_i) \equiv 0$ in $\mathbb{R}^d \setminus \overline{K}$. We have also predicted bounds on the sum of the first N eigenvalues, the counting function, the Riesz means, and the first-term asymptotic expansion of the partition function which was found to be

$$\begin{aligned}Z(t) &= \int_0^\infty e^{-\mathscr{E}t}d\mathcal{N}(\mathscr{E}) \\ &= \frac{1}{(2\pi)^d}|\Omega|\frac{(2\Gamma(1+1/2s))^d}{t^{d/2s}} + o\left(t^{-d/2s}\right),\end{aligned} \tag{3}$$

where the counting function $\mathcal{N}(\cdot)$ is[1]

$$\mathcal{N}(\mathscr{E}) = \sharp\left\{n \in \mathbb{Z}_+^d : \|n\|^{2s} \le \left(\frac{L}{\pi}\right)^{2s}\mathscr{E}(1+o(1))\right\}. \tag{4}$$

The present paper extends result (3) to all orders in $t^{-\beta}$, $\beta > 0$, in the short-time limit, and provides a novel result for a polyhedron's partition function which was lacking from the literature even for the ordinary Laplacian. This objective is achieved by using two different methods. The first method relies on the explicit knowledge of the Dirichlet spectrum (20) of the fractional operator and the Euler-Maclaurin summation formula (25). These two ingredients enable us to express the asymptotic behavior of the partition function (29) in terms of the volume of K and the rth quermassintegral. The second method, commonly known as the method of images, suggests that the heat kernel of the cavity, in the short time limit, receives contributions only from the heat kernels of the unbounded space with virtual source points generated by reflecting anticlockwise (or clockwise) the initial point source through the hyperplanes bounding the cavity. Note that only virtual source points clustering around each vertex of the cavity contribute; see (41). Tracing and integrating the heat kernel over the cavity, we recover results identical to those derived by the first method. It is worth noting that this method, though widely known and well established in classical electromagnetism, has not been used before to solve problems falling into this category.

The paper is organized into five sections. Section 2 reviews fundamental notions of convex geometry and applies them to a d-dimensional hyperrectangular parallelepiped in Euclidean space. Section 3 computes the asymptotic behavior of the partition function for the fractional Dirichlet operator by applying the first method. Section 4 uses the alternative approach of images for solving the same problem. Section 5 concludes the work by posing some open problems.

2. Geometric Preliminaries of Convex Bodies in $\mathbb{R}^d$

Let $C(\mathbb{K})$ be the family of all convex bodies (nonempty, compact, and convex sets) in the d-dimensional Euclidean vector space. We denote by $B^d = \{x \in \mathbb{R}^d : \|x\|_2 \leq 1\}$ the unit ball and $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\|_2 = 1\}$ the unit sphere. The Lebesgue measure on $\mathbb{R}^d$ is denoted by λ_d and the spherical Lebesgue measure on $\mathbb{S}^{d-1}$ is denoted by σ_{d-1}. In particular we have $\omega_d := \lambda_d(B^d) = \pi^{d/2}/\Gamma(1 + d/2)$ and $O_{d-1} := \sigma_{d-1}(\mathbb{S}^{d-1}) = d\omega_d = 2\pi^{d/2}/\Gamma(d/2)$. If $K \in C(\mathbb{K})$, then the parallel body K_ρ at distance $\rho > 0$ is given by

$$K_\rho := \left\{x \in \mathbb{R}^d : \operatorname{dist}(x, K) \leq \rho\right\}, \tag{5}$$

where $\operatorname{dist}(\cdot, K)$ denotes the Euclidean distance from K. According to Steiner's formula the volume of K_ρ is given by

$$\lambda_\rho\left(K_\rho\right) = \sum_{m=0}^{d} C_d^m W_m(K)\rho^m = \sum_{j=0}^{d} \omega_{d-j} V_j(K)\rho^{d-j}, \qquad \rho \geq 0,\ C_d^m = \binom{d}{m}, \tag{6}$$

where $W_m : C(\mathbb{K}) \to \mathbb{R}$ is the mth *quermassintegral*, or mean cross-sectional measure, introduced by Minkowski, and $V_j : C(\mathbb{K}) \to \mathbb{R}$ is the jth intrinsic volume[2]. The W_m in (6) is defined by [5]

$$W_m(K) = \frac{(d-m)O_{m-1}\cdots O_0}{dO_{d-2}\cdots O_{d-m-1}} \int_{G_{m,d-m}} V\left(K'_{d-m}\right) dL_{m[O]}, \tag{7}$$

where $V(K'_{d-m})$ denotes the volume of the convex set of all intersection points of the $(d-m)$-plane, passing through the fixed point O, with the m-planes orthogonal to it and $dL_{m[O]}$ is the invariant volume element of the Grassmannian manifold $G_{m,d-m}$ which is the set of m-dimensional planes in $\mathbb{R}^{d-m}$. Setting $m = d-1$ into (7) one can define another functional, the so-called mean breadth of K, as follows:

$$\bar{b} = \frac{2}{O_{d-1}} \int_{G_{d-1,1}} V\left(K'_1\right) dL_{d-1[O]} = \frac{2}{\omega_d} W_{d-1}(K). \tag{8}$$

If the boundary ∂K of a convex set is a hypersurface Σ of class C^2, the quermassintegrals W_m can be expressed by means of the integrals of mean curvature of ∂K. The mth integral of mean curvature $M_m(\Sigma)$ is defined by

$$M_m(\Sigma) = \frac{1}{C_{d-1}^m} \int_\Sigma S_m^{(d-1)}(\kappa)\, dA, \tag{9}$$

where $S_m^{(d-1)}(\kappa) = \sum_{1\leq i_1<\cdots<i_m\leq d-1} \kappa_{i_1}\cdots\kappa_{i_m}$ is the mth-elementary symmetric function of the $(d-1)$ principal curvatures and dA is the area element of Σ. The volume of the parallel body K_ρ can be then written as

$$\lambda_\rho\left(K_\rho\right) = V(K) + \sum_{m=0}^{d-1} \frac{C_{d-1}^m}{m+1} M_m(\partial K)\rho^{m+1} \tag{10}$$

and by comparison with (6) we end up with

$$M_m(\partial K) = dW_{m+1}(K). \tag{11}$$

This relation is well defined as long as ∂K is C^2. If K does not have a smooth boundary, then we compute

$$\lim_{\rho\to 0^+} M_m\left(\partial K_\rho\right) = dW_{m+1}(K). \tag{12}$$

In the present paper we focused on the hyperrectangular parallelepiped with edges a_j, $j = 1, \ldots, d$. The mth intrinsic volume and quermassintegral are given by

$$\begin{aligned} V_m(K) &= S_m^{(d)}(a), \\ V_0(a) &= 1, \\ W_m(K) &= \frac{\omega_m}{C_{d-1}^m} V_{d-m}(K). \end{aligned} \tag{13}$$

The mean breadth of K can be computed by combining (9) with (8) for the parallel body K_ρ and taking the $\rho \to 0^+$ limit in the end. A hyperrectangular parallelepiped domain

has a total of $2d$ faces, 2^d vertices, and $2^{d-1}d$ edges. The mean curvature is defined by

$$H_{d-2}(R) = \frac{1}{d-1} S_{d-2}^{(d-1)}(R) = \frac{1}{d-1} \sum_{1 \le i_1 < \cdots < i_{d-2} \le d-1} \frac{1}{R_{i_1} \cdots R_{i_{d-2}}}, \tag{14}$$

where R_i are the principal radii of curvature of ∂K_ρ. The boundary of ∂K_ρ consists of hyperplanes at the faces, hyperspheres at the vertices, and hypercylinders at the edges of K. The values of H for ∂K_ρ are

$$H_{d-2}(R) = \begin{cases} 0, & \text{hyperplanes}, \\ \dfrac{1}{\rho^{d-2}}, & \text{hyperspheres}, \\ \dfrac{1}{(d-1)\rho^{d-2}}, & \text{lateral face of hypercylinders}. \end{cases} \tag{15}$$

A careful calculation gives

$$W_{d-1}(K) = \lim_{\rho \to 0^+} W_{d-1}(\partial K_\rho) = \frac{1}{d} \omega_{d-1} \sum_{i=1}^{d} a_i \tag{16}$$

and therefore

$$\bar{b} = \frac{2}{d} \frac{\omega_{d-1}}{\omega_d} V_1(K). \tag{17}$$

3. The Partition Function for the Operator $\mathscr{D}_{2s}$ on K

Let $\mathbf{X}(t) = \{X_i(t)\}_{i=1}^d$, $t > 0$, be a collection of independent, one-dimensional, symmetric $2s$-stable processes in $\mathbb{R}$ and denote by $\{P_K(t)\}_{t \ge 0}$ the semigroup on $\mathscr{L}^2(K)$ of $\mathbf{X}(t)$ killed upon exiting K. Its transition density $p_K(x, t; y)$ satisfies

$$P_K(t) f(x) = \int_K p_K(x, t; y) f(y)\, dy. \tag{18}$$

In [3, 6], performing a slight modification, the following proposition has been proved.

Proposition 1. *On the open and bounded hyperrectangular parallelepiped $K \subset \mathbb{R}^d$ of side lengths a_i, $i = 1, \ldots, d$, the eigenvalues for the homogeneous Dirichlet problem*

$$\left(\sum_{i=1}^{d} \left(-\partial_i^2\right)^s \psi_n \right)(x) = \mathscr{E}_n(s) \psi_n(x), \quad \text{in } K; \ \mathscr{E}_n = \frac{E_n}{D_{2s}}, \tag{19}$$
$$\psi_n(x) = 0 \quad \text{on } \mathbb{R}^d \setminus \overline{K}$$

are given by

$$\mathscr{E}_n(s) = \sum_{i=1}^{d} \left| \frac{n_i \pi}{a_i} - \frac{(1-s)\pi}{2a_i} \right|^{2s} + \sum_{i=1}^{d} O\left(\frac{1}{n_i}\right), \quad n \in \mathbb{Z}_+^d, \tag{20}$$

where $\{\psi_n\}_{n=1}^\infty$ forms an orthonormal basis in $L^2(K)$ with $\psi_n(x) = c_n \prod_{j=1}^d \psi_{n_j}(x_j)$, none of the indices n_j vanishes, and $D_{2s} = (\hbar^2/[M])^s$ is a constant with dimensions $[D_{2s}] = [M]^s [L]^{4s} / [T]^{2s}$.

Arranging the positive, real, and discrete spectrum of $\mathscr{D}_{2s}$ in increasing order (including multiplicities), we have

$$0 < \mathscr{E}_1(K) < \mathscr{E}_2(K) \le \mathscr{E}_3(K) \le \cdots, \quad \lim_{n \to \infty} \mathscr{E}_n(K) = \infty. \tag{21}$$

The simplicity of $\mathscr{E}_{n_j}$ (the eigenvalue in the j-dimension) is conjectured to hold for $s \in (0, 1)$, proved for $s = 1/2$ and $s \in [1/2, 1)$ in [7], [6], respectively.

Taking into account the previous proposition, the generator of the semigroup acts on the orthonormal set ψ_n as

$$e^{-t\mathscr{D}_{2s}|_K} \psi_n(x) = e^{-t\mathscr{E}_n} \psi_n(x), \quad x \in K, \ t > 0. \tag{22}$$

The transition density is then given by

$$p_K(x, t; y) = \sum_{n \in \mathbb{Z}_+^d} e^{-t\mathscr{E}_n} \overline{\psi}_n(x) \psi_n(y) \tag{23}$$

and the partition function (or trace of the heat kernel), using (23), satisfies

$$\begin{aligned} Z_K(t, s) \equiv \operatorname{Tr}\left(e^{-t\mathscr{D}_{2s}}\right) &= \int_K p_K(x, t; x)\, dx \\ &= \sum_{n \in \mathbb{Z}_+^d} e^{-t\mathscr{E}_n} \int_K |\psi_n(x)|^2\, dx \\ &= \prod_{j=1}^{d} \left(\sum_{n_j \ge 1} e^{-t\mathscr{E}_{n_j}} \right) = \prod_{j=1}^{d} Z_j(t, s), \quad t > 0. \end{aligned} \tag{24}$$

Thus from (24) we observe that the total partition function is written as the disjoint product of partition functions for each spatial dimension. The sum $\sum_n \exp(-t\mathscr{E}_n)$ will be calculated utilizing the Euler-Maclaurin summation formula [8] which states the following.

Theorem 2. *Suppose f is a decreasing function with continuous derivatives up to order p. Then*

$$\begin{aligned}\sum_{k=n+1}^{m} f(k) &= \int_n^m f(u)\,du \\ &+ \sum_{l=1}^{p} (-1)^l \frac{B_l(0)}{l!}\left\{f^{(l-1)}(m) - f^{(l-1)}(n)\right\} \\ &+ \frac{(-1)^{p-1}}{p!}\int_n^m B_p(u-[u])\, f^{(p)}(u)\,du,\end{aligned} \tag{25}$$

where $B_p(u)$ are the Bernoulli polynomials and $B_l(0)$ the Bernoulli numbers.

Applying (25) in one dimension, in the zero time limit, we find for $h(s) = (1-s)/2$

$$\sum_{n_j=1}^{\infty} e^{-[(\pi/a_j)(n_j-h(s))]^{2s}t} \overset{\text{as } t\to 0^+}{\sim} \frac{a_j}{\pi t^{1/2s}}\Gamma\left(1+\frac{1}{2s}\right) - \frac{s}{2}, \quad s\in(0,1]. \tag{26}$$

The functions $f(t,u,s) = e^{-[(\pi/a)(u-h(s))]^{2s}t}$ belong to the following spaces:

$$f \in \begin{cases} C_u^1\left((0,\infty)^2 \times \left(\frac{1}{2},1\right)\right) \\ C_u^\infty\left((0,\infty)^2 \times \left\{\frac{1}{2}\right\}\right) \\ C^0\left((0,\infty)^2 \times \left(0,\frac{1}{2}\right)\right), \end{cases} \tag{27}$$

where the subscript u indicates that the functions are differentiable with respect to u. The case $s = 1/2$ can be summed exactly since it turns out to be a progression. The result is

$$\begin{aligned}Z_j\left(t,\frac{1}{2}\right) &= \frac{\left(\cosh\left((\pi/4a_j)t\right) - \sinh\left((\pi/4a_j)t\right)\right)}{\sinh\left((\pi/2a_j)t\right)} \\ &\overset{t\to 0^+}{\sim} \frac{a_j}{\pi t} - \frac{1}{4}.\end{aligned} \tag{28}$$

Substituting (26) into (24), for $s\in[1/2,1)$ we obtain

$$\begin{aligned}Z_K(t,s) &\overset{t\to 0^+}{\sim} \prod_{j=1}^{d}\left(\frac{a_j}{\pi t^{1/2s}}\Gamma\left(1+\frac{1}{2s}\right) - \frac{s}{2}\right) \\ &= \sum_{m=0}^{d}\left(\frac{1}{\pi t^{1/2s}}\Gamma\left(1+\frac{1}{2s}\right)\right)^m \left(-\frac{s}{2}\right)^{d-m} V_m(K) \\ &= \left(\frac{1}{\pi t^{1/2s}}\Gamma\left(1+\frac{1}{2s}\right)\right)^d V_d(K) \\ &+ \sum_{r=0}^{d-1}\left(\frac{1}{\pi t^{1/2s}}\Gamma\left(1+\frac{1}{2s}\right)\right)^{d-r}\left(-\frac{s}{2}\right)^r \\ &\cdot \frac{d-r}{O_{d-r-1}}\binom{d}{d-r} W_{d-r}(K),\end{aligned} \tag{29}$$

where $V_m(K)$, $W_{d-r}(K)$ are given by (13).

Remark 3. (1) The term $V_d(K) = \prod_{j=1}^{d} a_j$ is the d-dimensional volume of the hyperrectangle parallelepiped and $V_1(K)$ can be written in terms of the mean breadth of K as

$$V_1(K) = \frac{d\omega_d}{2\omega_{d-1}}\bar{b} \tag{30}$$

using (17).

(2) In two dimensions, the trace of the heat kernel for the ordinary Laplacian ($s = 1$), provided that the domain is a rectangle of sides a_1, a_2, is given by

$$Z_K(1,t) = \frac{1}{4}\left[\Theta\left(\frac{\pi t}{a_1^2}\right) - 1\right]\left[\Theta\left(\frac{\pi t}{a_2^2}\right) - 1\right], \tag{31}$$

where $\Theta(x)$ is the familiar Riemann theta function[3]

$$\Theta(x) = \vartheta\begin{bmatrix}0\\0\end{bmatrix}(0,-ix) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}. \tag{32}$$

The Poisson summation formula[4] can be casted into the form [9]

$$\Theta(x) = \frac{1}{\sqrt{x}}\Theta\left(\frac{1}{x}\right) \tag{33}$$

and utilized to determine the asymptotic behavior of the partition function in the $t \to 0^+$ limit. Therefore, we reproduce the well-known result

$$Z_K(t,s=1) \sim \frac{1}{4\pi t}V_2(K) - \frac{1}{8\sqrt{\pi t}}V_1(K) + \frac{1}{4}, \tag{34}$$

where $V_2(K)$ is the surface area of K and $V_1(K)$ is the length of the boundary ∂K. An independent calculation using (29) gives identical result. Kac, in [10], extended the dimensionless corner correction for a closed polygon with obtuse angles through a complicated integral. Later it was reported in [11] that D. B. Ray obtained the correction

$$\sum_{i=1}^{n}\frac{\pi^2 - \phi_i^2}{24\pi\phi_i}, \quad 0 < \phi_i < 2\pi \tag{35}$$

for arbitrary angles by expressing Green's function as a Kontorovich-Lebedev transform. It is noteworthy that if we approximate a circle by an inscribed regular polygon then (35) becomes

$$\sum_{i=1}^{n} \frac{\pi^2 - \phi_i^2}{24\pi\phi_i} = \frac{1}{6}\left(\frac{n-1}{n-2}\right). \tag{36}$$

In (36) as the number of edges of the polygon tends to infinity the sum converges to the topological invariant constant 1/6. For simply connected, open with compact boundary two-dimensional Riemannian manifolds $(\mathscr{M}, g)$, where g is the metric tensor, the partition function contains the t-independent term given by [11]

$$\frac{E}{6} = \frac{1}{12\pi}\int_{\mathscr{M}} \mathrm{Ric}\sqrt{\det g}\, dx, \tag{37}$$

where E is the Euler characteristic and Ric is the Ricci scalar of the manifold $\mathscr{M}$.

4. A New Alternative Approach Based on the Image Method

The transition function p_K (or elementary solution) is the solution of the following diffusion problem in $K = (0, L) \subset \mathbb{R}$ described by

$$\frac{\partial p_K(x,t;y)}{\partial t} = \left(-\frac{d^2}{dx^2}\right)^s p_K(x,t;y), \quad \text{in } K \times \mathbb{R}_+ \tag{38}$$

with initial and Dirichlet boundary conditions

$$\lim_{t\to 0^+}\int_K p_K(x,t;y)\,dx = \int_K \delta(|x-y|)\,dx = 1, \tag{39a}$$

$$\lim_{x\to q\in\partial K} p_K(x,t;y) = 0, \quad \forall y \in K. \tag{39b}$$

The physical context of (39a) is the existence of a point source initially described by a generalized Dirac-δ function while (39b) dictates that the process is killed upon reaching the boundary ∂K. The solution of (38) with the initial condition (39a) on $\mathbb{R}$ is given by [12]

$$p_{\mathbb{R}}(x,t;y) = \frac{1}{2\pi} \cdot \int_{\mathbb{R}} e^{-t|k-\mathrm{sign}(k)(\pi h(s)/L)|^{2s} + i(k-\mathrm{sign}(k)(\pi h(s)/L))|x-y|} dk, \tag{40}$$

$$h(s) = \frac{1-s}{2},$$

where the appearance of the function $\mathrm{sign}(k)(\pi h(s)/L)$ will be apparent shortly. Note that due to the translational invariance of the integral it can be absorbed in k and written in the usual way.

In general the asymptotic behavior of the Dirichlet partition function in the $t \to 0^+$ limit is dominated by the diagonal elements of the heat kernel which receive contributions from the heat kernels of the unbounded space

Table 1

The virtual image points of y			
Left fixed point		Right fixed point	
Group element	Location of virtual image point	Group element	Location of virtual image point
R_{a_2}	$-y$	R_{a_1}	$-y+2L$
$R_{a_1}\cdot R_{a_2}$	$y+2L$	$R_{a_2}\cdot R_{a_1}$	$y-2L$
$R_{a_2}\cdot R_{a_1}\cdot R_{a_2}$	$-y-2L$	$R_{a_1}\cdot R_{a_2}\cdot R_{a_1}$	$-y+4L$
$(R_{a_1}\cdot R_{a_2})^2$	$y+4L$	$(R_{a_2}\cdot R_{a_1})^2$	$y-4L$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

for the virtual source points clustering around each vertex of the polytope K. The corresponding expression is

$$\lim_{x\to y} p_K^{v}(x,t;y) \overset{t\to 0^+}{\sim} \sum_{\rho=0}^{|\mathscr{G}_v|} (-1)^{\rho}\, p_{\mathbb{R}}\left(\left\|\left(I - R_{a_{\rho_v}}\right) y\right\|, t\right), \tag{41}$$

where $|\mathscr{G}_v|$ is the order of the reflection group at vertex v. Thus integrating (41) over a suitable subdomain of K and then taking the $t \to 0^+$ limit we recover the desired result.

In $d = 1$, $\mathscr{G}$ coincides with the infinite dihedral group Dih_∞ with defining relations

$$R_{a_1}^2 = R_{a_2}^2 = I, \tag{42}$$

where R_{a_i}'s are involuntary transformations and represent reflections with respect to the boundary points of K. For every $y \in K$ we generate the following two infinite sequences of virtual source points depending on whether we start reflection from the left or the right fixed point of the isometry R_i (as shown in Table 1). In Figure 1, we graphically depict the virtual domains in which the corresponding image source points belong to.

The solution, using (41), is given by

$$p_K(\|x-y\|, t) = \sum_{n\in\mathbb{Z}} p_{\mathbb{R}}\left(\left\|x-y-2nL_j\right\|, t\right) - \sum_{n\in\mathbb{Z}} p_{\mathbb{R}}\left(\left\|x+y-2nL_j\right\|, t\right). \tag{43}$$

One can check that both the initial and boundary conditions are fulfilled and moreover the $x \to y$ limit of (43) gives

$$\lim_{x\to y} p_K(\|x-y\|, t) = p_{\mathbb{R}}(0,t) + 2\sum_{n\in\mathbb{N}\setminus\{0\}} p_{\mathbb{R}}(\|2nL\|, t) - \sum_{n\in\mathbb{Z}} p_{\mathbb{R}}(\|2(y-nL)\|, t) \tag{44}$$

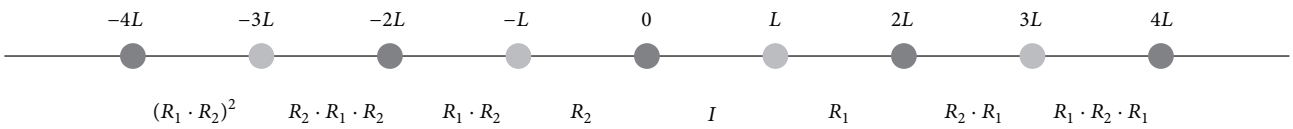

Figure 1: The fundamental region $K = (0, L)$ and the virtual domains generated by the elements $R_i \equiv R_{a_i}$ of the infinite dihedral group.

and the contribution of each term to the partition function $Z_K(t) = \int_K p_K(y, t; y)dy$ is

$$\begin{aligned} &\int_0^L p_{\mathbb{R}}(0,t)\,dy = \frac{L}{\pi t^{1/2s}}\Gamma\left(1+\frac{1}{2s}\right) + h(s) \\ &2\lim_{t\to 0^+}\left(\sum_{n\in\mathbb{N}\setminus\{0\}}\int_0^L p_{\mathbb{R}}(\|2nL\|,t)\,dy\right) = \sum_{n\in\mathbb{N}\setminus\{0\}}\delta(n) \\ &\quad = 0 \\ &\lim_{t\to 0^+}\left(\sum_{n\in\mathbb{Z}}\int_0^L p_{\mathbb{R}}(\|2(y-nL)\|,t)\,dy\right) = \frac{1}{2}. \end{aligned} \tag{45}$$

Therefore,

$$Z_K(t) \overset{t\to 0^+}{\sim} \frac{L}{\pi t^{1/2s}}\Gamma\left(1+\frac{1}{2s}\right) + \left(-\frac{s}{2}\right) \tag{46}$$

as promised.

The diffusion initial-boundary value problem for $K = (0, L_1) \times (0, L_2)$ with $s = 1$ can also be solved applying the method of images in a more general setting [13]. Let θ be the angle between two intersecting mirrors in $\mathbb{R}^2$. If we require the absence of a virtual mirror between the two given ones, after successive reflections of the fundamental domain, then $\theta = \pi/q$ where $q \in \mathbb{N} \setminus \{1\}$. This remark facilitates the enumeration of bounded tessellations of the plane through reflections [14]. One of the admissible polygons apart from the congruent equilateral triangles, the isosceles right triangles and the bisected equilateral triangles, is the rectangle. Generally speaking, if the angle $\theta = \pi/m$, $m \in \mathbb{N} \setminus \{1\}$, then there is a unique, up to isomorphism, group generated by two involutions $R_{\alpha_1}, R_{\alpha_2}$ such that their product $R_{\alpha_1} \cdot R_{\alpha_2}$ has order m. The group is denoted by Dih_{2m}, called the dihedral group of order $2m$ and has the presentation

$$\mathrm{Dih}_{2m} = \left\langle R_{\alpha_1}, R_{\alpha_2} \mid R_{\alpha_1}^2 = R_{\alpha_2}^2 = \left(R_{\alpha_1} \cdot R_{\alpha_2}\right)^m = I \right\rangle. \tag{47}$$

In our case $m = 2$ and each wedge produces the transition function

$$\begin{aligned} \lim_{x\to y} p_K(\|x-y\|,t) = {} & p_{\mathbb{R}}(0,t) \\ & - p_{\mathbb{R}}\left(\left\|\left(I - R_{\alpha_1}\right)y\right\|,t\right) \\ & + p_{\mathbb{R}}\left(\left\|\left(I - R_{\alpha_2}\cdot R_{\alpha_1}\right)y\right\|,t\right) \\ & - p_{\mathbb{R}}\left(\left\|\left(I - R_{\alpha_2}\right)y\right\|,t\right). \end{aligned} \tag{48}$$

Integrating (48) over a suitable subdomain $D_i \subset K$ once, multiplying the result by a factor of four in order to take into account the contribution from all four wedges of the rectangle, and taking the $t \to 0^+$ limit we recover (34).

5. Discussion

The nonlocal fractional operator $\mathscr{D}_{2s}$, considered in this paper, is the infinitesimal generator of time translations for a symmetric Lévy process with index (or characteristic exponent) s, killed upon exiting the cavity. The resulting partition function turns out to depend not only on time t and the dimensionality d of the space but also on s. If one investigates the same problem considering processes with stable law parameters, the skewness β, the scale c, and the location τ, then extra contributions should be expected.

In two dimensions, the topological term given by (35), for the ordinary diffusion ($s = 1$), predicts the value 1/4. Nevertheless, the calculation of the partition function for the fractional operator suggests the s-dependent value $(-s/2)^2$. This is an interesting result that signals the connection of the corner corrections of a bounded domain with the s-stable Lévy process. In the same spirit, relation (37) requires further investigation in order to reveal a possible similar dependence. Unfortunately, the absence of a relation between (37) and the corresponding term of the partition function, in the case of a symmetric Lévy flight, puts severe constraints in this direction.

Another challenging open problem would be to predict the partition function for an open manifold with convex and compact boundary knowing the sequence of polyhedral partition functions which approximates it. This issue is related to the geometric measure theory and fully resolving it will lead to the discovery of new results in higher dimensions than those presented in [11].

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Endnotes

1. We denote by $\|x\|^{2s} = \sum_{i=1}^d |x_i|^{2s}$ the $2s$-norm and by $\|\cdot\|_2$ the Euclidean norm.

2. The word intrinsic here means that the quantity under study does not depend on the dimension of the ambient space.

3. The Jacobi theta function is defined by

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau) = \sum_{n\in\mathbb{Z}} e^{i\pi(n+a)^2\tau+2i\pi(n+a)(z+b)}. \qquad (*)$$

4. The Poisson formula states

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2 A+2\pi nsA} = \frac{1}{\sqrt{A}} e^{\pi s^2 A} \sum_{m\in\mathbb{Z}} e^{-\pi m^2 A^{-1}-2i\pi ms} \qquad (**)$$

and can be proved using $\sum_m e^{2i\pi rm} = \sum_n \delta(r-n)$.

References

[1] R. L. Frank and L. Geisinger, "Two-term spectral asymptotics for the Dirichlet Laplacian on a bounded domain," in *Mathematical Results in Quantum Physics*, pp. 138–147, World Science Publisher, Hackensack, NJ, USA, 2011.

[2] R. Bañuelos and S. S. Yolcu, "Heat kernel of non-local operators," *Journal of the London Mathematical Society*, vol. 87, no. 2, pp. 304–318, 2013.

[3] A. N. Hatzinikitas, "Spectral Properties of the Dirichlet operator $\sum_{i=1}^{d}(-\partial_i^2)^s$ on domains in d-dimensional Euclidean space," *Journal of Mathematical Physics*, vol. 54, Article ID 103501, 2013.

[4] E. Di Nezza, G. Palatucci, and E. Valdinoci, "Hitchhiker's guide to the fractional Sobolev spaces," *Bulletin des Sciences Mathématiques*, vol. 136, no. 5, pp. 521–573, 2012.

[5] L. Santaló, *Integral Geometry and Geometric Probability*, Cambridge University Press, 2nd edition, 2004.

[6] M. Kwaśnicki, "Eigenvalues of the fractional Laplace operator in the interval," *Journal of Functional Analysis*, vol. 262, no. 5, pp. 2379–2402, 2012.

[7] T. Kulczycki, M. Kwaśnicki, J. Małecki, and A. Stos, "Spectral properties of the Cauchy process on half-line and interval," *Proceedings of the London Mathematical Society*, vol. 101, no. 2, pp. 589–622, 2010.

[8] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Cambridge University Press, 2000.

[9] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vol. 1, Interscience, New York, NY, USA, 1953.

[10] M. Kac, "Can one hear the shape of a drum?" *The American Mathematical Monthly*, vol. 73, no. 4, part 2, pp. 1–23, 1966.

[11] H. P. McKean and I. M. Singer, "Curvature and the eigenvalues of the Laplacian," *Journal of Differential Geometry*, vol. 1, no. 1, pp. 43–69, 1967.

[12] A. Hatzinikitas and J. K. Pachos, "One dimensional stable probability density functions for rational index $0 < \alpha \leq 2$," *Annals of Physics*, vol. 323, no. 12, pp. 3000–3019, 2008.

[13] A. N. Hatzinikitas, "The short-time limitof the Dirichlet partition function and the image method," http://arxiv.org/abs/1410.1631.

[14] H. S. M. Coxeter, *Regular Polytopes*, Dover Publications, New York, NY, USA, 3rd edition, 1973.