

Research Article

End Point Results in Metric Spaces Endowed with a Graph

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We introduce the notion of end point of multivalued mappings in the setting of metric space endowed with a graph and prove some existence results in this context. The mappings are assumed to satisfy certain generalized multivalued almost G -contractive type inequalities. Further, the consequences of the corresponding results in the cases of single-valued mappings are also discussed with examples.

1. Introduction and Mathematical Preliminaries

One of the most famous fixed point theorems is the Banach contraction principle, which initiated a new era of research in fixed point theory due to its immense applicability in major areas of mathematics like numerical analysis and differential/integral equations. This important principle was used by Boyd and Wong [1] to investigate the fixed point results of nonlinear contraction maps.

Study of fixed point results in partially ordered sets has been a very well-motivated area of research because of its ease of compatibility in modelling various problems and in finding new convergence schemes. The first attempt in this direction was carried out by Ran and Reurings [2] where they combined the Banach Contraction Principle and the Knaster-Tarski fixed point theorem. Ran and Reurings considered a class of mappings $f : X \rightarrow X$, with (X, d) as a complete metric space and a partial order \leq . The mappings they considered were continuous, monotone with respect to the partial order \leq . Those mappings also satisfy a Banach contraction inequality for every pair $(x, y) \in X \times X$ such that $x \leq y$. When for some $x_0 \in X$, the inequality $x_0 \leq f(x_0)$ is satisfied, they proved that the Picard sequence $\{f^n(x_0)\}$ would converge to a fixed point of f . Ran and Reurings also combined this

interesting result with the Schauder fixed point theorem and applied it to obtain some existence and uniqueness results to nonlinear matrix equations.

Neito and Rodríguez-López ([3, 4]) extended the results of Ran and Reurings to the functions which were not necessarily continuous. They also applied their results to obtain a theorem on the existence of a unique solution for periodic boundary problems relative to ordinary differential equations.

The additional structure of ordering was introduced in metrizable uniform spaces by Turinici [5]. Later, fixed point studies were attempted in partially ordered metric spaces in a good number of works as, for instance, in [6–9]. Multivalued fixed point results were deduced in such contexts in works like [10–14].

The use of graph was introduced in metric fixed point theory by Jachymski [15]. It is in furtherance of metric fixed point theory in partially ordered spaces in that the partial orders themselves introduce a directed graph in metric spaces. A version of the Banach contraction mapping principle was established in these spaces where contraction inequality needs to be satisfied on the edges of the graph. In fact this is a general feature of the line of research which is that the contractions can be restricted to parts of spaces for the purpose of fixed points to exist. There are several results on fixed points

of functions defined on these structures, some of the important references from these works being [16–20].

In the following we describe the mathematical background materials which are necessary for establishing the results in this papers.

Let (X, d) be a metric space. We consider the following classes of subsets of the metric space X :

$$\begin{aligned} N(X) &= \{A : A \text{ is a non-empty subset of } X\}, \\ B(X) &= \{A : A \text{ is a non-empty bounded subset of } X\}. \end{aligned} \tag{1}$$

For $A, B \in B(X)$, the functions D and δ are defined as follows:

$$\begin{aligned} D(A, B) &= \inf \{d(a, b) : a \in A, b \in B\}, \\ \delta(A, B) &= \sup \{d(a, b) : a \in A, b \in B\}. \end{aligned} \tag{2}$$

If $A = \{a\}$, then we write $D(A, B) = D(a, B)$ and $\delta(A, B) = \delta(a, B)$. Also in addition, if $B = \{b\}$, then $D(A, B) = d(a, b)$ and $\delta(A, B) = d(a, b)$. Obviously, $D(A, B) \leq \delta(A, B)$. For all $A, B, C \in B(X)$, the definition of $\delta(A, B)$ yields the following:

$$\begin{aligned} \delta(A, B) &= \delta(B, A), \\ \delta(A, B) &\leq \delta(A, C) + \delta(C, B), \\ \delta(A, B) &= 0, \quad \text{iff } A = B = \{a\}, \\ \delta(A, A) &= \text{diam } A \end{aligned} \tag{3}$$

(see [21]) δ -distance is not a metric like the Hausdorff distance [22] but shares all the properties of a metric except that it is possible to have $\delta(A, A) \neq 0$.

Definition 1. Let X be a nonempty set and T a self-map on X . By a fixed point of T , we mean an element x of X such that $x = Tx$. The set of all fixed points of T is denoted as $\text{Fix}(T)$.

Definition 2. Let X be a nonempty set and $T : X \rightarrow N(X)$ be multivalued mapping. Then $x \in X$ is called a fixed point of T if $x \in Tx$.

Definition 3. Let X be a nonempty set and $T : X \rightarrow N(X)$ be multivalued mapping. Then $x \in X$ is called an end point of T if $\{x\} = Tx$. The set of all end points of T is denoted as $\text{End}(T)$.

Remark 4. Every end point of a mapping $T : X \rightarrow N(X)$ is a fixed point of the mapping T but the converse is not true. So $\text{End}(T) \subseteq \text{Fix}(T)$.

Fixed point theory for multivalued operators is an important topic of set-valued analysis. There are several works on fixed point theory of set-valued maps which have utilized δ -distance [10–13, 21, 23, 24].

Berinde [25] introduced a new class of self-mappings (usually almost contractions) that satisfy a simple but general contraction condition that includes any strict contraction, Kannan [26] and Zamfirescu [27] mappings, and a large

class of quasi-contractions. Almost contractions and their generalizations were further considered in several works like [12, 13, 28–31]. Our aim in this paper is to deduce some fixed point theorems for certain multivalued almost contractions using δ -distance.

Let X be a nonempty set and $\Delta := \{(x, x) : x \in X\}$. Let G be a directed graph such that its vertex set $V(G)$ coincides with X ; that is, $V(G) = X$ and the edge set $E(G)$ contains all loops; that is, $\Delta \subseteq E(G)$. Assume that G has no parallel edges. We can identify G with the pair $G(V(G), E(G))$. By G^{-1} we denote the conversion of a graph G , that is, the graph obtained from G by reversing the directions of the edges. Thus we have

$$\begin{aligned} V(G^{-1}) &= V(G), \\ E(G^{-1}) &= \{(x, y) \in X \times X : (y, x) \in E(G)\}. \end{aligned} \tag{4}$$

Let \tilde{G} denote the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat \tilde{G} as a directed graph for which the set of its edges is symmetric. Under this convention,

$$\begin{aligned} V(\tilde{G}) &= V(G), \\ E(\tilde{G}) &= E(G) \cup E(G^{-1}). \end{aligned} \tag{5}$$

A graph $S(V(S), E(S))$ is called a subgraph of the graph $G(V(G), E(G))$ if $V(S) \subseteq V(G)$ and $E(S) \subseteq E(G)$.

Definition 5. If x and y are vertices in a graph G , then a path in G from x to y of length m ($m \in \mathbb{N}$) is a sequence $(x_i)_{i=0}^m$ of $m + 1$ vertices such that $x_0 = x$, $x_m = y$, and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, \dots, m$.

A graph G is connected if there is a path between any two vertices. G is weakly connected if \tilde{G} is connected. If G is such that $E(G)$ is symmetric and x is a vertex in G , then the subgraph G_x consisting of all edges and vertices which are contained in some path beginning at x is called the component of G containing x . In this case $V(G_x) = [x]_G$, where $[x]_G$ is the equivalence class of the following relation R defined on $V(G)$ by the rule:

$$yRz \text{ if there is a path in } G \text{ from } y \text{ to } z. \tag{6}$$

Definition 6. Let $T : X \rightarrow X$ be a mapping. Then $\text{Fix}(T)$, $X_T(G)$, $X^T(G)$, and $X_T^T(G)$ are defined as follows:

$$\begin{aligned} \text{Fix}(T) &:= \{x \in X : x = Tx\}, \\ X_T(G) &:= \{x \in X : (x, Tx) \in E(G)\}, \\ X^T(G) &:= \{x \in X : (Tx, x) \in E(G)\}, \\ X_T^T(G) &:= \{x \in X : (x, Tx), (Tx, x) \in E(G)\} \\ &= X_T(G) \cap X^T(G). \end{aligned} \tag{7}$$

Definition 7. Let $T : X \rightarrow N(X)$ be a multivalued mapping. Then $\text{Fix}(T)$, $\text{End}(T)$, $X_T(G)$, $X^T(G)$, and $X_T^T(G)$ are defined as follows:

$$\begin{aligned} \text{Fix}(T) &:= \{x \in X : x \in Tx\}, \\ \text{End}(T) &:= \{x \in X : \{x\} = Tx\}, \\ X_T(G) &:= \{x \in X : (x, u) \in E(G) \text{ for some } u \in Tx\}, \\ X^T(G) &:= \{x \in X : (u, x) \in E(G) \text{ for some } u \in Tx\}, \quad (8) \\ X_T^T(G) &:= \{x \in X : (x, u), (u, x) \in E(G) \text{ for some } u \in Tx\} \\ &= X_T(G) \cap X^T(G). \end{aligned}$$

The essential feature of Jachymski’s work [15] is that the contraction inequality needs to be satisfied only on certain edges of the graph. This opened a new direction in fixed point theory in which a significant amount of works has appeared. Our work is in furtherance of this line of research.

2. Main Results

We say a metric space (X, d) is endowed with a directed graph G , if G is a directed graph such that $V(G) = X$ and $\Delta \subseteq E(G)$.

In the following we define generalized almost G -contraction for single-valued mappings and generalized multivalued almost G -contraction of types (A), (B), and (C). Assume that (X, d) is a metric space, and G is a directed graph such that $V(G) = X$ and $\Delta \subseteq E(G)$.

Definition 8. A mapping $T : X \rightarrow X$ is a generalized almost G -contraction if for all $x, y \in X$ with $(x, y) \in E(G)$

$$\begin{aligned} (1) \quad & (Tx, Ty) \in E(G), \\ (2) \quad & d(Tx, Ty) \leq \alpha \max\{d(x, y), d(x, Tx), d(y, Ty), (d(x, Ty) + d(y, Tx))/2\} + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \end{aligned}$$

where $\alpha \in (0, 1)$ and $L \geq 0$.

Definition 9. A mapping $T : X \rightarrow B(X)$ is a generalized multivalued almost G -contraction of type (A) if for all $x, y \in X$ with $(x, y) \in E(G)$

$$\begin{aligned} (i) \quad & u \in Tx \text{ implies that there exists } v \in Ty \text{ such that } (u, v) \in E(G), \\ (ii) \quad & \delta(Tx, Ty) \leq \alpha \max\{d(x, y), D(x, Tx), D(y, Ty), (D(x, Ty) + D(y, Tx))/2\} + L \min\{D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx)\}, \end{aligned}$$

where $\alpha \in (0, 1)$ and $L \geq 0$.

Definition 10. A mapping $T : X \rightarrow B(X)$ is a generalized multivalued almost G -contraction of type (B) if for all $x, y \in X$ with $(x, y) \in E(G)$

$$(i) \quad v \in Ty \text{ implies that there exists } u \in Tx \text{ such that } (u, v) \in E(G),$$

$$(ii) \quad \delta(Tx, Ty) \leq \alpha \max\{d(x, y), D(x, Tx), D(y, Ty), (D(x, Ty) + D(y, Tx))/2\} + L \min\{D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx)\},$$

where $\alpha \in (0, 1)$ and $L \geq 0$.

Definition 11. A mapping $T : X \rightarrow B(X)$ is a generalized multivalued almost G -contraction of type (C) if for all $x, y \in X$ with $(x, y) \in E(G)$

$$\begin{aligned} (i) \quad & u \in Tx \text{ implies that there exists } v \in Ty \text{ such that } (u, v) \in E(G) \text{ and } v \in Ty \text{ implies that there exists } u \in Tx \text{ such that } (u, v) \in E(G), \\ (ii) \quad & \delta(Tx, Ty) \leq \alpha \max\{d(x, y), D(x, Tx), D(y, Ty), (D(x, Ty) + D(y, Tx))/2\} + L \min\{D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx)\}, \end{aligned}$$

where $\alpha \in (0, 1)$ and $L \geq 0$.

Proposition 12. *If T is generalized multivalued almost G -contraction of type (B) then T is generalized multivalued almost G^{-1} -contraction of type (A).*

Proof. Suppose that T is generalized multivalued almost G -contraction of type (B). Let $x, y \in X$ with $(x, y) \in E(G)$. Then $(y, x) \in E(G^{-1})$.

- (1) By the condition (i) of the Definition 10, $v \in Ty$ implies that there exists $u \in Tx$ such that $(u, v) \in E(G)$. Now $(u, v) \in E(G)$ implies that $(v, u) \in E(G^{-1})$. Therefore, if $x, y \in X$ with $(y, x) \in E(G^{-1})$, then $v \in Ty$ implies that there exists $u \in Tx$ such that $(v, u) \in E(G^{-1})$.
- (2) By the condition (ii) of the Definition 10, for all $x, y \in X$ with $(x, y) \in E(G)$

$$\begin{aligned} \delta(Tx, Ty) &\leq \alpha \max \left\{ d(x, y), D(x, Tx), D(y, Ty), \right. \\ &\quad \left. \frac{D(x, Ty) + D(y, Tx)}{2} \right\} + L \min \{D(x, Tx), \\ &\quad D(y, Ty), D(x, Ty), D(y, Tx)\}, \end{aligned} \quad (9)$$

where $\alpha \in (0, 1)$ and $L \geq 0$.

Since $(x, y) \in E(G)$ implies that $(y, x) \in E(G^{-1})$ and δ and d are symmetric, it follows that for all $x, y \in X$ with $(y, x) \in E(G^{-1})$

$$\begin{aligned} \delta(Ty, Tx) &\leq \alpha \max \left\{ d(y, x), D(y, Ty), D(x, Tx), \right. \\ &\quad \left. \frac{D(y, Tx) + D(x, Ty)}{2} \right\} + L \min \{D(y, Ty), \\ &\quad D(x, Tx), D(y, Tx), D(x, Ty)\}, \end{aligned} \quad (10)$$

where $\alpha \in (0, 1)$ and $L \geq 0$. □

Then it follows by Definition 9 that T is generalized multivalued almost G^{-1} -contraction of type (A).

Proposition 13. *If T is generalized multivalued almost G -contraction of type (C) then T is generalized multivalued almost \bar{G} -contraction of type (A).*

Proof. Suppose that T is generalized multivalued almost G -contraction of type (C). By Definitions 9 and 10 and Proposition 12 it follows that T is both generalized multivalued almost G -contraction of type (A) and generalized multivalued almost G^{-1} -contraction of type (A). Hence T is generalized multivalued almost \bar{G} -contraction of type (A). \square

Definition 14. The triple (X, d, G) is said to be regular if

- (1) for any sequence $\{x_n\}$ in X with $x_n \rightarrow x$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$,
- (2) for any sequence $\{x_n\}$ in X with $x_n \rightarrow x$ and $(x_{n+1}, x_n) \in E(G)$ for all $n \in \mathbb{N}$, $(x, x_n) \in E(G)$ for all $n \in \mathbb{N}$.

Theorem 15. *Let (X, d) be a complete metric space endowed with a directed graph G and $T : X \rightarrow B(X)$ be a generalized multivalued almost G -contraction of type (A). Suppose that the triple (X, d, G) is regular. Then the following statements hold:*

- (1) For any $x \in X_T(G)$, $T|_{[x]_{\bar{G}}}$ has an end point.
- (2) If $X_T(G) \neq \emptyset$ and G is weakly connected, then T has an end point in X .
- (3) If $X^* := \bigcup\{[x]_{\bar{G}} : x \in X_T(G)\}$, then $T|_{X^*}$ has an end point.
- (4) $\text{End}(T) \neq \emptyset$ if and only if $X_T(G) \neq \emptyset$.

Proof. (1) Let $x_0 \in X_T(G)$. By the definition of $X_T(G)$, there exists $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$. Since T is generalized multivalued almost G -contraction of type (A), there exists $x_2 \in Tx_1$ such that $(x_1, x_2) \in E(G)$. Again, by similar logic, there exists $x_3 \in Tx_2$ such that $(x_2, x_3) \in E(G)$. Continuing this process we construct a sequence $\{x_n\}$ in X such that $\forall n \geq 0$

$$\begin{aligned} x_{n+1} &\in Tx_n, \\ (x_n, x_{n+1}) &\in E(G). \end{aligned} \quad (11)$$

Since $(x_n, x_{n+1}) \in E(G)$ and T is generalized multivalued almost G -contraction of type (A), we have, for all $n \geq 0$,

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &\leq \delta(Tx_n, Tx_{n+1}) \leq \alpha \max \left\{ d(x_n, x_{n+1}), \right. \\ &D(x_n, Tx_n), D(x_{n+1}, Tx_{n+1}), \\ &\left. \frac{D(x_n, Tx_{n+1}) + D(x_{n+1}, Tx_n)}{2} \right\} + L \\ &\cdot \min \{D(x_n, Tx_n), D(x_{n+1}, Tx_{n+1}), D(x_n, Tx_{n+1}), \end{aligned}$$

$$\begin{aligned} &D(x_{n+1}, Tx_n)\} \leq \alpha \max \left\{ d(x_n, x_{n+1}), d(x_n, x_{n+1}), \right. \\ &d(x_{n+1}, x_{n+2}), \frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})}{2} \left. \right\} + L \\ &\cdot \min \{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+2}), \\ &d(x_{n+1}, x_{n+1})\} \leq \alpha \max \left\{ d(x_n, x_{n+1}), \right. \\ &d(x_{n+1}, x_{n+2}), \frac{d(x_n, x_{n+2})}{2} \left. \right\}. \end{aligned} \quad (12)$$

Since $d(x_n, x_{n+2})/2 \leq \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}$, it follows that

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &\leq \delta(Tx_n, Tx_{n+1}) \\ &\leq \alpha \max \{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}. \end{aligned} \quad (13)$$

Suppose that $d(x_n, x_{n+1}) < d(x_{n+1}, x_{n+2})$, for some positive integer n . So, $0 < d(x_n, x_{n+1})$.

Then it follows from (13) that

$$0 < d(x_{n+1}, x_{n+2}) \leq \alpha d(x_{n+1}, x_{n+2}), \quad (14)$$

which is a contradiction since $\alpha \in (0, 1)$. Therefore, $d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1})$, for all $n \geq 0$ and hence we have from (13) that

$$d(x_{n+1}, x_{n+2}) \leq \delta(Tx_n, Tx_{n+1}) \leq \alpha d(x_n, x_{n+1}). \quad (15)$$

By repeated application of (15), we have

$$d(x_{n+1}, x_{n+2}) \leq \alpha^{n+1} d(x_0, x_1). \quad (16)$$

For arbitrary $m, n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots \\ &\quad + d(x_{m-1}, x_m) \\ &\leq [\alpha^n + \alpha^{n+1} + \alpha^{n+2} + \dots + \alpha^{m-1}] d(x_0, x_1) \\ &\leq \frac{\alpha^n}{1 - \alpha} d(x_0, x_1) \rightarrow 0, \end{aligned} \quad (17)$$

as $n \rightarrow \infty$ (since $\alpha \in (0, 1)$),

which implies that $\{x_n\}$ is a Cauchy sequence. From the completeness of X , there exists a $z \in X$ such that

$$x_n \rightarrow z, \quad \text{as } n \rightarrow \infty. \quad (18)$$

Since $\{x_n\}$ is a sequence in X such that $x_n \rightarrow z$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, using the regular property of (X, d, G) we have that

$$(x_n, z) \in E(G), \quad \forall n \geq 0. \quad (19)$$

Since $(x_n, z) \in E(G)$ for all $n \geq 0$ and T is generalized multi-valued almost G -contraction of type (A), we have

$$\begin{aligned} \delta(x_{n+1}, Tz) &\leq \delta(Tx_n, Tz) \leq \alpha \max \left\{ d(x_n, z), \right. \\ &D(x_n, Tx_n), D(z, Tz), \left. \frac{D(x_n, Tz) + D(z, Tx_n)}{2} \right\} \\ &+ L \min \{D(x_n, Tx_n), D(z, Tz), D(x_n, Tz), \\ &D(z, Tx_n)\} \leq \alpha \max \left\{ d(x_n, z), d(x_n, x_{n+1}), \right. \\ &D(z, Tz), \left. \frac{D(x_n, Tz) + d(z, x_{n+1})}{2} \right\} + L \\ &\cdot \min \{d(x_n, x_{n+1}), D(z, Tz), D(x_n, Tz), \\ &d(z, x_{n+1})\}. \end{aligned} \tag{20}$$

Taking the limit as $n \rightarrow \infty$ in the above inequality and using (18), we have

$$\delta(z, Tz) \leq \alpha D(z, Tz) \leq \alpha \delta(z, Tz). \tag{21}$$

Since $\alpha \in (0, 1)$, the above inequality implies that $\delta(z, Tz) = 0$ or that $\{z\} = Tz$. Moreover, z is an end point of T . By (19), it is obvious that $(x_0, z) \in E(G)$ and so $z \in [x_0]_{\bar{G}}$. Hence $T|_{[x_0]_{\bar{G}}}$ has an end point.

(2) Let $X_T(G) \neq \emptyset$ and G is weakly connected. Since G is weakly connected, $[x]_{\bar{G}} = X$ for every $x \in X$. Since $X_T(G) \neq \emptyset$, there exists $x_0 \in X_T(G)$. Then $[x_0]_{\bar{G}} = X$. So by (1), T has an end point in X .

(3) Let $X^* := \bigcup \{[x]_{\bar{G}} : x \in X_T(G)\}$. By (1) and (2), $T|_{X^*}$ has an end point.

(4) Let $\text{End}(T) \neq \emptyset$. Then there exists at least one element $x \in \text{End}(T)$. Now $x \in \text{End}(T)$ means $\{x\} = Tx$. So, $x \in Tx$. Now $\Delta \subseteq E(G)$ implies that $(x, x) \in E(G)$. Therefore, we have $x \in Tx$ and $(x, x) \in E(G)$, which implies that $x \in X_T(G)$. Hence $X_T(G) \neq \emptyset$. Conversely suppose that $X_T(G) \neq \emptyset$. Then by (1), $\text{End}(T) \neq \emptyset$. \square

Theorem 16. Let (X, d) be a complete metric space and $T : X \rightarrow B(X)$ be a generalized multivalued almost G -contraction of type (B). Suppose that the triple (X, d, G) is regular. Then the following statements hold:

- (1) For any $x \in X^T(G)$, $T|_{[x]_{\bar{G}}}$ has an end point.
- (2) If $X^T(G) \neq \emptyset$ and G is weakly connected, then T has an end point in X .
- (3) If $X^* := \bigcup \{[x]_{\bar{G}} : x \in X^T(G)\}$, then $T|_{X^*}$ has an end point.
- (4) $\text{End}(T) \neq \emptyset$ if and only if $X^T(G) \neq \emptyset$.

Proof. Let $x \in X^T(G)$. Then there exist $u \in Tx$ such that $(u, x) \in E(G)$. Now $(u, x) \in E(G)$ implies that $(x, u) \in E(G^{-1})$. Since $x \in X$ and $(x, u) \in E(G^{-1})$ for some $u \in Tx$,

we have $x \in X_T(G^{-1})$. Conversely, $y \in X_T(G^{-1})$ implies that $y \in X^T(G)$. So $X^T(G) = X_T(G^{-1})$. Therefore,

- (1) $X^T(G) \neq \emptyset$ implies that $X_T(G^{-1}) \neq \emptyset$,
- (2) G is weakly connected implies that G^{-1} is also weakly connected,
- (3) $[x]_{\bar{G}} = [x]_{\bar{G}^{-1}}$,
- (4) $X^* := \bigcup \{[x]_{\bar{G}} : x \in X^T(G)\} = \bigcup \{[x]_{\bar{G}^{-1}} : x \in X_T(G^{-1})\}$,
- (5) By Proposition 12, $T : X \rightarrow B(X)$ is a generalized multivalued almost G^{-1} -contraction of type (A).

Let $G^{-1} = W$. Then by Theorem 15 we have the required proof. \square

Theorem 17. Let (X, d) be a complete metric space and $T : X \rightarrow B(X)$ be a generalized multivalued almost G -contraction of type (C). Suppose that the triple (X, d, G) is regular. Then the following statements hold:

- (1) For any $x \in X_T^T(G)$, $T|_{[x]_{\bar{G}}}$ has an end point.
- (2) If $X_T^T(G) \neq \emptyset$ and G is weakly connected, then T has an end point in X .
- (3) If $X^* := \bigcup \{[x]_{\bar{G}} : x \in X_T^T(G)\}$, then $T|_{X^*}$ has an end point.
- (4) $\text{End}(T) \neq \emptyset$ if and only if $X_T^T(G) \neq \emptyset$.

Proof. Arguing similarly as in the proof of Theorem 16, we have that $X_T^T(G) = X_T(\bar{G})$. Therefore,

- (1) $X_T^T(G) \neq \emptyset$ implies that $X_T(\bar{G}) \neq \emptyset$,
- (2) G is weakly connected implies that \bar{G} is connected,
- (3) $X^* := \bigcup \{[x]_{\bar{G}} : x \in X_T^T(G)\} = \bigcup \{[x]_{\bar{G}} : x \in X_T(\bar{G})\}$,
- (4) By Proposition 13, $T : X \rightarrow B(X)$ is a generalized multivalued almost \bar{G} -contraction of type (A).

Let $\bar{G} = W$. Then by Theorem 15 we have the required proof. \square

Example 18. Let $X = \{v_1 = (0, 0), v_2 = (-1/2, -1/2), v_3 = (0, -1)\}$ be a subset of R^2 (R is the set of all real numbers) and G be a directed graph with $V(G) = X$ and $E(G) = \{(v_1, v_1), (v_2, v_2), (v_3, v_3), (v_2, v_1), (v_3, v_1)\}$. Let d be metric on X defined as

$$d(x, y) = \max \{|p - u|, |q - v|\}, \tag{22}$$

for $x = (p, q), y = (u, v) \in X$.

Let $T : X \rightarrow B(X)$ be defined as follows:

$$Tx = \begin{cases} \{v_1\}, & \text{if } x = v_1, \\ \{v_1, v_2\}, & \text{if } x = v_3, \\ \{v_1\}, & \text{if } x = v_2. \end{cases} \tag{23}$$

Let $L > 1$ and $\alpha \in (1/2, 1)$.

All the conditions of Theorem 15 are satisfied and it is seen that v_1 is an end point of T .

3. Consequences in Single-Valued Cases

In this section we obtain some consequences of the corresponding results of Section 2 in the cases of single-valued mappings.

Theorem 19. *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a generalized almost G -contraction. Suppose that the triple (X, d, G) is regular. Then following statements hold:*

- (1) For any $x \in X_T(G)$, $T|_{[x]_{\overline{G}}}$ has a fixed point.
- (2) If $X_T(G) \neq \emptyset$ and G is weakly connected, then T has a fixed point in X .
- (3) If $X^* := \bigcup\{[x]_{\overline{G}} : x \in X_T(G)\}$, then $T|_{X^*}$ has a fixed point.
- (4) $\text{Fix}(T) \neq \emptyset$ if and only if $X_T(G) \neq \emptyset$.

Proof. We know that, for every $x \in X$, $\{x\} \in B(X)$. We define a multivalued mapping $S : X \rightarrow B(X)$ as $Sx = \{Tx\}$, for $x \in X$.

- (1) Since $T : X \rightarrow X$ is a generalized almost G -contraction, it follows by Definitions 8 and 9 that S is a generalized multivalued almost G -contraction of type (A).
- (2) $X_T(G) = X_S(G)$. So, $X_T(G) \neq \emptyset$ implies that $X_S(G) \neq \emptyset$.
- (3) $X^* := \bigcup\{[x]_{\overline{G}} : x \in X_T(G)\} = \bigcup\{[x]_{\overline{G}} : x \in X_S(G)\}$.
- (4) Every fixed point of T is an end point of S and every end point of S is a fixed point of T . Hence $\text{Fix}(T) = \text{End}(S)$.

So, all the conditions of Theorem 15 are satisfied and hence by application of the Theorem 15 we have required proof. \square

Theorem 20. *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a generalized almost G -contraction. Suppose that the triple (X, d, G) is regular. Then the following statements hold:*

- (1) For any $x \in X^T(G)$, $T|_{[x]_{\overline{G}}}$ has a fixed point.
- (2) If $X^T(G) \neq \emptyset$ and G is weakly connected, then T has a fixed point in X .
- (3) If $X^* := \bigcup\{[x]_{\overline{G}} : x \in X^T(G)\}$, then $T|_{X^*}$ has a fixed point.
- (4) $\text{Fix}(T) \neq \emptyset$ if and only if $X^T(G) \neq \emptyset$.

Theorem 21. *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a generalized almost G -contraction. Suppose that the triple (X, d, G) is regular. Then the following statements hold:*

- (1) For any $x \in X_T^T(G)$, $T|_{[x]_{\overline{G}}}$ has a fixed point.
- (2) If $X_T^T(G) \neq \emptyset$ and G is weakly connected, then T has a fixed point in X .

(3) If $X^* := \bigcup\{[x]_{\overline{G}} : x \in X_T^T(G)\}$, then $T|_{X^*}$ has a fixed point.

(4) $\text{Fix}(T) \neq \emptyset$ if and only if $X_T^T(G) \neq \emptyset$.

Example 22. Let $X = [0, 1] \cup [a, b]$, where $a, b \in \mathbb{R}$ with $\sqrt{2} < a < b$, and let G be a directed graph with $V(G) = X$ and $E(G) = \{(x, y) : x, y \in [0, 1] \text{ and } x \geq y\} \cup \{(x, y) : x, y \in [a, b] \text{ and } x \leq y\}$. Let d be metric the usual metric on X .

Let $T : X \rightarrow X$ be defined as

$$Tx = \begin{cases} \frac{x}{2}, & \text{if } x \in [0, 1], \\ x + \frac{1}{x} - \frac{1}{b}, & \text{for } x \in [a, b]. \end{cases} \quad (24)$$

Let $\alpha \in (0, 1)$ be such that $1 - 1/b^2 \leq \alpha < 1$ and $L > 1$.

Then all the conditions of Theorem 19 are satisfied and it is seen that the set of fixed points of T is nonempty. Here $\text{Fix}(T) = \{0, b\}$.

Remark 23. The results we obtain here are for functions without any continuity requirement. We assume in our results that (X, d, G) is regular. If we omit this condition in our theorem and, instead, assume the function T to be continuous then also the conclusions of the theorem are valid.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contributions

The authors contributed equally to this work.

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