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Research Article

Multiple-Term Refinements of Young Type Inequalities

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Recently, a multiple-term refinement of Young's inequality has been proved. In this paper, we show its reverse refinement. Moreover, we will present multiple-term refinements of Young's inequality involving Kantorovich constants. Finally, we will apply scalar inequalities to operators.

1. Introduction

The classical Young inequality states that if $a, b > 0$ and $0 \le \nu \le 1$, then

$$(1-\nu)a + \nu b \ge a^{1-\nu} b^{\nu}. \tag{1}$$

For $0 \le \nu \le 1$, we define three functions $r_0(\nu)$, $r_1(\nu)$, and $R_0(\nu)$ by

$$\begin{aligned} r_0(\nu) &= \min\{\nu, 1-\nu\}, \\ r_1(\nu) &= \min\{2r_0(\nu), 1-2r_0(\nu)\}, \\ R_0(\nu) &= 1 - r_0(\nu). \end{aligned} \tag{2}$$

In [1] and the references there, the following improvements of Young inequality and its reverse are discussed:

$$\begin{aligned} &(1-\nu)a + \nu b \ge a^{1-\nu} b^{\nu} + r_0(\nu)\left(\sqrt{a} - \sqrt{b}\right)^2, \\ &(1-\nu)a + \nu b \le a^{1-\nu} b^{\nu} + \left(1 - r_0(\nu)\right)\left(\sqrt{a} - \sqrt{b}\right)^2, \\ &(1-\nu)a + \nu b \ge a^{1-\nu} b^{\nu} + r_0(\nu)\left(\sqrt{a} - \sqrt{b}\right)^2 + r_1(\nu) \\ &\quad \cdot \left(\left(\sqrt{a} - \sqrt[4]{ab}\right)^2 \chi_{(0,1/2)}(\nu) \right. \\ &\quad \left. + \left(\sqrt{b} - \sqrt[4]{ab}\right)^2 \chi_{(1/2,1)}(\nu)\right), \\ &(1-\nu)a + \nu b \le a^{1-\nu} b^{\nu} + R_0(\nu)\left(\sqrt{a} - \sqrt{b}\right)^2 - r_1(\nu) \\ &\quad \cdot \left(\left(\sqrt{b} - \sqrt[4]{ab}\right)^2 \chi_{(0,1/2)}(\nu) \right. \\ &\quad \left. + \left(\sqrt{a} - \sqrt[4]{ab}\right)^2 \chi_{(1/2,1)}(\nu)\right), \end{aligned} \tag{3}$$

$$\begin{aligned} &(1-\nu)a + \nu b \le a^{1-\nu} b^{\nu} + R_0(\nu)\left(\sqrt{a} - \sqrt{b}\right)^2 - r_1(\nu) \\ &\quad \cdot \left(\left(\sqrt{b} - \sqrt[4]{ab}\right)^2 \chi_{(0,1/2)}(\nu) \right. \\ &\quad \left. + \left(\sqrt{a} - \sqrt[4]{ab}\right)^2 \chi_{(1/2,1)}(\nu)\right), \end{aligned} \tag{4}$$

where χ_I is the characteristic function defined by

$$\chi_I(x) = \begin{cases} 1, & \text{if } x \in I \\ 0, & \text{if } x \notin I. \end{cases} \tag{5}$$

Another form of Young type inequalities discussed in [1] is as follows:

$$\begin{aligned} &((1-\nu)a + \nu b)^2 \ge \left(a^{1-\nu} b^{\nu}\right)^2 + r_0^2(\nu)(a-b)^2 + r_1(\nu) \\ &\quad \cdot \left(\left(a - \sqrt{ab}\right)^2 \chi_{(0,1/2)}(\nu) + \left(b - \sqrt{ab}\right)^2 \chi_{(1/2,1)}(\nu)\right), \\ &((1-\nu)a + \nu b)^2 \le \left(a^{1-\nu} b^{\nu}\right)^2 + R_0^2(\nu)(a-b)^2 - r_1(\nu) \\ &\quad \cdot \left(\left(b - \sqrt{ab}\right)^2 \chi_{(0,1/2)}(\nu) + \left(a - \sqrt{ab}\right)^2 \chi_{(1/2,1)}(\nu)\right). \end{aligned} \tag{6}$$

Other types of improvements of the Young inequality is to use Kantorovich constants. Wu and Zhao [2] showed

$$\begin{aligned}(1-\nu)a+\nu b &\geq K_1(a,b)^{r_1(\nu)}a^{1-\nu}b^{\nu}\\ &\quad+r_0(\nu)\left(\sqrt{a}-\sqrt{b}\right)^2,\\ (1-\nu)a+\nu b &\leq K_1(a,b)^{-r_1(\nu)}a^{1-\nu}b^{\nu}\\ &\quad+R_0(\nu)\left(\sqrt{a}-\sqrt{b}\right)^2,\end{aligned}\tag{7}$$

where

$$K_1(a,b)=\frac{\left(\sqrt{a}+\sqrt{b}\right)^2}{4\sqrt{ab}}.\tag{8}$$

Note that (7) improves (3), since $K_1(a,b)\geq 1$ for all $a,b>0$. In [3], Liao and Wu improved (4) as follows:

$$\begin{aligned}(1-\nu)a+\nu b &\geq K_2(a,b)^{r_2(\nu)}a^{1-\nu}b^{\nu}+r_0(\nu)\\ &\quad\cdot\left(\sqrt{a}-\sqrt{b}\right)^2+r_1(\nu)\\ &\quad\cdot\left(\left(a-\sqrt{ab}\right)^2\chi_{(0,1/2)}(\nu)+\left(b-\sqrt{ab}\right)^2\chi_{(1/2,1)}(\nu)\right),\\ (1-\nu)a+\nu b &\leq K_2(a,b)^{-r_2(\nu)}a^{1-\nu}b^{\nu}+R_0(\nu)\\ &\quad\cdot\left(\sqrt{a}-\sqrt{b}\right)^2-r_1(\nu)\\ &\quad\cdot\left(\left(b-\sqrt{ab}\right)^2\chi_{(0,1/2)}(\nu)+\left(a-\sqrt{ab}\right)^2\chi_{(1/2,1)}(\nu)\right),\end{aligned}\tag{9}$$

where

$$\begin{aligned}K_2(a,b)&=\frac{\left(\sqrt[4]{a}+\sqrt[4]{b}\right)^2}{4\sqrt[4]{ab}},\\ r_2(\nu)&=\min\{2r_1(\nu),1-2r_1(\nu)\}.\end{aligned}\tag{10}$$

The constants of the form $(m+M)^2/4mM$ are called Kantorovich constants.

Throughout the paper, we will use the following functions.

Definition 1. One defines the sequence $\{r_n(\nu)\}$ of functions on $[0,1]$ as follows:

$$\begin{aligned}r_0(\nu)&=\min\{\nu,1-\nu\},\\ r_n(\nu)&=\min\{2r_{n-1}(\nu),1-2r_{n-1}(\nu)\},\end{aligned}\tag{11}$$

for $n\in\mathbb{N}$.

Definition 2. For $l,k\in\mathbb{N}$ and $a,b>0$, we define the functions $g_{l,k}(a,b)$ by

$$\begin{aligned}g_{l,k}(a,b)&=\left(\sqrt{a^{1-(k-1)/2^l}b^{(k-1)/2^l}}-\sqrt{a^{1-k/2^l}b^{k/2^l}}\right)^2\\ &=a^{1-(k-1)/2^l}b^{(k-1)/2^l}+a^{1-k/2^l}b^{k/2^l}\\ &\quad-2a^{1-(2k-1)/2^{l+1}}b^{(2k-1)/2^{l+1}}.\end{aligned}\tag{12}$$

The following multiple-term refinement of Young inequality has been recently proved in [1]. In the next section, we will present a different and simpler proof of it.

Theorem 3. *Let $a,b>0$ and N be any positive integer. Then*

$$\begin{aligned}&(1-\nu)a+\nu b\\ &\quad\geq a^{1-\nu}b^{\nu}+r_0(\nu)\left(\sqrt{a}-\sqrt{b}\right)^2\\ &\qquad+\sum_{l=1}^{N-1}r_l(\nu)\sum_{k=1}^{2^l}g_{l,k}(a,b)\,\chi_{((k-1)/2^l,k/2^l)}(\nu),\end{aligned}\tag{13}$$

for $0\leq\nu\leq 1$.

The above is a simplified expression of the original one in [1] which is written in rather complicated notation, but they are essentially identical. Note that the first inequalities of (3) and (4) are obtained from (13) with $N=1$ and 2, respectively.

The object of this paper is to show

(1) a reverse of (13) which generalizes the second inequalities of (3) and (4),

(2) multiple-term refinements of Young inequality involving Kantorovich constants which generalize (7) and (9),

(3) operator inequalities related to Young inequality.

2. Multiple-Term Reverse of Young Inequality

From now on, we will fix $a,b>0$ and use the following functions:

$$\begin{aligned}\mu(\nu)&=(1-\nu)a+\nu b-a^{1-\nu}b^{\nu},\\ \alpha_N(\nu)&=\sum_{l=0}^{N-1}r_l(\nu)\sum_{k=1}^{2^l}g_{l,k}(a,b)\,\chi_{((k-1)/2^l,k/2^l)}(\nu),\\ \beta_N(\nu)&=\left(\sqrt{a}-\sqrt{b}\right)^2\\ &\quad-\sum_{l=0}^{N-1}r_l(\nu)\sum_{k=1}^{2^l}g_{l,k}(b,a)\,\chi_{((k-1)/2^l,k/2^l)}(\nu),\end{aligned}\tag{14}$$

for $\nu\in[0,1]$ and an integer $N>0$. As we will see (Lemma 4), $r_l(k/2^l)=0$ for any integer k with $0\leq k\leq 2^l$. Thus the interval of the characteristic function in $\alpha_N(\nu)$ or $\beta_N(\nu)$ can include boundary points. For example, $\chi_{((k-1)/2^l,k/2^l)}$ may be replaced by $\chi_{((k-1)/2^l,k/2^l]}$ or $\chi_{[(k-1)/2^l,k/2^l]}$.

We can express $r_0(\nu)$ and $r_1(\nu)$ as multipart functions as follows:

$$r_0(\nu)=\begin{cases}\nu, & 0\leq\nu\leq\dfrac{1}{2}\\ 1-\nu, & \dfrac{1}{2}<\nu\leq 1,\end{cases}$$

$$r_1(v) = \begin{cases} 2v, & 0 \le v \le \frac{1}{4} \\ 1-2v, & \frac{1}{4} < v \le \frac{1}{2} \\ 2v-1, & \frac{1}{2} < v \le \frac{3}{4} \\ 2-2v, & \frac{3}{4} < v \le 1. \end{cases} \tag{15}$$

For any $l \ge 0$, we can formulate $r_l(v)$ explicitly.

Lemma 4. *Let $l \ge 0$ and $1 \le k \le 2^l$ be integers. If $(k-1)/2^l \le v \le k/2^l$, then*

$$r_l(v) = \begin{cases} 2^l v - k + 1, & \frac{k-1}{2^l} \le v \le \frac{2k-1}{2^{l+1}} \\ k - 2^l v, & \frac{2k-1}{2^{l+1}} < v \le \frac{k}{2^l}. \end{cases} \tag{16}$$

Proof. We prove it by induction on l. The case $l = 0$ is obvious. Assume that $(k-1)/2^{l+1} \le v \le (k-1)/2^{l+1} + 1/2^{l+2}$. If $k = 2m-1$ is odd, then $(m-1)/2^l \le v \le (m-1)/2^l + 1/2^{l+2} < (2m-1)/2^{l+1}$ and $r_l(v) = 2^l v - m + 1$ by induction. Since $v \le (2k-1)/2^{l+2}$, one has

$$\begin{aligned} r_l(v) &= 2^l v - \frac{k-1}{2} \le \frac{2k-1}{4} - \frac{k-1}{2} = \frac{1}{4}, \\ r_{l+1}(v) &= \min\{2r_l(v), 1-2r_l(v)\} = 2r_l(v) \\ &= 2^{l+1}v - k + 1. \end{aligned} \tag{17}$$

If $k = 2m$ is even, then $(2m-1)/2^{l+1} \le v \le m/2^l - 1/2^{l+2} < m/2^l$ and $r_l(v) = m - 2^l v$ by induction. Since $v \le (2k-1)/2^{l+2}$, we have

$$\begin{aligned} r_l(v) &= \frac{k}{2} - 2^l v \ge \frac{k}{2} - \frac{2k-1}{4} = \frac{1}{4}, \\ r_{l+1}(v) &= 1 - 2r_l(v) = 2^{l+1}v - k + 1. \end{aligned} \tag{18}$$

Using the same argument, we can show that if $(k-1)/2^{l+1} + 1/2^{l+2} < v \le k/2^{l+1}$, then $r_{l+1}(v) = k - 2^{l+1}v$. We omit the detailed proof. □

Lemma 5. *For a positive integer N, $\alpha_N(v)$ is the linear interpolation of $\mu(v)$ at $v = k/2^N$ for $k = 0, 1, \ldots, 2^N$.*

Proof. Since $r_l(v)$ is a line segment on each interval $[(k-1)/2^{l+1}, k/2^{l+1}]$ for $1 \le k \le 2^{l+1}$, $\alpha_N(v)$ is a line segment on $[(k-1)/2^N, k/2^N]$ for $1 \le k \le 2^N$. Thus it suffices to show

$$\alpha_N\left(\frac{k}{2^N}\right) = \mu\left(\frac{k}{2^N}\right), \quad 1 \le k < 2^N. \tag{19}$$

Note that since $\alpha_N(v) = \mu(v) = 0$ at $v = 0, 1$, (19) holds for $k = 0, 2^N$. We will prove (19) by induction on N. Since $\alpha_1(v) = r_0(v)(\sqrt{a} - \sqrt{b})^2$, one has

$$\alpha_1\left(\frac{1}{2}\right) = \frac{1}{2}\left(\sqrt{a} - \sqrt{b}\right)^2 = \mu\left(\frac{1}{2}\right). \tag{20}$$

Assume that (19) holds and $1 \le k < 2^{N+1}$. If $k = 2m$ is even, then

$$\begin{aligned} \alpha_{N+1}\left(\frac{k}{2^{N+1}}\right) &= \alpha_{N+1}\left(\frac{m}{2^N}\right) \\ &= \alpha_N\left(\frac{m}{2^N}\right) \\ &\quad + r_N\left(\frac{m}{2^N}\right)\sum_{k=1}^{2^N} g_{N,k}(a,b)\,\chi_{((k-1)/2^N, k/2^N)}\left(\frac{m}{2^N}\right) \\ &= \alpha_N\left(\frac{m}{2^N}\right), \quad \left(\text{since } r_N\left(\frac{m}{2^N}\right) = 0\right) \\ &= \mu\left(\frac{k}{2^{N+1}}\right) \quad (\text{by induction}). \end{aligned} \tag{21}$$

If $k = 2m-1$ is odd, then

$$\begin{aligned} \alpha_{N+1}\left(\frac{k}{2^{N+1}}\right) &= \alpha_N\left(\frac{m}{2^N} - \frac{1}{2^{N+1}}\right) \\ &\quad + r_N\left(\frac{m}{2^N} - \frac{1}{2^{N+1}}\right) g_{N,m}(a,b). \end{aligned} \tag{22}$$

Since $\alpha_N(v)$ is the line segment joining $((m-1)/2^N, \mu((m-1)/2^N))$ and $(m/2^N, \mu(m/2^N))$ on $[(m-1)/2^N, m/2^N]$, we have

$$\alpha_N\left(\frac{m}{2^N} - \frac{1}{2^{N+1}}\right) = \frac{1}{2}\mu\left(\frac{m-1}{2^N}\right) + \frac{1}{2}\mu\left(\frac{m}{2^N}\right). \tag{23}$$

By Lemma 4, $r_N(m/2^N - 1/2^{N+1}) = 1/2$. Noting that $a^{1-v}b^v = (1-v)a + vb - \mu(v)$, we can write $g_{N,m}(a,b)$ by

$$\begin{aligned} g_{N,m}(a,b) &= a^{1-(m-1)/2^N} b^{(m-1)/2^N} + a^{1-m/2^N} b^{m/2^N} \\ &\quad - 2a^{1-(2m-1)/2^{N+1}} b^{(2m-1)/2^{N+1}} \\ &= -\mu\left(\frac{m-1}{2^N}\right) - \mu\left(\frac{m}{2^N}\right) \\ &\quad + 2\mu\left(\frac{2m-1}{2^{N+1}}\right). \end{aligned} \tag{24}$$

Thus, from (22), we deduce that

$$\alpha_{N+1}\left(\frac{k}{2^{N+1}}\right) = \mu\left(\frac{2m-1}{2^{N+1}}\right) = \mu\left(\frac{k}{2^{N+1}}\right). \tag{25}$$

□

Remark 6. Since $\mu(v)$ is concave, $\mu(v) \ge \alpha_N(v)$ by Lemma 5, which proves Theorem 3 in a much simpler way, where the original proof is done by mathematical induction on N.

Lemma 7. *Let $\lambda(v)$ be the reflection of $\mu(v)$ about the point $(1/2, \mu(1/2))$; that is,*

$$\begin{aligned} \lambda(v) &= 2\mu\left(\frac{1}{2}\right) - \mu(1-v), \\ &= \left(\sqrt{a} - \sqrt{b}\right)^2 - va - (1-v)b + a^v b^{1-v}, \\ &= (1-v)a + vb + a^v b^{1-v} - 2\sqrt{ab}. \end{aligned} \tag{26}$$

Then each of the following is true.

(1) $\lambda(v) \geq \mu(v)$ *for all* $0 \leq v \leq 1$.

(2) $\beta_N(v)$ *is the linear interpolation of* $\lambda(v)$ *at* $v = k/2^N$ *at* $k = 0, \ldots, 2^N$.

Proof. Let $f(v) = \lambda(v) - \mu(v)$. Since

$$f'(v) = a^v b^{1-v}\left(1 - \left(\frac{a}{b}\right)^{1-2v}\right)\ln\frac{a}{b}, \tag{27}$$

one derives that $f'(v) < 0$ for $v \in (0, 1/2)$, $f'(v) > 0$ for $v \in (1/2, 1)$, and $f'(1/2) = 0$. Thus we have $\lambda(v) - \mu(v) \geq \lambda(1/2) - \mu(1/2) = 0$ for all $0 \leq v \leq 1$. So part (1) is proved.

For the second part, it suffices to show that $\beta_N(v)$ is the reflection of $\alpha_N(v)$ about the point $(1/2, \alpha_N(1/2))$; that is,

$$2\alpha_N\left(\frac{1}{2}\right) - \alpha_N(1-v) = \beta_N(v). \tag{28}$$

Noting that $r_l(v) = r_l(1-v)$ and $g_{l,2^l-k+1}(a,b) = g_{l,k}(b,a)$, one gets that

$$\begin{aligned}
&2\alpha_N\left(\frac{1}{2}\right) - \alpha_N(1-v)\\
&\quad = 2\mu\left(\frac{1}{2}\right)\\
&\qquad - \sum_{l=0}^{N-1} r_l(v)\sum_{k=1}^{2^l} g_{l,k}(a,b)\,\chi_{((k-1)/2^l,k/2^l)}(1-v)\\
&\quad = \left(\sqrt{a}-\sqrt{b}\right)^2\\
&\qquad - \sum_{l=0}^{N-1} r_l(v)\sum_{k=1}^{2^l} g_{l,k}(a,b)\,\chi_{(1-k/2^l,1-(k-1)/2^l)}(v)\\
&\quad = \left(\sqrt{a}-\sqrt{b}\right)^2\\
&\qquad - \sum_{l=0}^{N-1} r_l(v)\sum_{k=1}^{2^l} g_{l,2^l-k+1}(a,b)\,\chi_{((k-1)/2^l,k/2^l)}(v)\\
&\quad = \beta_N(v).
\end{aligned} \tag{29}$$

□

Theorem 8. *For any integer* $N \geq 1$ *and* $0 \leq v \leq 1$, *one has*

$$\alpha_N(v) \leq \mu(v) \leq \beta_N(v). \tag{30}$$

That is,

$$\begin{aligned}
&(1-v)a + vb\\
&\quad \geq a^{1-v}b^v + \sum_{l=0}^{N-1} r_l(v)\sum_{k=1}^{2^l} g_{l,k}(a,b)\,\chi_{((k-1)/2^l,k/2^l)}(v)\\
&\quad = a^{1-v}b^v + r_0(v)\left(\sqrt{a}-\sqrt{b}\right)^2\\
&\qquad + \sum_{l=1}^{N-1} r_l(v)\sum_{k=1}^{2^l} g_{l,k}(a,b)\,\chi_{((k-1)/2^l,k/2^l)}(v),\\
&(1-v)a + vb\\
&\quad \leq a^{1-v}b^v + \left(\sqrt{a}-\sqrt{b}\right)^2\\
&\qquad - \sum_{l=0}^{N-1} r_l(v)\sum_{k=1}^{2^l} g_{l,k}(b,a)\,\chi_{((k-1)/2^l,k/2^l)}(v)\\
&\quad = a^{1-v}b^v + R_0(v)\left(\sqrt{a}-\sqrt{b}\right)^2\\
&\qquad - \sum_{l=1}^{N-1} r_l(v)\sum_{k=1}^{2^l} g_{l,k}(b,a)\,\chi_{((k-1)/2^l,k/2^l)}(v).
\end{aligned} \tag{31}$$

Proof. As mentioned in Remark 6, $\alpha_N(v) \leq \mu(v)$ follows from the concavity of $\mu(v)$. Since $\mu(v) \leq \lambda(v)$ and $\beta_N(v)$ is the linear interpolation of $\lambda(v)$ which is convex on $[0,1]$, $\mu(v) \leq \lambda(v) \leq \beta_N(v)$. □

Note that the second inequalities of (3) and (4) can be obtained from $\mu(v) \leq \beta_N(v)$ with $N = 1$ and 2, respectively. Moreover, (6) follows from the following result with $N = 2$.

Corollary 9. *For any integer* $N \geq 1$ *and* $0 \leq v \leq 1$, *one has*

$$\begin{aligned}
&((1-v)a + vb)^2\\
&\quad \geq \left(a^{1-v}b^v\right)^2 + r_0^2(v)(a-b)^2\\
&\qquad + \sum_{l=1}^{N-1} r_l(v)\sum_{k=1}^{2^l} g_{l,k}\left(a^2,b^2\right)\chi_{((k-1)/2^l,k/2^l)}(v),\\
&((1-v)a + vb)^2\\
&\quad \leq \left(a^{1-v}b^v\right)^2 + R_0^2(v)(a-b)^2\\
&\qquad - \sum_{l=1}^{N-1} r_l(v)\sum_{k=1}^{2^l} g_{l,k}\left(b^2,a^2\right)\chi_{((k-1)/2^l,k/2^l)}(v).
\end{aligned} \tag{32}$$

Proof. Replacing a and b by their squares in Theorem 8, we obtain that

$$\begin{aligned}
&(1-v)a^2 + vb^2\\
&\quad \geq \left(a^{1-v}b^v\right)^2 + r_0(v)(a-b)^2\\
&\qquad + \sum_{l=1}^{N-1} r_l(v)\sum_{k=1}^{2^l} g_{l,k}\left(a^2,b^2\right)\chi_{((k-1)/2^l,k/2^l)}(v),\\
&(1-v)a^2 + vb^2\\
&\quad \leq \left(a^{1-v}b^v\right)^2 + R_0(v)(a-b)^2\\
&\qquad - \sum_{l=1}^{N-1} r_l(v)\sum_{k=1}^{2^l} g_{l,k}\left(b^2,a^2\right)\chi_{((k-1)/2^l,k/2^l)}(v).
\end{aligned} \tag{33}$$

Since $r_0(v) - r_0^2(v) = R_0(v) - R_0^2(v) = v(1-v)$ for all v, we derive from the above that

$$\begin{aligned}
&(1-v)a^2 + vb^2 \\
&\quad \geq v(1-v)(a-b)^2 + \left(a^{1-v}b^v\right)^2 + r_0^2(v)(a-b)^2 \\
&\qquad + \sum_{l=1}^{N-1} r_l(v) \sum_{k=1}^{2^l} g_{l,k}\left(a^2, b^2\right) \chi_{((k-1)/2^l, k/2^l)}(v), \\
&(1-v)a^2 + vb^2 \\
&\quad \leq v(1-v)(a-b)^2 + \left(a^{1-v}b^v\right)^2 + R_0^2(v)(a-b)^2 \\
&\qquad - \sum_{l=1}^{N-1} r_l(v) \sum_{k=1}^{2^l} g_{l,k}\left(b^2, a^2\right) \chi_{((k-1)/2^l, k/2^l)}(v).
\end{aligned} \tag{34}$$

Hence (32) follows from the identity

$$\begin{aligned}
((1-v)a + vb)^2 &= (1-v)a^2 + vb^2 \\
&\quad - v(1-v)(a-b)^2.
\end{aligned} \tag{35}$$

□

3. Young Inequalities Involving Kantorovich Constants

In this section, we will discuss multiple-term improvements of Young inequality involving Kantorovich constants. For a nonnegative integer n, we define $K_n(a,b)$ by

$$K_n(a,b) = \frac{\left(a^{1/2^n} + b^{1/2^n}\right)^2}{4(ab)^{1/2^n}}. \tag{36}$$

Lemma 10. *For $0 \leq v \leq 1$, one has*

$$(1-v)a + vb \geq K_0(a,b)^{r_0(v)} a^{1-v} b^v. \tag{37}$$

Proof. Replacing $a^{-1}b$ by t, the inequality is equivalent to

$$1 - v + vt \geq K_0(1,t)^{r_0(v)} t^v, \tag{38}$$

for $t > 0$. Taking the natural logarithm, it suffices to show that

$$f(t) \equiv \ln(1 - v + vt) - r_0(v) \ln K_0(1,t) - v \ln t \geq 0. \tag{39}$$

A direct computation shows that

$$\begin{aligned}
&(t+1)(1-v+vt) f'(t) \\
&\quad = \left(1 - t^{-1}\right)\left(v(1-v)(t+1) - r_0(v)(1-v+vt)\right) \\
&\quad = \begin{cases} v(1-2v)(t-1), & 0 \leq v \leq \dfrac{1}{2} \\ (1-v)(2v-1)\left(1-t^{-1}\right), & \dfrac{1}{2} < v \leq 1. \end{cases}
\end{aligned} \tag{40}$$

Thus $f(t) \geq f(1) = 0$ for any $t > 0$. □

Lemma 11. *For a positive integer N and $0 \leq v \leq 1$, define $s_N(t)$ by*

$$\begin{aligned}
s_N(t) &= 1 - v + vt - \sum_{l=0}^{N-1} r_l(v) \\
&\quad \cdot \sum_{k=1}^{2^l} \left(\sqrt{t^{(k-1)/2^l}} - \sqrt{t^{k/2^l}}\right)^2 \chi_{((k-1)/2^l, k/2^l)}(v),
\end{aligned} \tag{41}$$

for $t > 0$. Then one has

$$\begin{aligned}
&s_N(t) \\
&\quad = \sum_{k=1}^{2^N} \left(\left(k - 2^N v\right) t^{(k-1)/2^N} + \left(2^N v - k + 1\right) t^{k/2^N}\right) \\
&\quad \cdot \chi_{((k-1)/2^N, k/2^N]}(v).
\end{aligned} \tag{42}$$

Proof. As mentioned in the previous section, since $r_l(k/2^l) = 0$ for $0 \leq k \leq 2^l$, $\chi_{((k-1)/2^l, k/2^l)}$ in the definition of $s_N(t)$ may be replaced by $\chi_{((k-1)/2^l, k/2^l]}$. We prove (42) by induction on N. We have

$$\begin{aligned}
s_1(t) &= 1 - v + vt - r_0(v)\left(1 - \sqrt{t}\right)^2 \\
&= \left(1 - 2v + 2v\sqrt{t}\right)\chi_{(0,1/2]} \\
&\quad + \left((2-2v)\sqrt{t} + (2v-1)t\right)\chi_{(1/2,1]}.
\end{aligned} \tag{43}$$

Suppose that (42) holds. Then

$$\begin{aligned}
&s_{N+1}(t) = s_N(t) - r_N(v) \sum_{k=1}^{2^N} \left(\sqrt{t^{(k-1)/2^N}} - \sqrt{t^{k/2^N}}\right)^2 \\
&\quad \cdot \chi_{((k-1)/2^N, k/2^N]}(v) = \sum_{k=1}^{2^N} l \left(\left(k - 2^N v\right) t^{(k-1)/2^N} \right. \\
&\quad + \left(2^N v - k + 1\right) t^{k/2^N} \\
&\quad \left. - r_N(v)\left(\sqrt{t^{(k-1)/2^N}} - \sqrt{t^{k/2^N}}\right)^2\right) \\
&\quad \cdot \chi_{((k-1)/2^N, k/2^N]}(v) \\
&\quad = \sum_{k=1}^{2^N} \left(\left(2k - 2^{N+1} v - 1\right) t^{(k-1)/2^N} \right. \\
&\quad \left. + \left(2^{N+1} v - 2k + 2\right) t^{(2k-1)/2^{N+1}}\right) \\
&\quad \cdot \chi_{((k-1)/2^N, (2k-1)/2^{N+1}]}(v) \\
&\quad + \sum_{k=1}^{2^N} \left(\left(2k - 2^{N+1} v\right) t^{(2k-1)/2^{N+1}} \right. \\
&\quad \left. + \left(2^{N+1} v - 2k + 1\right) t^{k/2^N}\right) \chi_{((2k-1)/2^{N+1}, k/2^N]}(v).
\end{aligned} \tag{44}$$

Replacing $2k-1$ in the first summation and $2k$ in the second summation, respectively, by m, one has

$$\begin{aligned} s_{N+1}(t) &= \sum_{\substack{m=1\\ m,\text{odd}}}^{2^{N+1}-1} \left(\left(m-2^{N+1}v\right)t^{(m-1)/2^{N+1}} \right.\\ &\quad \left. + \left(2^{N+1}v-m+1\right)t^{m/2^{N+1}}\right)\chi_{((m-1)/2^{N+1},m/2^{N+1}]}(v)\\ &\quad + \sum_{\substack{m=2\\ m,\text{even}}}^{2^{N+1}} \left(\left(m-2^{N+1}v\right)t^{(m-1)/2^{N+1}} \right.\\ &\quad \left. + \left(2^{N+1}v-m+1\right)t^{m/2^{N+1}}\right)\chi_{((m-1)/2^{N+1},m/2^{N+1}]}(v)\\ &= \sum_{k=1}^{2^{N+1}} \left(\left(k-2^{N+1}v\right)t^{(k-1)/2^{N+1}} \right.\\ &\quad \left. + \left(2^{N+1}v-k+1\right)t^{k/2^{N+1}}\right)\chi_{((k-1)/2^{N+1},k/2^{N+1}]}(v). \end{aligned} \tag{45}$$

Thus (42) holds for all positive integers N. □

The following shows a multiple-term refinement of Young inequality involving Kantorovich constants.

Theorem 12. *For $a,b>0$, $0\le v\le 1$, and $N\in\mathbb{N}$, one has*

$$\begin{aligned} &(1-v)a+vb\\ &\quad \ge K_N(a,b)^{r_N(v)}a^{1-v}b^v\\ &\qquad + \sum_{l=0}^{N-1} r_l(v)\sum_{k=1}^{2^l} g_{l,k}(a,b)\,\chi_{((k-1)/2^l,k/2^l)}(v). \end{aligned} \tag{46}$$

Proof. Putting $t=a^{-1}b$, (46) can be rewritten as

$$\begin{aligned} 1-v+vt &\ge K_N(1,t)^{r_N(v)}t^v + \sum_{l=0}^{N-1} r_l(v)\\ &\quad \cdot \sum_{k=1}^{2^l}\left(t^{(k-1)/2^l}+t^{k/2^l}-2t^{(2k-1)/2^{l+1}}\right)\chi_{((k-1)/2^l,k/2^l)}(v). \end{aligned} \tag{47}$$

By Lemma 11, the above can be expressed by

$$\begin{aligned} &\sum_{k=1}^{2^N}\left(\left(k-2^Nv\right)t^{(k-1)/2^N}+\left(2^Nv-k+1\right)t^{k/2^N}\right)\\ &\quad \cdot \chi_{((k-1)/2^N,k/2^N]}(v) \ge K_N(1,t)^{r_N(v)}t^v. \end{aligned} \tag{48}$$

Thus it suffices to show that if $(k-1)/2^N<v\le k/2^N$, then

$$\begin{aligned} &\left(k-2^Nv\right)t^{(k-1)/2^N}+\left(2^Nv-k+1\right)t^{k/2^N}\\ &\quad \ge K_N(1,t)^{r_N(v)}t^v, \end{aligned} \tag{49}$$

for $t>0$. Replacing t by t^{2^N} and letting $\mu=2^Nv$, the above is equivalent to

$$(k-\mu)t^{k-1}+(\mu-k+1)t^k \ge K_0(1,t)^{r_N(v)}t^\mu. \tag{50}$$

Let $s=\min\{k-\mu,\mu-k+1\}$. By Lemma 10, one gets that

$$\begin{aligned} &(k-\mu)t^{k-1}+(\mu-k+1)t^k\\ &\quad \ge K_0\left(t^{k-1},t^k\right)^s t^{(k-\mu)(k-1)+(\mu-k+1)k} = K_0(1,t)^s t^\mu. \end{aligned} \tag{51}$$

Since

$$r_N(v)=\begin{cases} 2^Nv-k+1, & \dfrac{k-1}{2^N}\le v\le \dfrac{2k-1}{2^{N+1}}\\ k-2^Nv, & \dfrac{2k-1}{2^{N+1}}<v\le\dfrac{k}{2^N} \end{cases} \tag{52}$$

by Lemma 4, it follows that if $(k-1)/2^N<v\le k/2^N$, then $s=\min\{k-2^Nv,2^Nv-k+1\}=r_N(v)$ and therefore (50) holds. □

Note that (46) can be written as

$$\begin{aligned} (1-v)a+vb &\ge K_N(a,b)^{r_N(v)}a^{1-v}b^v\\ &\quad + r_0(v)\left(\sqrt{a}-\sqrt{b}\right)^2\\ &\quad + \sum_{l=1}^{N-1} r_l\sum_{k=1}^{2^l} g_{l,k}(a,b)\,\chi_{((k-1)/2^l,k/2^l)}(v), \end{aligned} \tag{53}$$

which gives the first inequalities of (7) and (9) with $N=1$ and 2, respectively.

Now we consider a reverse inequality corresponding to Theorem 12. A given inequality of the form $(1-v)a+vb\ge\xi(v,a,b)$ can be utilized to derive its reverse in many cases. For example, replacing v by $1-v$ in

$$(1-v)a+vb \ge a^{1-v}b^v + r_0(v)\left(\sqrt{a}-\sqrt{b}\right)^2, \tag{54}$$

which is the first inequality of (3), we obtain

$$\begin{aligned} (1-v)a+vb &\le a+b-a^vb^{1-v}-r_0(v)\left(\sqrt{a}-\sqrt{b}\right)^2\\ &= 2\sqrt{ab}-a^vb^{1-v}+R_0(v)\left(\sqrt{a}-\sqrt{b}\right)^2. \end{aligned} \tag{55}$$

Since $2\sqrt{ab}\le a^vb^{1-v}+a^{1-v}b^v$, the above implies the second inequality of (3). Similarly, replacing v by $1-v$ in

$$\begin{aligned} (1-v)a+vb &\ge a^{1-v}b^v + r_0(v)\left(\sqrt{a}-\sqrt{b}\right)^2 + r_1(v)\\ &\quad \cdot\left(\left(\sqrt{a}-\sqrt[4]{ab}\right)^2\chi_{(0,1/2)}(v)\right.\\ &\quad \left. + \left(\sqrt{b}-\sqrt[4]{ab}\right)^2\chi_{(1/2,1)}(v)\right), \end{aligned} \tag{56}$$

which is the first inequality of (4), we get

$$\begin{aligned}(1-\nu)a+\nu b &\le a+b-a^{\nu}b^{1-\nu}-r_0(\nu)\left(\sqrt{a}-\sqrt{b}\right)^2\\ &-r_1(\nu)\left(\left(\sqrt{b}-\sqrt[4]{ab}\right)^2\chi_{(0,1/2)}(\nu)\right.\\ &\left.+\left(\sqrt{a}-\sqrt[4]{ab}\right)^2\chi_{(1/2,1)}(\nu)\right)=2\sqrt{ab}-a^{\nu}b^{1-\nu}\\ &+R_0(\nu)\left(\sqrt{a}-\sqrt{b}\right)^2-r_1(\nu)\\ &\cdot\left(\left(\sqrt{a}-\sqrt[4]{ab}\right)^2\chi_{(1/2,1)}(\nu)\right.\\ &\left.+\left(\sqrt{b}-\sqrt[4]{ab}\right)^2\chi_{(0,1/2)}(\nu)\right).\end{aligned} \tag{57}$$

Since $2\sqrt{ab}\le a^{\nu}b^{1-\nu}+a^{1-\nu}b^{\nu}$, the above implies the second inequality of (4). In the same way, the first inequality in Theorem 8 can be used to derive

$$\begin{aligned}&(1-\nu)a+\nu b\\ &\quad\le 2\sqrt{ab}-a^{\nu}b^{1-\nu}+R_0(\nu)\left(\sqrt{a}-\sqrt{b}\right)^2\\ &\qquad-\sum_{l=1}^{N-1}r_l(\nu)\sum_{k=1}^{2^l}g_{l,k}(b,a)\,\chi_{((k-1)/2^l,k/2^l)}(\nu),\end{aligned} \tag{58}$$

which is stronger than the second inequality in the theorem. Based on such an observation, we can show a reverse inequality corresponding to Theorem 12 as follows.

Theorem 13. *For $a,b>0$, $0\le\nu\le1$, and $N\in\mathbb{N}$, one has*

$$\begin{aligned}&(1-\nu)a+\nu b\\ &\quad\le 2\sqrt{ab}-K_N(a,b)^{r_N(\nu)}a^{\nu}b^{1-\nu}\\ &\qquad+R_0(\nu)\left(\sqrt{a}-\sqrt{b}\right)^2\\ &\qquad-\sum_{l=1}^{N-1}r_l(\nu)\sum_{k=1}^{2^l}g_{l,k}(b,a)\,\chi_{((k-1)/2^l,k/2^l)}(\nu)\\ &\quad\le K_N(a,b)^{-r_N(\nu)}a^{1-\nu}b^{\nu}+R_0(\nu)\left(\sqrt{a}-\sqrt{b}\right)^2\\ &\qquad-\sum_{l=1}^{N-1}r_l(\nu)\sum_{k=1}^{2^l}g_{l,k}(b,a)\,\chi_{((k-1)/2^l,k/2^l)}(\nu).\end{aligned} \tag{59}$$

Proof. Replacing ν by $1-\nu$ and k by 2^l-k+1 in (46), we have

$$\begin{aligned}&(1-\nu)a+\nu b\\ &\quad\le a+b-K_N(a,b)^{r_N(\nu)}a^{\nu}b^{1-\nu}\\ &\qquad-\sum_{l=0}^{N-1}r_l(\nu)\sum_{k=1}^{2^l}g_{l,k}(b,a)\,\chi_{((k-1)/2^l,k/2^l)}(\nu)\\ &\quad=a+b-K_N(a,b)^{r_N(\nu)}a^{\nu}b^{1-\nu}\\ &\qquad-r_0(\nu)\left(\sqrt{a}-\sqrt{b}\right)^2\\ &\qquad-\sum_{l=1}^{N-1}r_l(\nu)\sum_{k=1}^{2^l}g_{l,k}(b,a)\,\chi_{((k-1)/2^l,k/2^l)}(\nu)\\ &\quad=2\sqrt{ab}-K_N(a,b)^{r_N(\nu)}a^{\nu}b^{1-\nu}\\ &\qquad+R_0(\nu)\left(\sqrt{a}-\sqrt{b}\right)^2\\ &\qquad-\sum_{l=1}^{N-1}r_l(\nu)\sum_{k=1}^{2^l}g_{l,k}(b,a)\,\chi_{((k-1)/2^l,k/2^l)}(\nu)\\ &\quad\le K_N(a,b)^{-r_N(\nu)}a^{1-\nu}b^{\nu}+R_0(\nu)\left(\sqrt{a}-\sqrt{b}\right)^2\\ &\qquad-\sum_{l=1}^{N-1}r_l(\nu)\sum_{k=1}^{2^l}g_{l,k}(b,a)\,\chi_{((k-1)/2^l,k/2^l)}(\nu),\end{aligned} \tag{60}$$

where the last inequality results from

$$2\sqrt{ab}\le K_N(a,b)^{r_N(\nu)}a^{\nu}b^{1-\nu}+K_N(a,b)^{-r_N(\nu)}a^{1-\nu}b^{\nu}. \tag{61}$$

□

Note that the second inequalities of (7) and (9) follow from the above theorem with $N=1$ and 2, respectively.

4. Operator Inequalities

From now on, we use uppercase letters for invertible positive operators on a Hilbert space and lowercase letters for real numbers. The following notations will be used:

(i) $A\ge B$ ($A>B$) denotes that $A-B$ is a positive (invertible positive) operator.

(ii) $A\ge0$ ($A>0$) denotes that A is a positive (invertible positive) operator.

For $A,B>0$ and $0\le\nu\le1$, the ν-arithmetic and ν-geometric means of A and B are defined, respectively, by

$$\begin{aligned}A\nabla_{\nu}B&=(1-\nu)A+\nu B,\\ A\sharp_{\nu}B&=A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\nu}A^{1/2}.\end{aligned} \tag{62}$$

In the case $\nu=1/2$, we will omit the ν-value in them. For example, $A\nabla B$ denotes $A\nabla_{1/2}B$.

The operator version of (1) is well known as follows:

$$A\nabla_{\nu}B\ge A\sharp_{\nu}B, \tag{63}$$

for A and B positive invertible operators and $0\le\nu\le1$ (see [4, 5] for more matrix Young inequalities). To show operator inequalities corresponding to their scalar versions, we will use the operator monotonicity of continuous functions; that is, if f is a real valued continuous function defined on the spectrum of a self-adjoint operator A, then $f(t)\ge0$ for every t in the spectrum of A implies that $f(A)$ is a positive operator.

From now on, $g_{l,k}(A,B)$ will denote the operator version of $g_{l,k}(a,b)$ defined in Definition 2. That is,

$$g_{l,k}(A,B) = A\sharp_{(k-1)/2^l}B + A\sharp_{k/2^l}B - 2A\sharp_{(2k-1)/2^{l+1}}B, \quad (64)$$

for $A, B > 0$.

Theorem 14. *Let $A, B > 0$ and $0 \le \nu \le 1$. Then*

$$\begin{aligned} A\nabla_\nu B &\ge A\sharp_\nu B + 2r_0(\nu)(A\nabla B - A\sharp B) \\ &\quad + \sum_{l=1}^{N-1} r_l(\nu) \sum_{k=1}^{2^l} g_{l,k}(A,B)\, \chi_{((k-1)/2^l,k/2^l)}(\nu), \\ A\nabla_\nu B &\le A\sharp_\nu B + 2R_0(\nu)(A\nabla B - A\sharp B) \\ &\quad - \sum_{l=1}^{N-1} r_l(\nu) \sum_{k=1}^{2^l} g_{l,k}(B,A)\, \chi_{((k-1)/2^l,k/2^l)}(\nu). \end{aligned} \quad (65)$$

Proof. For any $x > 0$, we have

$$\begin{aligned} &(1-\nu) + \nu x \\ &\quad \ge x^\nu + r_0(\nu)\left(1-\sqrt{x}\right)^2 \\ &\qquad + \sum_{l=1}^{N-1} r_l(\nu) \sum_{k=1}^{2^l} g_{l,k}(1,x)\, \chi_{((k-1)/2^l,k/2^l)}(\nu), \\ &(1-\nu) + \nu x \\ &\quad \le x^\nu + R_0(\nu)\left(1-\sqrt{x}\right)^2 \\ &\qquad - \sum_{l=1}^{N-1} r_l(\nu) \sum_{k=1}^{2^l} g_{l,k}(x,1)\, \chi_{((k-1)/2^l,k/2^l)}(\nu), \end{aligned} \quad (66)$$

by Theorem 8. Thus, for any positive operator X, we have

$$\begin{aligned} &(1-\nu) I + \nu X \\ &\quad \ge X^\nu + r_0(\nu)\left(I + X - 2\sqrt{X}\right) \\ &\qquad + \sum_{l=1}^{N-1} r_l(\nu) \sum_{k=1}^{2^l} g_{l,k}(I,X)\, \chi_{((k-1)/2^l,k/2^l)}(\nu), \\ &(1-\nu) I + \nu X \\ &\quad \le X^\nu + R_0(\nu)\left(I + X - 2\sqrt{X}\right) \\ &\qquad - \sum_{l=1}^{N-1} r_l(\nu) \sum_{k=1}^{2^l} g_{l,k}(X,I)\, \chi_{((k-1)/2^l,k/2^l)}(\nu), \end{aligned} \quad (67)$$

by the operator monotonicity of continuous functions, where I is the identity operator. Note that since $A\sharp_\nu B = B\sharp_{1-\nu}A$, we can express $g_{l,k}(I,X)$ and $g_{l,k}(X,I)$ by

$$\begin{aligned} g_{l,k}(I,X) &= I\sharp_{(k-1)/2^l}X + I\sharp_{k/2^l}X - 2I\sharp_{(2k-1)/2^{l+1}}X \\ &= X^{(k-1)/2^l} + X^{k/2^l} - 2X^{(2k-1)/2^{l+1}}, \\ g_{l,k}(X,I) &= X\sharp_{(k-1)/2^l}I + X\sharp_{k/2^l}I - 2X\sharp_{(2k-1)/2^{l+1}}I \\ &= I\sharp_{1-(k-1)/2^l}X + I\sharp_{1-k/2^l}X \\ &\quad - 2I\sharp_{1-(2k-1)/2^{l+1}}X \\ &= X^{1-(k-1)/2^l} + X^{1-k/2^l} - 2X^{1-(2k-1)/2^{l+1}}. \end{aligned} \quad (68)$$

Letting $X = A^{-1/2}BA^{-1/2}$ and then multiplying all terms by A on both sides, (67) yields (65), where $g_{l,k}(B,A)$ can be obtained as follows:

$$\begin{aligned} &A^{1/2} g_{l,k}\left(A^{-1/2}BA^{-1/2}, I\right) A^{1/2} \\ &\quad = A\sharp_{1-(k-1)/2^l}B + A\sharp_{1-k/2^l}B - 2A\sharp_{1-(2k-1)/2^{l+1}}B \\ &\quad = B\sharp_{(k-1)/2^l}A + B\sharp_{k/2^l}A - 2B\sharp_{(2k-1)/2^{l+1}}A \\ &\quad = g_{l,k}(B,A). \end{aligned} \quad (69)$$

□

The following shows matrix inequalities corresponding to Corollary 9.

Theorem 15. *Let A, B, X be $n \times n$ complex matrices. If A and B are positive semidefinite, then*

$$\begin{aligned} &\|(1-\nu)AX + \nu XB\|_2^2 \\ &\quad \ge \left\|A^{1-\nu}XB^\nu\right\|_2^2 + r_0^2(\nu)\|AX - XB\|_2^2 \\ &\qquad + \sum_{l=1}^{N-1} r_l(\nu) \sum_{k=1}^{2^l} h_{l,k}(A,B;X)\, \chi_{((k-1)/2^l,k/2^l)}(\nu), \\ &\|(1-\nu)AX + \nu XB\|_2^2 \\ &\quad \le \left\|A^{1-\nu}XB^\nu\right\|_2^2 + R_0^2(\nu)\|AX - XB\|_2^2 \\ &\qquad - \sum_{l=1}^{N-1} r_l(\nu) \sum_{k=1}^{2^l} h_{l,2^l-k+1}(A,B;X)\, \chi_{((k-1)/2^l,k/2^l)}(\nu), \end{aligned} \quad (70)$$

where $\|\cdot\|_2$ denotes the Frobenius norm and

$$\begin{aligned} &h_{l,k}(A,B;X) \\ &\quad = \left\|A^{1-(k-1)/2^l}XB^{(k-1)/2^l} - A^{1-k/2^l}XB^{k/2^l}\right\|_2^2. \end{aligned} \quad (71)$$

Proof. Since A and B are positive semidefinite, there exist unitary matrices U and V such that $A = U\,\mathrm{diag}(a_1,\ldots,a_n)U^*$ and $B = V\,\mathrm{diag}(b_1,\ldots,b_n)V^*$, where $\mathrm{diag}(c_1,\ldots,c_n)$ denotes the $n\times n$ diagonal matrix whose ith diagonal entry is c_i. Letting $Y = U^*XV = (y_{ij})_n$, we can show the following directly:

$$\begin{aligned} AX - XB &= U\left(\left(a_i - b_j\right) y_{ij}\right) V^*, \\ (1-\nu)AX + \nu XB &= U\left(\left((1-\nu)a_i + \nu b_j\right) y_{ij}\right) V^*, \end{aligned}$$

$$\begin{aligned} &A^{1-\nu}XB^{\nu} = U\left(a_i^{1-\nu}b_j^{\nu}y_{ij}\right)V^*, \\ &A^{1-(k-1)/2^l}XB^{(k-1)/2^l} - A^{1-k/2^l}XB^{k/2^l} \\ &\quad = U\left(\left(a_i^{1-(k-1)/2^l}b_j^{(k-1)/2^l} - a_i^{1-k/2^l}b_j^{k/2^l}\right)y_{ij}\right)V^*. \end{aligned} \tag{72}$$

Since $\|\cdot\|_2$ is unitarily invariant, one obtains that

$$\begin{aligned} &\|(1-\nu)AX + \nu XB\|_2^2 - \left\|A^{1-\nu}XB^{\nu}\right\|_2^2 \\ &\quad = \sum_{i,j=1}^{n}\left(\left((1-\nu)a_i + \nu b_j\right)\left|y_{ij}\right|\right)^2 \\ &\qquad - \sum_{i,j=1}^{n}\left(a_i^{1-\nu}b_j^{\nu}\left|y_{ij}\right|\right)^2 \\ &\quad = \sum_{i,j=1}^{n}\left(\left((1-\nu)a_i + \nu b_j\right)^2 - \left(a_i^{1-\nu}b_j^{\nu}\right)^2\right)\left|y_{ij}\right|^2. \end{aligned} \tag{73}$$

Thus, by (32), we have

$$\begin{aligned} &\|(1-\nu)AX + \nu XB\|_2^2 \geq \left\|A^{1-\nu}XB^{\nu}\right\|_2^2 + r_0^2(\nu) \\ &\quad \cdot \|AX - XB\|_2^2 + \sum_{l=1}^{N-1} r_l(\nu) \\ &\quad \cdot \sum_{k=1}^{2^l}\sum_{i,j=1}^{n} g_{l,k}\left(a_i^2, b_j^2\right)\left|y_{ij}\right|^2 \chi_{((k-1)/2^l,k/2^l)}(\nu), \\ &\|(1-\nu)AX + \nu XB\|_2^2 \leq \left\|A^{1-\nu}XB^{\nu}\right\|_2^2 + R_0^2(\nu) \\ &\quad \cdot \|AX - XB\|_2^2 - \sum_{l=1}^{N-1} r_l(\nu) \\ &\quad \cdot \sum_{k=1}^{2^l}\sum_{i,j=1}^{n} g_{l,k}\left(b_j^2, a_i^2\right)\left|y_{ij}\right|^2 \chi_{((k-1)/2^l,k/2^l)}(\nu). \end{aligned} \tag{74}$$

Since

$$\begin{aligned} g_{l,k}\left(a_i^2, b_j^2\right) &= \left(a_i^{1-(k-1)/2^l}b_j^{(k-1)/2^l} - a_i^{1-k/2^l}b_j^{k/2^l}\right)^2, \\ g_{l,k}\left(b_j^2, a_i^2\right) &= \left(a_i^{(k-1)/2^l}b_j^{1-(k-1)/2^l} - a_i^{k/2^l}b_j^{1-k/2^l}\right)^2, \end{aligned} \tag{75}$$

we have

$$\begin{aligned} \sum_{i,j=1}^{n} g_{l,k}\left(a_i^2, b_j^2\right)\left|y_{ij}\right|^2 &= h_{l,k}(A, B; X), \\ \sum_{i,j=1}^{n} g_{l,k}\left(b_j^2, a_i^2\right)\left|y_{ij}\right|^2 &= h_{l,2^l-k+1}(A, B; X). \end{aligned} \tag{76}$$

□

Next we consider operator inequalities related to Kantorovich constants.

Theorem 16. *Let $A, B > 0$ and h be a real number with $h \geq 1$. If either $hA \leq B$ or $hB \leq A$, then*

$$\begin{aligned} &A\nabla_\nu B \geq K_N(1,h)^{r_N}(\nu)\, A\sharp_\nu B + r_0(\nu)(A + B - 2A\sharp B) \\ &\quad + \sum_{l=1}^{N-1} r_l(\nu) \\ &\quad \cdot \sum_{k=1}^{2^l}\left(A\sharp_{(k-1)/2^l}B + A\sharp_{k/2^l}B - 2A\sharp_{(2k-1)/2^{l+1}}B\right) \\ &\quad \cdot \chi_{((k-1)/2^l,k/2^l)}(\nu), \\ &A\nabla_\nu B \leq 2A\sharp B - K_N(1,h)^{r_N(\nu)}\, B\sharp_\nu A + R_0(\nu)(A + B \\ &\quad - 2A\sharp B) - \sum_{l=1}^{N-1} r_l(\nu) \\ &\quad \cdot \sum_{k=1}^{2^l}\left(B\sharp_{(k-1)/2^l}A + B\sharp_{k/2^l}A + B\sharp_{(2k-1)/2^{l+1}}A\right) \\ &\quad \cdot \chi_{\chi_{((k-1)/2^l,k/2^l)}}(\nu) \leq K_N(1,h)^{-r_N(\nu)}\, A\sharp_\nu B + R_0(\nu)(A \\ &\quad + B - 2A\sharp B) - \sum_{l=1}^{N-1} r_l(\nu) \\ &\quad \cdot \sum_{k=1}^{2^l}\left(B\sharp_{(k-1)/2^l}A + B\sharp_{k/2^l}A + B\sharp_{(2k-1)/2^{l+1}}A\right) \\ &\quad \cdot \chi_{\chi_{((k-1)/2^l,k/2^l)}}(\nu). \end{aligned} \tag{77}$$

Proof. Letting $t = a^{-1}b$ in Theorem 12, we have

$$\begin{aligned} 1 - \nu + \nu t &\geq K_N(1,t)^{r_N(\nu)}\, t^{\nu} \\ &\quad + \sum_{l=0}^{N-1} r_l(\nu)\sum_{k=1}^{2^l} g_{l,k}(1,t)\,\chi_{((k-1)/2^l,k/2^l)}(\nu), \end{aligned} \tag{78}$$

for any $t > 0$. Since $K_N(1,t)$ is an increasing function on $(1,\infty)$ and $K_N(1,t^{-1}) = K_N(1,t)$ for $t > 0$, $K_N(1,h) \leq K_N(1,t)$ for $t \geq h$ or $t \leq 1/h$. Thus, from (78), we have

$$\begin{aligned} 1 - \nu + \nu t &\geq K_N(1,h)^{r_N(\nu)}\, t^{\nu} \\ &\quad + \sum_{l=0}^{N-1} r_l(\nu)\sum_{k=1}^{2^l} g_{l,k}(1,t)\,\chi_{((k-1)/2^l,k/2^l)}(\nu), \end{aligned} \tag{79}$$

for $t \geq h$ or $t \leq 1/h$. Replacing t by $X = A^{-1/2}BA^{-1/2}$ which satisfies $hI \leq X$ or $X \leq (1/h)I$, we get that

$$\begin{aligned} &(1-\nu)I + \nu X \\ &\quad \geq K_N(1,h)^{r_N(\nu)}\, X^{\nu} \\ &\qquad + \sum_{l=0}^{N-1} r_l(\nu)\sum_{k=1}^{2^l} g_{l,k}(I, X)\,\chi_{((k-1)/2^l,k/2^l)}(\nu). \end{aligned} \tag{80}$$

Multiplying the inequality by $A^{1/2}$ on both sides, we have

$$\begin{aligned} A\nabla_\nu B \geq{}& K_N(1,h)^{r_N(\nu)} A\sharp_\nu B + \sum_{l=0}^{N-1} r_l(\nu) \\ &\cdot \sum_{k=1}^{2^l}\left(A\sharp_{(k-1)/2^l}B + A\sharp_{k/2^l}B - 2A\sharp_{(2k-1)/2^{l+1}}B\right) \\ &\cdot \chi_{((k-1)/2^l,k/2^l)}(\nu). \end{aligned} \tag{81}$$

Since the zeroth term ($l=0$) in the summation is $r_0(\nu)(A+B-2A\sharp B)$, we finish the proof of the first inequality of this theorem.

We can prove the second one in the same way. Letting $t = a^{-1}b$ in Theorem 13, we have

$$\begin{aligned} 1-\nu+\nu t \leq{}& 2\sqrt{t} - K_N(1,t)^{r_N(\nu)} t^{1-\nu} \\ &+ R_0(\nu)\left(t+1-2\sqrt{t}\right) \\ &- \sum_{l=1}^{N-1} r_l(\nu) \sum_{k=1}^{2^l} g_{l,k}(t,1)\, \chi_{((k-1)/2^l,k/2^l)}(\nu) \\ \leq{}& K_N(1,t)^{-r_N(\nu)} t^\nu + R_0(\nu)\left(t+1-2\sqrt{t}\right) \\ &- \sum_{l=1}^{N-1} r_l(\nu) \sum_{k=1}^{2^l} g_{l,k}(t,1)\, \chi_{((k-1)/2^l,k/2^l)}(\nu), \end{aligned} \tag{82}$$

for any $t>0$. In particular, if $t \geq h$ or $t \leq 1/h$, then

$$\begin{aligned} 1-\nu+\nu t \leq{}& 2\sqrt{t} - K_N(1,h)^{r_N(\nu)} t^{1-\nu} \\ &+ R_0(\nu)\left(t+1-2\sqrt{t}\right) \\ &- \sum_{l=1}^{N-1} r_l(\nu) \sum_{k=1}^{2^l} g_{l,k}(t,1)\, \chi_{((k-1)/2^l,k/2^l)}(\nu) \\ \leq{}& K_N(1,h)^{-r_N(\nu)} t^\nu + R_0(\nu)\left(t+1-2\sqrt{t}\right) \\ &- \sum_{l=1}^{N-1} r_l(\nu) \sum_{k=1}^{2^l} g_{l,k}(t,1)\, \chi_{((k-1)/2^l,k/2^l)}(\nu). \end{aligned} \tag{83}$$

Replacing t by $X = A^{-1/2}BA^{-1/2}$, one has

$$\begin{aligned} &(1-\nu)I + \nu X \\ &\quad\leq 2\sqrt{X} - K_N(1,h)^{r_N(\nu)} X^{1-\nu} \\ &\qquad + R_0(\nu)\left(X+I-2\sqrt{X}\right) \\ &\qquad - \sum_{l=1}^{N-1} r_l(\nu) \sum_{k=1}^{2^l} g_{l,k}(X,I)\, \chi_{((k-1)/2^l,k/2^l)}(\nu) \\ &\quad\leq K_N(1,h)^{-r_N(\nu)} X^\nu + R_0(\nu)\left(X+I-2\sqrt{X}\right) \\ &\qquad - \sum_{l=1}^{N-1} r_l(\nu) \sum_{k=1}^{2^l} g_{l,k}(X,I)\, \chi_{((k-1)/2^l,k/2^l)}(\nu). \end{aligned} \tag{84}$$

Finally, multiplying each term by $A^{1/2}$ on both sides, we get that

$$\begin{aligned} A\nabla_\nu B \leq{}& 2A\sharp B - K_N(1,h)^{r_N(\nu)} A\sharp_{1-\nu}B + R_0(\nu)(A+B \\ &- 2A\sharp B) - \sum_{l=1}^{N-1} r_l(\nu) \\ &\cdot \sum_{k=1}^{2^l}\left(A\sharp_{1-(k-1)/2^l}B + A\sharp_{1-k/2^l}B + A\sharp_{1-(2k-1)/2^{l+1}}B\right) \\ &\cdot \chi_{((k-1)/2^l,k/2^l)}(\nu) \leq K_N(1,h)^{-r_N(\nu)} A\sharp_\nu B + R_0(\nu) \\ &\cdot (A+B-2A\sharp B) - \sum_{l=1}^{N-1} r_l(\nu) \\ &\cdot \sum_{k=1}^{2^l}\left(A\sharp_{1-(k-1)/2^l}B + A\sharp_{1-k/2^l}B + A\sharp_{1-(2k-1)/2^{l+1}}B\right) \\ &\cdot \chi_{((k-1)/2^l,k/2^l)}(\nu). \end{aligned} \tag{85}$$

Since $A\sharp_{1-\mu}B = B\sharp_\mu A$, the above shows the second inequality of this theorem. □

In the following, we consider special values of ν.

Corollary 17. *Let $A, B > 0$ and $0 \leq \nu \leq 1$. If $\nu = k_0/2^N$ for some $1 \leq k_0 < 2^N$, then*

$$\begin{aligned} A\nabla_\nu B \geq{}& A\sharp_\nu B + \sum_{l=0}^{N-1} r_l(\nu) \\ &\cdot \sum_{k=1}^{2^l}\left(A\sharp_{(k-1)/2^l}B + A\sharp_{k/2^l}B - 2A\sharp_{(2k-1)/2^{l+1}}B\right) \\ &\cdot \chi_{((k-1)/2^l,k/2^l)}(\nu), \\ A\nabla_\nu B \leq{}& 2A\sharp B - B\sharp_\nu A + R_0(\nu)(A+B-2A\sharp B) \\ &- \sum_{l=1}^{N-1} r_l(\nu) \\ &\cdot \sum_{k=1}^{2^l}\left(B\sharp_{(k-1)/2^l}A + B\sharp_{k/2^l}A + B\sharp_{(2k-1)/2^{l+1}}A\right) \\ &\cdot \chi_{((k-1)/2^l,k/2^l)}(\nu) \leq A\sharp_\nu B + R_0(\nu)(A+B-2A\sharp B) \\ &- \sum_{l=1}^{N-1} r_l(\nu) \\ &\cdot \sum_{k=1}^{2^l}\left(B\sharp_{(k-1)/2^l}A + B\sharp_{k/2^l}A + B\sharp_{(2k-1)/2^{l+1}}A\right) \\ &\cdot \chi_{((k-1)/2^l,k/2^l)}(\nu). \end{aligned} \tag{86}$$

If $v = (2k_0 - 1)/2^{N+1}$ *for some* $1 \le k_0 \le 2^N$, *then*

$$\begin{aligned} A\nabla_v B &\ge \frac{1}{4}\left(A\sharp_{v+2^{-N}}B + A\sharp_{v-2^{-N}}B + 2A\sharp_v B\right) \\ &\quad + \sum_{l=0}^{N-1} r_l(v) \\ &\quad \cdot \sum_{k=1}^{2^l}\left(A\sharp_{(k-1)/2^l}B + A\sharp_{k/2^l}B - 2A\sharp_{(2k-1)/2^{l+1}}B\right) \\ &\quad \cdot \chi_{((k-1)/2^l,k/2^l)}(v), \\ A\nabla_v B &\le 2A\sharp B - \frac{1}{4}\left(B\sharp_{v+2^{-N}}A + B\sharp_{v-2^{-N}}A + 2B\sharp_v A\right) \\ &\quad + R_0(v)(A + B - 2A\sharp B) - \sum_{l=1}^{N-1} r_l(v) \\ &\quad \cdot \sum_{k=1}^{2^l}\left(B\sharp_{(k-1)/2^l}A + B\sharp_{k/2^l}A + B\sharp_{(2k-1)/2^{l+1}}A\right) \\ &\quad \cdot \chi_{((k-1)/2^l,k/2^l)}(v). \end{aligned} \tag{87}$$

Proof. If $v = k_0/2^N$, then $r_N(v) = 0$. Thus from (78) and (82), we have

$$\begin{aligned} 1 - v + vt &\ge t^v + \sum_{l=0}^{N-1} r_l \sum_{k=1}^{2^l} g_{l,k}(1,t)\,\chi_{((k-1)/2^l,k/2^l)}(v), \\ 1 - v + vt &\le 2\sqrt{t} - t^{1-v} + R_0(v)\left(t + 1 - 2\sqrt{t}\right) \\ &\quad - \sum_{l=1}^{N-1} r_l(v) \sum_{k=1}^{2^l} g_{l,k}(t,1)\,\chi_{((k-1)/2^l,k/2^l)}(v) \\ &\le t^v + R_0(v)\left(t + 1 - 2\sqrt{t}\right) \\ &\quad - \sum_{l=1}^{N-1} r_l(v) \sum_{k=1}^{2^l} g_{l,k}(t,1)\,\chi_{((k-1)/2^l,k/2^l)}(v). \end{aligned} \tag{88}$$

Using the same argument as in the proof of Theorem 16, it is easy to derive the desired operator inequalities from the above.

Meanwhile, if $v = (2k_0 - 1)/2^{N+1}$, then $r_N(v) = 1$. Thus, from (78) and the first inequality of (82), we have

$$\begin{aligned} 1 - v + vt &\ge \frac{1}{4}\left(t^{v+1/2^N} + t^{v-1/2^N} + 2t^v\right) \\ &\quad + \sum_{l=0}^{N-1} r_l(v) \sum_{k=1}^{2^l} g_{l,k}(1,t)\,\chi_{((k-1)/2^l,k/2^l)}(v), \\ 1 - v + vt &\le 2\sqrt{t} - \frac{1}{4}\left(t^{1-v+1/2^N} + t^{1-v-1/2^N} + 2t^{1-v}\right) \\ &\quad + R_0(v)\left(t + 1 - 2\sqrt{t}\right) \\ &\quad - \sum_{l=1}^{N-1} r_l(v) \sum_{k=1}^{2^l} g_{l,k}(t,1)\,\chi_{((k-1)/2^l,k/2^l)}(v). \end{aligned} \tag{89}$$

Letting $t = X = A^{-1/2}BA^{-1/2}$, we can obtain the desired operator inequalities. □

Competing Interests

The author declares that there is no competing interests regarding the publication of this paper.

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