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Research Article

Locally Defined Operators in the Space of Functions of Bounded Riesz-Variation

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We study the locally defined operator on the spaces of bounded Riesz p-variation functions and we prove that those operators are the Nemytskii operator.

1. Introduction

We have an closed interval I of the real line and let $\mathscr{X} = \mathscr{X}(I), \mathscr{Y} = \mathscr{Y}(I)$ be function spaces $f : I \to \mathbb{R}$. An operator $K : \mathscr{X} \to \mathscr{Y}$ is called a *locally defined operator*, or $(\mathscr{X}, \mathscr{Y})$-*local operator*, briefly, a *local operator* [1], if for every open interval $J \subset \mathbb{R}$ and for all functions $f, g \in \mathscr{X}$, the implication

$$f|_{J\cap I} = g|_{J\cap I} \Longrightarrow K(f)|_{J\cap I} = K(g)|_{J\cap I} \tag{1}$$

is true, where $f|_{J\cap I}$ denotes the restriction of f to $J \cap I$.

There is a vast literature on the problem treated here, mainly compiled of definitions of locally defined operators involving a measure space (cf., e.g., [2–5]). Also we proved that, in general, K is a composition (or Nemytskii) operator of the form $K(f)(x) = h(x, f(x))$ for a two-variable function h. Assuming additionally that K is continuous in measure, the generating function h can be replaced by a function satisfying the Caratheodory conditions (cf. [6]). The present paper concerns topological aspects of locally defined operators (cf. [1, 7–10]). For more knowledge on theory of the composition operators, see Appell and Zabrejko [11]. In [7] it was done is the case when $\mathscr{X} = C^n(I)$ and $\mathscr{Y} = C(I)$ or $\mathscr{Y} = C^1(I)$. Subsequently, this result has been extended by several authors: [8, 9, 12] (for spaces of Whitney differentiable functions), [10, 13] (for space of Hölder functions), [14] (for continuous and monotone functions), and [1] (for functions of bounded φ-variation in the sense Wiener). In the present paper we are interested in such operators in the context of bounded Riesz-variation functions. In particular, we show that if the operator K maps the space $RV_p(I, \mathbb{R})$ into itself and is locally defined, then K is a Nemytskii composition operator.

2. Notation and Preliminaries

In this section we present some necessary notations and definitions and recall some knowledge concerning the bounded Riesz-variation.

In the sequel, $\mathbb{N}$, $\mathbb{N}_0$, and $\mathbb{R}$ denote, respectively, the set of positive integers, nonnegative integers, and the set of real numbers.

Let $I = [a, b] \subset \mathbb{R}$; $\pi = \{t_i\}_{i=0}^m$ be partition of I, defined by $\pi : a = t_0 < t_1 < \cdots < t_m = b$. As usually, $\mathbb{R}^I$ denote the family of all functions $f : I \to \mathbb{R}$.

Given $1 \le p < \infty$, $f \in \mathbb{R}^I$ and a partition π of I, we define

$$v_p^R(f) = v_p^R(f, I) := \sup_\pi \sum_{i=1}^m \frac{|f(t_i) - f(t_{i-1})|^p}{|t_i - t_{i-1}|^{p-1}}, \tag{2}$$

where the supremum is taken over all partitions π of I. $v_p^R(f)$ is the classical p-variation of f in the sense of Riesz [15] in I. A function f is said to be of bounded p-variation if

$v_p^R(f,I)<\infty$. By $RV_p(I)$ we denote the Banach space of all functions $f\in\mathbb{R}^I$ of bounded p-variation equipped with the norm

$$\|f\|_p := |f(a)| + \left(v_p^R(f)\right)^{1/p}, \quad f\in RV_p(I). \tag{3}$$

Lemma 1. *Let $I=[a,b]\subset\mathbb{R}$ be an interval and let $(x_0,y_0)\in I\times\mathbb{R}$, $x_0<\sup(I)$ be fixed. Then for every sequence $(x_k,y_k)\in I\times\mathbb{R}$ satisfying the condition*

$$\begin{gathered}\lim_{k\to\infty}(x_k,y_k)=(x_0,y_0),\\ x_{k+1}<x_k;\ y_{k+1}<y_k,\ k\in\mathbb{N}\end{gathered} \tag{4}$$

there exists a function $\psi\in RV_p(I)$ such that, for all $k\in\mathbb{N}_0$,

$$\psi(x_k)=y_k. \tag{5}$$

Proof. Take an arbitrary sequence $(x_k,y_k)\in I\times\mathbb{R}$ satisfying (4) and define a sequence of functions $\psi_k: I\to\mathbb{R}$, $k\in\mathbb{N}$, by

$$\psi_k(x) := \begin{cases} y_0, & \text{for } x\in[a,x_0];\\ \dfrac{y_k-y_0}{x_k-x_0}(x-x_0)+y_0, & \text{for } x\in(x_0,x_k];\\ \dfrac{y_i-y_{i-1}}{x_i-x_{i-1}}(x-x_i)+y_i, & \text{for } x\in(x_i,x_{i-1}],\ i\in\{2,\dots,k\};\\ y_1, & \text{for } x\in(x_1,b]. \end{cases} \tag{6}$$

Let us observe that

$$\begin{gathered}\psi_k(x_0)=y_0,\\ \psi_k(x_k)=\psi_{k+\ell}(x_k)=y_k,\\ k,\ell\in\mathbb{N},\end{gathered} \tag{7}$$

and for every $x\in I\setminus\{x_k : k\in\mathbb{N}_0\}$ there exist $k_0\in\mathbb{N}$ such that

$$\psi_k(x)=\psi_{k_0}(x), \quad k\ge k_0,\ k\in\mathbb{N}. \tag{8}$$

Put

$$\psi(x)=\lim_{k\to\infty}\psi_k(x), \quad x\in I. \tag{9}$$

From (7) and (8), the function ψ is well defined. Moreover, ψ is nondecreasing and

$$\psi(x_k)=y_k, \quad \forall k\in\mathbb{N}_0, \tag{10}$$

and by (9), for each $\epsilon>0$, we obtain

$$|\psi_k(x)-\psi(x)|<\epsilon, \quad \forall x\in I, \tag{11}$$

so $\|\psi_k-\psi\|_\infty\le\epsilon$. Thus the sequence $(\psi_k)_{k\in\mathbb{N}}$ tends uniformly to ψ.

Now as $\psi_k\in RV_p(I)$ for all $k\in\mathbb{N}$ and ψ_k tends uniformly to ψ, then

$$\begin{aligned}\frac{|\psi(x_i)-\psi(x_{i-1})|^p}{|x_i-x_{i-1}|^{p-1}} &= \lim_{k\to\infty}\frac{|\psi_k(x_i)-\psi_k(x_{i-1})|^p}{|x_i-x_{i-1}|^{p-1}}\\ &\le \lim_{k\to\infty} v_p^R(\psi_k,I)<\infty;\end{aligned} \tag{12}$$

thus $v_p^R(\psi,I)<\infty$ and therefore $\psi\in RV_p(I)$. □

Similarly, we can get the following.

Remark 2. If $(x_0,y_0)\in I\times\mathbb{R}$, where $x_0>\inf(I)$ and $(x_k,y_k)\in I\times\mathbb{R}$ is a sequence satisfying the condition

$$\begin{gathered}\lim_{k\to\infty}(x_k,y_k)=(x_0,y_0),\\ x_k<x_{k+1};\ y_k\le y_{k+1},\ k\in\mathbb{N},\end{gathered} \tag{13}$$

then there exists a function $\psi\in RV_p(I)$ such that, for all $k\in\mathbb{N}_0$,

$$\psi(x_k)=y_k. \tag{14}$$

3. Locally Defined Operators

Now we can introduce the definition of the local defined operators of type $K: RV_p(I)\to C(I)$.

Definition 3 (see [1]). An operator $K: RV_p(I)\to C(I)$ is said to be locally defined, if, for every two functions $f,g\in RV_p(I)$ and for every open interval $J\subset\mathbb{R}$,

$$f|_{J\cap I}=g|_{J\cap I} \Longrightarrow K(f)|_{J\cap I}=K(g)|_{J\cap I}. \tag{15}$$

Theorem 4. *Let $1<p<\infty$. If a locally defined operator K maps $RV_p(I)$ into $C(I)$, then there exists a unique function $h: I\times\mathbb{R}\to\mathbb{R}$ such that, for all $f\in RV_p(I)$,*

$$K(f)(t)=h(t,f(t)), \quad t\in I. \tag{16}$$

Proof. We begin by showing that, for every $f,g\in RV_p(I)$ and, for every $x_0\in\operatorname{int}(I)$, the condition

$$f(x_0)=g(x_0) \tag{17}$$

implies that

$$K(f)(x_0)=K(g)(x_0). \tag{18}$$

To this end choose arbitrary $x_0\in\operatorname{int}(I)$ and take an arbitrary pair of functions $f,g\in RV_p(I)$ which fulfil (17) (i.e., $f(x_0)=g(x_0)$). The function $\varphi: I\to\mathbb{R}$, defined by

$$\varphi(x)=\begin{cases} f(x), & \text{for } x\in[a,x_0];\\ g(x), & \text{for } x\in(x_0,b],\end{cases} \tag{19}$$

belongs to $RV_p(I)$. Indeed, define the functions $f_1,g_1: I\to\mathbb{R}$ by

$$\begin{gathered} f_1(x)=\begin{cases} f(x)-f(x_0), & \text{for } x\in[a,x_0];\\ 0, & \text{for } x\in(x_0,b],\end{cases}\\ g_1(x)=\begin{cases} 0, & \text{for } x\in[a,x_0];\\ g(x)-g(x_0), & \text{for } x\in(x_0,b].\end{cases}\end{gathered} \tag{20}$$

Since $f,g\in RV_p(I)$, $v_p^R(f)<\infty$ and $v_p^R(g)<\infty$. Let $\pi=\{x_i\}_{i=0}^m$ be a partition of I such that $x_{\ell-1}\le x_0<x_\ell$ for some $1\le\ell\le m$. Then

$$\sum_{i=1}^m\frac{|f_1(x_i)-f_1(x_{i-1})|^p}{|x_i-x_{i-1}|^{p-1}}\le v_p^R(f). \tag{21}$$

Hence $v_p^R(f_1) < \infty$. By a similar reasoning, we have $v_p^R(g_1) < \infty$. Finally $f_1 + g_1 \in RV_p(I)$, as $RV_p(I)$ is a linear space. Thus

$$v_p^R(f_1 + g_1) < \infty. \tag{22}$$

Since, for all $x, x' \in I$

$$(f_1 + g_1)(x) - (f_1 + g_1)(x') = \psi(x) - \psi(x'), \tag{23}$$

the condition (22) implies that $\varphi \in RV_p(I)$.

As

$$\begin{aligned} f|_{(-\infty,x_0)\cap I} &= \varphi|_{(-\infty,x_0)\cap I}, \\ g|_{(x_0,\infty)\cap I} &= \psi|_{(x_0,\infty)\cap I}, \end{aligned} \tag{24}$$

by definition of a local operator, we get

$$\begin{aligned} K(f)|_{(-\infty,x_0)\cap I} &= K(\varphi)|_{(-\infty,x_0)\cap I}, \\ K(g)_{(x_0,\infty)\cap I} &= K(\varphi)_{(x_0,\infty)\cap I}. \end{aligned} \tag{25}$$

Therefore, by the continuity of $K(f)$, $K(g)$ and $K(\varphi)$ at x_0, we obtain

$$K(f)(x_0) = K(\varphi)(x_0) = K(g)(x_0). \tag{26}$$

Suppose now that x_0 is the left endpoint of the interval I (i.e., $x_0 = a$). There exists a sequence $(x_k, y_k) \in I \times \mathbb{R}$ such that $x_0 < x_{k+1} < x_k$, $y_0 \leq y_{k+1} < y_k$, $k \in \mathbb{N}$, and by the continuity of f and g at x_0

$$\lim_{k\to\infty} (x_k, y_k) = (x_0, y_0). \tag{27}$$

By Lemma 1 there exists a function $\psi \in RV_p(I)$ such that $\psi(x_k) = y_k$ for all $k \in \mathbb{N}_0$.

There is no loss of generality in supposing that $f(x_0) = g(x_0) = y_0$, $\psi(x_{2k-1}) = y_{2k-1} = g(x_{2k-1})$ and $\psi(x_{2k}) = y_{2k} = f(x_{2k})$, $k \in \mathbb{N}$.

According to the first part of the proof, we have

$$\begin{aligned} K(\psi)(x_{2k-1}) &= K(g)(x_{2k-1}), \\ K(\psi)(x_{2k}) &= K(f)(x_{2k}), \\ & \qquad k \in \mathbb{N}. \end{aligned} \tag{28}$$

Hence, by continuity of $K(\psi)$, $K(g)$, and $K(f)$ at x_0, letting $k \to \infty$, we get

$$K(f)(x_0) = K(\psi)(x_0) = K(g)(x_0). \tag{29}$$

When x_0 is the right endpoint of I, the argument is similar.

To define the function $h : I \times \mathbb{R} \to \mathbb{R}$ and fix arbitrarily an $y_0 \in \mathbb{R}$, let us define a function $P_{y_0} : I \to \mathbb{R}$ by

$$P_{y_0}(x) := y_0, \quad x \in I. \tag{30}$$

Of course P_{y_0}, as a constant function, belongs to $RV_p(I)$. For $x_0 \in I$, $y_0 \in \mathbb{R}$, put

$$h(x_0, y_0) := K(P_{y_0})(x_0). \tag{31}$$

Since, by (30), for all functions f,

$$f(x_0) = P_{f(x_0)}(x_0), \tag{32}$$

according to what has already been proved, we have

$$K(f)(x_0) = K(P_{f(x_0)})(x_0) = h(x_0, f(x_0)). \tag{33}$$

To prove the uniqueness of h, assume that $\overline{h} : I \times \mathbb{R} \to \mathbb{R}$ is such that

$$K(f)(x) = \overline{h}(x, f(x)) \tag{34}$$

for all $f \in RV_p(I)$ and $x \in I$. To show that $h = \overline{h}$ let us fix arbitrarily $x \in I$, $y \in \mathbb{R}$ and take $f \in RV_p(I)$ with $f(x) = y$. From (33), we have

$$\begin{aligned} \overline{h}(x, y) &= \overline{h}(x, f(x)) = K(f)(x) = h(x, f(x)) \\ &= h(x, y), \end{aligned} \tag{35}$$

which proves the uniqueness of h. □

Definition 5. Let $X \subset \mathbb{R}$ and a function $h : X \times \mathbb{R} \to \mathbb{R}$ be fixed. The mapping $H : \mathbb{R}^X \to \mathbb{R}^X$, given by

$$H(f)(x) := h(x, f(x)), \quad f \in \mathbb{R}^X, \ x \in X, \tag{36}$$

is said to be composition (Nemytskii or superposition) operator. The function h is referred to as the generator of the operator H.

As an immediate consequence of Theorem 4 we get the following.

Corollary 6. *Let $1 \leq p < \infty$. If a local operator K maps $RV_p(I)$ into $C(I)$, then it is a Nemytskii operator.*

Note that if a local operator K maps $RV_p(I)$ into itself then, obviously, K maps $RV_p(I)$ into $C(I)$. Therefore, by Theorem 4, we get the following.

Theorem 7. *Let $1 \leq p < \infty$. If a local operator K maps $RV_p(I)$ into itself, then there exists a unique function $h : I \times \mathbb{R} \to \mathbb{R}$ such that, for all $f \in RV_p(I)$,*

$$K(f)(x) = h(x, f(x)), \quad x \in I. \tag{37}$$

Corollary 8. *Let $1 \leq p < \infty$. If a local operator K maps $RV_p(I)$ into itself, then it is a Nemytskii operator.*

Under the additional assumption that the locally defined operator is uniformly continuous, we get a complete characterization of its generating function h. Namely, we have the following.

Theorem 9. *Let $1 \leq p < \infty$. If a local operator $K : RV_p(I) \to RV_p(I)$ is uniformly continuous, then there exists $f_1, f_2 \in RV_p(I)$ such that*

$$K(f)(x) = f_1(x) f(x) + f_2(x), \tag{38}$$

$f \in RV_p(I)$, $x \in I$.

Proof. From Theorem 7 there exists a unique function $h : I \times \mathbb{R} \to \mathbb{R}$ such that $K(f)(x) = h(x, f(x))$ for all $f \in RV_p(I)$, $x \in I$. Fix $(x_0, y_0) \in I \times \mathbb{R}$, take an arbitrary sequence $x_n \in I$ with $x_n \to x_0$, and let $P_{y_0} : I \to \mathbb{R}$ be defined by $P_{y_0}(x) = y_0$, $x \in I$. Since $h(x_0, y_0) = K(P_{y_0})(x_0)$,

$$\begin{aligned} &\left|h(x_n, y_0) - h(x_0, y_0)\right| \\ &\quad = \left|h\left(x_n, P_{y_0}(x_n)\right) - h\left(x_0, P_{y_0}(x_0)\right)\right| \\ &\quad = \left|K\left(P_{y_0}\right)(x_n) - K\left(P_{y_0}\right)(x_0)\right|; \end{aligned} \tag{39}$$

applying the continuity of $K(P_{y_0})$ at x_0, we get the continuity of h with respect to the first variable. Thus, by [16, Theorem 1] (with $\varphi(x) = \psi(x) = x^p$),

$$h(x, y) = f_1(x)\, y + f_2(x), \quad x \in I,\ y \in \mathbb{R}, \tag{40}$$

for some $f_1, f_2 : I \to \mathbb{R}$. Since $h(\cdot, y_0) = K(P_{y_0})(\cdot) \in RV_p(I)$ and $f_2(x) = h(x, 0)$, $f_1(x) = h(x, 1) - f_2(x)$, the functions $f_1, f_2 \in RV_p(I)$. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] M. Wrobel, "Locally defined operators in the space of functions of bounded φ-variation," *Real Analysis Exchange*, vol. 38, no. 1, pp. 79–94, 2013.

[2] K. Karták, "A generalization of the Carathéodory theory of differential equations," *Czechoslovak Mathematical Journal*, vol. 17, pp. 482–514, 1967.

[3] K. Karták, "On Carathéodory operators," *Czechoslovak Mathematical Journal*, vol. 17, no. 92, pp. 515–519, 1967.

[4] W. Kozlowski, "Nonlinear opeartors in funtion Banach spaces," *Commentationes Mathematicae. Prace Matematyczne*, vol. 22, pp. 85–103, 1980.

[5] I. V. Shragin, "On representation of a locally defined operator in the form of the Nemytskii operator," *Functional Differential Equations*, vol. 3, no. 3-4, pp. 447–452, 1996.

[6] J. Appell and P. P. Zabreiko, "Continuity properties of the superposition operators," *Journal of the Australian Mathematical Society. Series A*, vol. 47, no. 2, pp. 186–210, 1989.

[7] K. Lichawski, J. Matkowski, and J. Miś, "Locally defined operators in the space of differentiable functions," *Bulletin of the Polish Academy of Sciences. Mathematics*, vol. 37, no. 1–6, pp. 315–325, 1989.

[8] J. Matkowski and M. Wróbel, "Locally defined operators in the space of Whitney differentiable functions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 10, pp. 2933–2942, 2008.

[9] J. Matkowski and M. Wrobel, "Representation theorem for locally defined operators in the space of Whitney differentiable functions," *Manuscripta Mathematica*, vol. 129, no. 4, pp. 437–448, 2009.

[10] M. Wróbel, "Locally defined operators in the Hölder's spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 74, no. 1, pp. 317–323, 2011.

[11] J. Appell and P. P. Zabrejko, *Nonlinear Superposition Operators*, Cambridge University Press, Cambridge, UK, 1990.

[12] M. Wróbel, "Locally defined operators and a partial solution of a conjecture," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 1, pp. 495–506, 2010.

[13] J. Matkowski and M. Wrobel, *The Bounded Local Operators in the Banach Space of Hölder Functions*, vol. 15 of *Scientific Issues, Mathematics*, Jan Dlugosz University in Czestochowa, 2010.

[14] M. Wrobel, "Representation theorem for local operators in the space of continuous and monotone functions," *Journal of Mathematical Analysis and Applications*, vol. 372, no. 1, pp. 45–54, 2010.

[15] F. Riesz, "Untersuchungen über Systeme integrierbarer Funktionen," *Mathematische Annalen*, vol. 69, no. 4, pp. 449–497, 1910.

[16] A. Acosta, W. Aziz, J. Matkowski, and N. Merentes, "Uniformly continuous composition operator in the space of φ-variation functions in the sense of Riesz," *Fasciculi Mathematici*, vol. 43, pp. 1–7, 2010.