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Research Article

Different Characterizations of Large Submodules of QTAG-Modules

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A module M over an associative ring R with unity is a $QTAG$-module if every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules. The study of large submodules and its fascinating properties makes the theory of QTAG-modules more interesting. A fully invariant submodule L of M is large in M if $L + B = M$, for every basic submodule B of M. The impetus of these efforts lies in the fact that the rings are almost restriction-free. This motivates us to find the necessary and sufficient conditions for a submodule of a QTAG-module to be large and characterize them. Also, we investigate some properties of large submodules shared by Σ-modules, summable modules, σ-summable modules, and so on.

1. Introduction and Preliminaries

All the rings R considered here are associative with unity and modules M are unital $QTAG$-modules. An element $x \in M$ is uniform, if xR is a nonzero uniform (hence uniserial) module and, for any R-module M with a unique composition series, $d(M)$ denotes its composition length. For a uniform element $x \in M$, $e(x) = d(xR)$ and $H_M(x) = \sup\{d(yR/xR) \mid y \in M,\ x \in yR \text{ and } y \text{ uniform}\}$ are the exponent and height of x in M, respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k. For any arbitrary $x \in M$, $H(x) = k$ if $x \in H_k(M)$ but $x \notin H_{k+1}(M)$. M is h-divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$ and it is h-reduced if it does not contain any h-divisible submodule. In other words it is free from the elements of infinite height.

A submodule N of M is h-pure in M if $N \cap H_k(M) = H_k(N)$, for every integer $k \geq 0$. For a limit ordinal α, $H_\alpha(M) = \bigcap_{\rho<\alpha} H_\rho(M)$, for all ordinals $\rho < \alpha$, and it is α-pure in M if $H_\sigma(N) = H_\sigma(M) \cap N$ for all ordinals $\sigma < \alpha$ and it is an isotype if it is α-pure for every ordinal α. A submodule $B \subseteq M$ is a basic submodule of M, if B is h-pure in M, $B = \bigoplus B_i$, where each B_i is the direct sum of uniserial modules of length i and M/B is h-divisible. For a QTAG-module M, the σth–Ulm invariant of M, $f_M(\sigma)$ is the cardinal number $g(\mathrm{Soc}(H_\sigma(M))/\mathrm{Soc}(H_{\sigma+1}(M)))$ [1]. Several results which hold for TAG-modules also hold good for $QTAG$-modules [2].

A module M is summable if $\mathrm{Soc}(M) = \bigoplus_{\tau<\alpha} S_\alpha$, where S_α is the set of all elements of $H_\alpha(M)$ which are not in $H_{\alpha+1}(M)$, where τ is the length of M. A $QTAG$-module M is called σ-summable if $\mathrm{Soc}(M) = \bigcup_{n<\omega} M_n$, $M_n \subseteq M_{n+1}$ and, for every positive integer n, there is an ordinal α_n such that $M_n \cap H_{\alpha_n}(M) = 0$, $\alpha_n <$ length of M.

For any uniform element $x \in M$, there exist uniform elements $x_1, x_2, \ldots$ such that $xR \supseteq x_1R \supseteq x_2R \supseteq \cdots$ and $d(x_iR/x_{i+1}R) = 1$. Now the Ulm-sequence of x is defined as $U(x) = (H(x), H(x_1), H(x_2), \ldots)$. U sequences are defined as $U(x)$. This is analogous to the Ulm-sequences defined in groups [3]. These sequences are partially ordered because $U(x) \leq U(y)$ if $H(x_i) \leq H(y_i)$ for every i. For the sequence $n = (n_0, n_1, n_2, \ldots)$ of nonnegative, nondecreasing integers we may consider L as the submodule of M generated by the elements x of M for which $U(x) \geq n$. If f is an endomorphism of M, then $H(x) \leq H(f(x))$, and therefore

L is fully invariant. Therefore with every large submodule L of M we may associate a sequence $n(L)$.

2. Some Characterizations of Large Submodules

In this section we study and characterize the properties of fully invariant and large submodules of $QTAG$-modules. We also discuss the properties of large submodules inherited from the containing module.

We start with the facts which are true for any module. For a fully invariant submodule N of a $QTAG$-module M and an endomorphism f of M, it induces an endomorphism $\overline{f}$ of M/N such that $\overline{f}(x+N) = f(x)+N$. On the other hand for the endomorphism $\overline{f}$ of M/N induced by an endomorphism f of M and a fully invariant submodule $K/N \subseteq M/N$, $\overline{f}(x+N) = f(x)+N \in K/N$. That is, $f(x) \in K$ and K is fully invariant in M. For a fully invariant submodule $A \subseteq M = \bigoplus M_i$, $A = \bigoplus (A \cap M_i)$ and each $A \cap M_i$ is fully invariant in M_i.

For any sequence $n = (n_1, n_2, \ldots)$ we define $M(n)$ as the submodule of M, generated by the elements x for which $U(x) \geq n$. This submodule is a large submodule of M. In fact for every large submodule there is a sequence and, for every sequence, there is a large submodule [4].

For a $QTAG$-module M, consider the homomorphism $f : M \to M/M^1$. As $M^1 = \bigcap_{k=0}^{\infty} H_k(M)$, f is height preserving. This implies that $H(x) = H(f(x))$ and $U(x) = U(f(x))$ for all $x \in M$.

We conclude that L/M^1 is large in M/M^1 if and only if L is large in M. In a module M without elements of infinite height, consider a fully invariant submodule K of M, and $x \in \mathrm{Soc}(K)$ such that $n = H(x) \leq H(y)$ for every $y \in \mathrm{Soc}(K)$. Let $z \in \mathrm{Soc}(M)$, such that $H(z) \geq n$. Then there exists an endomorphism f of M such that $f(x) = z$; therefore $z \in K$ and $\mathrm{Soc}(K) = \mathrm{Soc}(H_n(M))$.

Remark 1. For any large submodule L of M, $\mathrm{Soc}(L) = \mathrm{Soc}(H_n(M))$ for some positive integer n.

Lemma 2. *Let N be submodule of M such that $\mathrm{Soc}(H_{n_k}(M)) \subseteq \mathrm{Soc}(H_k(N))$ for $k = 0, 1, 2, \ldots$, where the sequence of positive integers $n_0, n_1, n_2, \ldots$ is monotonically increasing. Then $M = N + B$ for any basic submodule B of M.*

Proof. Let $B = \bigoplus B_i$ be a basic submodule of M and $M = B_1 \oplus \cdots \oplus B_k \oplus (B_k^*, H_k(M))$ [5]. Then

$$\begin{aligned}\mathrm{Soc}(M) &= \mathrm{Soc}\left(B_1 + \cdots + B_k\right) \oplus \mathrm{Soc}\left(H_k(M)\right) \\ &= \mathrm{Soc}\left(B_1 \oplus \cdots \oplus B_{n_k}\right) \oplus \mathrm{Soc}\left(H_k(N)\right).\end{aligned} \tag{1}$$

Now suppose, for every $x \in M$, $e(x) \leq k$ implies that $x \in (B+N)$. Consider $x \in M$ such that $e(x) = k+1$; then there exists $y \in M$ such that $d(xR/yR) = k$. Now $y \in \mathrm{Soc}(M)$ and $y = b+z$, where $b \in B$ and $z \in \mathrm{Soc}(H_k(N))$, ensuring the existence of z' such that $d(z'R/zR) = k$. By the h-purity of B, there exists $b' \in B$, such that $d(b'R/bR) = k$. Now $e(x - b' - z') \leq k$, and thus $x - b' - z' \in B + N$ or $x \in B + N$ implying that $M = B + N$. □

The following remarks are significant to be stated.

Remark 3. Let L be a large submodule of an unbounded $QTAG$-module M without elements of infinite height M and B a proper basic submodule of M. Then

$$\frac{M}{B} = \frac{(B+L)}{B} \cong \frac{L}{(B \cap L)}; \tag{2}$$

therefore L is unbounded. Conversely for an unbounded fully invariant submodule L of M, $H_k(L)$ is fully invariant for all $k \in \mathbb{Z}^+$. As an immediate consequence of Lemma 2, L is a large submodule of M. We can say that the unbounded fully invariant submodules of M are exactly the large submodules of M.

Remark 4. If B_i is the direct sum of uniserial modules of length i and $x (\neq 0) \in B_i$, then

$$U(x) = \left\langle n_0, n_1, \ldots, n_{i-n_0-1}, \ldots, \infty \right\rangle, \tag{3}$$

where $n_0 = H(x)$ and $n_k = n_0 + k$, $0 \leq k \leq i - n_0 - 1$.

Remark 5. Let B_i be the direct sum of uniserial modules of length i and $x, y \in B_i$. Then there exists an endomorphism f of B_i with $f(x) = y$, if and only if $H(x) \leq H(y)$.

Remark 6. Let A be a fully invariant submodule of B_i, a direct sum of uniserial modules of length i. Then $A = H_{n_i}(B_i)$, where $n_i \leq i$. If $A = 0$, if $n_i = i$, and if $A \neq 0$, then $n_i = \min\{H(x),\ x \in A\}$.

Remark 7. If B_i and B_{i+j} are the direct sums of uniserial modules of length i and $i+j$, respectively, and $x \in B_i$, $y \in B_{i+j}$, then

(i) there exists a homomorphism $f : B_i \to B_{i+j}$ such that $f(x) = y$ if and only if $e(x) \geq e(y)$,

(ii) there exists a homomorphism $g : B_{i+j} \to B_i$ such that $g(y) = x$ if and only if $H(x) \geq H(y)$.

Theorem 8. *Let $B = \bigoplus_i B_i$, where each B_i is the direct sum of uniserial modules of length i. Then L is a fully invariant submodule of B if and only if $L = \bigoplus_i H_{n_i}(B_i)$, where $n_i \leq i$, for every $i \in \mathbb{Z}^+$ and $n_i \leq n_{i+j} \leq n_i + j$ for $i, j \in \mathbb{Z}^+$. A fully invariant submodule L is large in B if and only if $L = \bigoplus_i H_{n_i}(B_i)$; the above conditions hold and the sequence $\langle 1 - n_1, 2 - n_2, 3 - n_3, \ldots \rangle$ is unbounded if B is unbounded.*

Proof. Let L be a fully invariant submodule of B. Then

$$L = L \cap B = \bigoplus \left(B_i \cap L\right) = \bigoplus H_{n_i}\left(B_i\right) \tag{4}$$

by the facts mentioned above and Remark 6. Now $n_i \leq i$ for $i \in \mathbb{Z}^+$ and the first condition holds. If $L = 0$, then $H_{n_i}(B_i) = 0$ for every i; therefore $n_i = i$ for every i and the second condition holds. If $L \neq 0$, then there exists a least positive integer k such that $H_{n_k}(B_k) \neq 0$. Then $H_{n_i}(B_i) \neq 0$ for all $i \geq k$, where $B_i \neq 0$. Since $\mathrm{Soc}(B_k) = \mathrm{Soc}(H_{k-1}(B_k)) \subseteq L$, this implies that $\mathrm{Soc}(H_{k-1}(B)) \subseteq \mathrm{Soc}(L)$. Again $\mathrm{Soc}(B_i) = \mathrm{Soc}(H_{k-1}(B_i))$ for $i \geq k$; we have $\mathrm{Soc}(B_i) \subseteq L \cap B_i = H_{n_i}(B_i)$. Now suppose $L \neq 0$ and $B_i \neq 0 \neq B_{i+j}$. If $H_{n_{i+j}}(B_{i+j}) = 0$, then $H_{n_i}(B_i) = 0$ and $n_i = i$, $n_{i+j} = i + j = n_i + j$ and the second condition holds. We assume that $H_{n_{i+j}}(B_{i+j}) \neq 0$. Consider $x \in B_i$ such that $H(x) \geq n_{i+j}$ and $y \in H_{n_{i+j}}(B_{i+j})$ such that $H(y) = n_{i+j}$. Now, by Remark 7, there exists an endomorphism g of B mapping y onto x. Hence $x \in L$ and $H_{n_{i+j}}(B_i) \subseteq L \cap B_i = H_{n_i}(B_i)$; thus $n_i \leq n_{i+j}$.

Now suppose $H_{n_i}(B_i) = 0$. Then $n_i = i$ so $n_{i+j} \leq i + j = n_i + j$. If $H_{n_i}(B_i) \neq 0$ and $y \in B_{i+j}$ such that $H(y) \geq n_i + j$, we may choose $x \in B_i$ such that $H(x) = n_i$. Then $e(x) = i - n_i$ and $e(y) \leq i + j - (n_i + j) = i - n_i$. By Remark 7, there exists an endomorphism f of B with $f(x) = y$. Thus $y \in L$ and we have $H_{n_{i+j}}(B_{i+j}) \subseteq L \cap B_{i+j} = H_{n_{i+j}}(B_{i+j})$; therefore $n_{i+j} \leq n_i + j$.

If $B_i \neq 0 \neq B_{i+j}$, then $n_i \leq n_{i+j} \leq n_i + j$ but if $B_i = 0$, we may define n_i so that this inequality holds for all i. Thus all fully invariant submodules of B are the direct sums of $H_{n_i}(B_i)$. If L is a large submodule of B and B is unbounded, then, by Lemma 2, L is also unbounded. Therefore $\langle 1 - n_1, 2 - n_2, 3 - n_3, \ldots\rangle$ must be unbounded.

For the converse, suppose $L = \bigoplus H_{n_i}(B_i)$, where $n_i \leq i$ for all $i \in \mathbb{Z}^+$ and $n_i \leq n_{i+j} \leq n_i + j$ for all $i, j \in \mathbb{Z}^+$. To establish the full invariance of L, we consider any $i \in \mathbb{Z}^+$ and $x \in H_{n_i}(B_i)$. We have to show that for any endomorphism f of B, $f(x) \in L$. Consider $x \neq 0$, such that $f(x) = x_1 + \cdots + x_l$, where $x_r \in B_r$ and $H(x) \leq H(f(x)) = \min(H(x_k))$, $1 \leq k \leq l$, $e(x) \geq e(f(x)) = \max\{e(x_k) \mid 1 \leq k \leq l\}$. If $k \leq i$, then $H(x_k) \geq H(x) \geq n_i$ so $x_k \in H_{n_i}(B_k) \subseteq H_{n_k}(B_k)$, because $n_k \leq n_i$; hence $x_k \in L$. If $k = i + j$, then $e(x_k) \leq e(x) \leq i - n_i = i + j - (n_i + j) \leq i + j - n_{i+j}$ because $n_{i+k} \leq n_i + k$. Thus $x_k \in H_{i+j-n_{i+j}}(B_{i+j}) = H_{n_{i+j}}(B_{i+j}) \subseteq L$.

This implies that L is a fully invariant submodule of B. If B is unbounded and $\langle 1-n_1, 2-n_2, 3-n_3, \ldots\rangle$ is also unbounded, then L is unbounded and is therefore a large submodule of B by Remark 3. □

Corollary 9. *If L is a large submodule of a QTAG-module M, then M/L is a direct sum of uniserial modules.*

Proof. For any basic submodule B of M,

$$\frac{M}{L} = \frac{(B+L)}{L} \cong \frac{B}{(B \cap L)} = \frac{\bigoplus_i B_i}{\bigoplus_i H_{n_i}(B_i)} \cong \bigoplus_i \left(\frac{B_i}{H_{n_i}(B_i)}\right) \tag{5}$$

and the result follows. □

Corollary 10. *For any large submodule L of M, $L^1 = M^1$.*

Proof. Since M/L is a direct sum of uniserial modules, $(M/L)^1 = 0$ or $M^1 = L^1$. □

Theorem 11. *Let N be h-pure submodule of a QTAG-module M and L a large submodule of N. Then there exists a large submodule L' of M such that $L' \cap N = L$. If M/N is h-divisible, then L' is the closure of L in M and is therefore uniquely determined by L and $M/L' \cong N/L$.*

Proof. Let $L = N(n)$ and $L' = M(n)$. Since N is h-pure in M and $n = \langle n_1, n_2, \ldots\rangle$ is a U-sequence for N, we have that n is a U-sequence for M. Thus L' is a large submodule of M.

If $x \in L$, then $U_M(x) = U_N(x) \geq n$; therefore $x \in L' \cap N$ and $L \subseteq L' \cap N$. Conversely if $y \in L' \cap N$, then $U_N(y) = U_M(y) \geq n$ implies that $y \in L$ or $L' \cap N \subseteq L$. Thus $L = L' \cap N$.

Let M/N be h-divisible and L' a large submodule of M with $L' \cap N = L$. Then

$$\frac{L'}{L} = \frac{L'}{(L' \cap N)} \cong \frac{(N + L')}{N} = \frac{M}{N}. \tag{6}$$

That is, L'/L is h-divisible. But $M/L' \cong (M/L)/(L'/L)$, where L'/L is a direct summand of M/L; we have $M/L \cong (L'/L) \oplus (M/L')$ and M/L' is a direct sum of uniserial modules [6]. Now $M/L' \cong N/L$, thus

$$\frac{M}{L'} = \frac{(N + L')}{L'} \cong \frac{N}{(N \cap L')} = \frac{N}{L}. \tag{7}$$

□

Now we characterize large submodules in terms of *Ulm* invariants.

Theorem 12. *Let L be a submodule of a QTAG-module M. Then L is a large submodule of M if and only if $L = \sum_{k=1}^{\infty} H_{n_k}(H^{k-n_k}(M))$, where*

(i) $n_k \leq k$, $k \in \mathbb{Z}^+$,

(ii) $n_k \leq n_{k+1} \leq n_k + 1$,

(iii) *the sequence $\langle 1-n_1, 2-n_2, 3-n_3, \ldots\rangle$ is unbounded if M is unbounded and the Ulm-invariants of L are given by $f_L(n) = \sum_k (f_M(k-1))$, $k - n_k - 1 = n$, for all $n \in \mathbb{Z}^+$.*

Proof. Suppose $L = \sum_{k=1}^{\infty} H_{n_k}(H^{k-n_k}(M))$. Since $H_{n_k}(H^{k-n_k}(M))$'s are fully invariant submodules, their sum is again fully invariant submodule of M. If M is bounded, then L is large. If M is unbounded, then, by the third condition, for each $j \in \mathbb{Z}^+$, there exists a positive integer i such that $i - n_i > j$ or $i > n_i + j$.

Since, $i > n_i + j$ and $\mathrm{Soc}(H_i(M)) \subseteq \mathrm{Soc}(H_{j+n_i}(M)) \subseteq H_j(L)$, If $x \in \mathrm{Soc}(H_{j+n_i}(M))$, there exists $y \in M$ such that $d(yR/xR) = j + n_i$, where $e(x) = 1$. Now $y \in H^{j+n_i+1}(M)$, where $i - n_i > j$ or $i - n_i \geq j + 1$; thus $i \geq n_i + j + 1$ and $y \in H^i(M)$.

If $d(yR/zR) = n_i$, then $z \in H_{n_i}(H^{i-n_i}(M)) \subseteq L$ and $x \in H_j(L)$ because $d(yR/xR) = j + n_i$ and $d(zR/xR) = j$. Now, by Lemma 2, $L + B = M$, for every basic submodule B of M, and L is a large submodule of M.

Conversely suppose L is a large submodule of M. Then for any basic submodule B of M, $L \cap B$ is a large submodule of B and, by Theorem 8, $L \cap B = \bigoplus_k H_{n_k}(B_k)$, where $k \in \mathbb{Z}^+$ and n_k's satisfy the given conditions.

Now,

$$H_{n_j}\left(B_j\right)=H_{n_j}\left(H^{j-n_j}\left(B_j\right)\right)\subseteq H_{n_j}\left(H^{j-n_j}\left(B\right)\right),\quad \text{for every } j\in\mathbb{Z}^{+} \tag{8}$$

and, for each $j\in\mathbb{Z}^{+}$,

$$H_{n_j}\left(H^{j-n_j}\left(B\right)\right)\subseteq \Sigma H_{n_k}\left(H^{k-n_k}\left(B\right)\right). \tag{9}$$

This implies that $\bigoplus H_{n_k}(B_k)\subseteq \Sigma H_{n_k}(H^{k-n_k}(B))$.

For the converse, consider $x\in H_{n_j}(H^{j-n_j}(B))$, where $x=x_1+x_2+\cdots+x_m$, $x_i\in B_i$. Then $H(x_i)\geq H(x)\geq n_j$ for x_i, $1\leq i\leq m$ and $e(x_i)\leq e(x)\leq j-n_i$ for all x_i's. Now, for $i\leq j$, we have $H(x_i)\geq n_j\geq n_i$ and $x_i\in H_{n_i}(B_i)$.

If $i=j+l$ for $l\in\mathbb{Z}^{+}$, then $e(x_i)\leq j-n_j=j+l-(n_j+l)\leq j+l-n_{j+l}$. (by the given condition). Therefore

$$x_i\in\left(H^{j+l-(n_{j+l})}\left(B_{j+l}\right)\right)=H_{n_{j+l}}\left(B_{j+l}\right)=H_{n_i}\left(B_i\right) \tag{10}$$

and $H_{n_j}(H^{j-n_j}(B))\subseteq\bigoplus_k H_{n_k}(B_k)$. Let $L'=\sum H_{n_k}(H^{k-n_k}(M))$. Now $L'\cap B=\sum H_{n_k}(H^{k-n_k}(B))=L\cap B$. Since B is h-pure in M and M/B is h-divisible, $L'=L$, by Theorem 11. Again $L\cap B$ is a basic submodule of L; thus $f_L(n)=f_{L\cap B}(n)$, for all $n\in\mathbb{Z}^{+}$.

If $L\cap B=\bigoplus(L\cap B_i)_i$, where $(L\cap B)_i$ is the direct sum of uniserial modules of length i, then $f_{(L\cap B)}(n)=g((L\cap B)_{n+1})=g(\bigoplus H_{n_k}(B_k))$, where $H_{n_k}(B_k)$ is a direct sum of uniserial modules of length $n+1$.
Again,

$$\begin{aligned} f_{L\cap B}(n)&=g\left(\bigoplus_k H_{n_k}\left(B_k\right)\right)=\sum_k\left(g\left(H_{n_k}\left(B_k\right)\right)\right)\\ &=\sum_k\left(g\left(B_k\right)\right)=\sum_k\left(f_B\left(k-1\right)\right)\\ &=\sum_k\left(f_M\left(k-1\right)\right),\quad\text{where } k-n_k-1=n. \end{aligned} \tag{11}$$

And the proof is complete. □

3. Properties of Large Submodules of *QTAG*-Modules

In this section we compare the structures of $QTAG$-modules and their large submodules. We investigate the characteristics of $QTAG$-modules which are preserved by their large submodules. We start with the Σ-modules, that is, the modules whose high submodules are direct sums of uniserial modules [7]. Then we study summable, σ-summable, $(\omega+1)$-projective, and h-pure complete $QTAG$-modules.

Singh [8] proved that a $QTAG$-module M is a direct sum of uniserial submodules if and only if M is the union of an ascending sequence of submodules M_n, $n=1,2,3,\ldots$, such that, for every n, there exists $k_n>0$ and $H_M(x)\leq k_n$ for all $x\in M_n$.

This helps us to prove the following.

Theorem 13. *A QTAG-module is a Σ-module if and only if $\mathrm{Soc}(M)=\bigcup_{k=1}^{\infty}M_k$, where $M_k\subset M_{k+1}$ and for every $k\in N$, $M_k\cap H_k(M)=\mathrm{Soc}(M^1)$.*

Proof. Since M is a Σ-module, it contains a high submodule N such that N is a direct sum of uniserial modules.

Again N is a high submodule [9] of M if and only if N is h-pure in M and $\mathrm{Soc}(M)=\mathrm{Soc}(N)+\mathrm{Soc}(M^1)$. Therefore by the above result [8], $\mathrm{Soc}(N)=\bigcup_{k=1}^{\infty}N_k$, $N_k\subseteq N_{k+1}$, and $N_k\cap H_k(N)=0$, and we deduce $\mathrm{Soc}(M)=\bigcup_{k=1}^{\infty}(N_k+\mathrm{Soc}(M^1))$. If we put $M_k=\mathrm{Soc}(M^1)+N_k$, then $M_k\subseteq M_{k+1}$ and $(\mathrm{Soc}(M^1)+N_k)\cap\mathrm{Soc}(M^1)+\mathrm{Soc}(H_k(N))=\mathrm{Soc}(M^1)+(N_k\cap H_k(N))=\mathrm{Soc}(M^1)$, because $H_k(N)$ is a high submodule of $H_k(M)$.

For the converse if $\mathrm{Soc}(N)=\bigcup_{k=1}^{\infty}(M_k\cap N)=\bigcup_{k=1}^{\infty}N_k$, where we put $N_k=N\cap M_k$, then $N_k\subseteq N_{k+1}$. Also

$$\begin{aligned} N_k\cap H_k\left(N\right)&=M_k\cap H_k\left(M\right)=M_k\cap\left(H_k\left(M\right)\cap N\right)\\ &=M_k\cap H_k\left(M\right)\cap N=M^1\cap N=0. \end{aligned} \tag{12}$$

Therefore N is a direct sum of uniserial modules and M is a Σ-module. □

Now we may prove the following.

Theorem 14. *A QTAG-module M is a Σ- module if and only if its large submodule L is a Σ-module.*

Proof. Since $L^1=M^1$ [6], there is a natural number m such that $\mathrm{Soc}(L)=\mathrm{Soc}(H_m(M))$ and $\mathrm{Soc}(H_n(L))=\mathrm{Soc}(H_{t_n}(M))$ for every $n<\omega$ and some t_n such that $m\leq t_n<\omega$. If M is a Σ-module, then, by Theorem 13, $\mathrm{Soc}(M)$ is the union of ascending chain of submodules M_k such that $M_k\subseteq M_{k+1}$ and $M_k\cap H_k(M)=\mathrm{Soc}(M^1)$ for every $k\in N$.

This implies that $\mathrm{Soc}(L)=\bigcup_{k<\omega}(M_k\cap L)$ and $M_k\cap L\subseteq M_{k+1}\cap L$. Therefore

$$\begin{aligned} M_k\cap L\cap H_k\left(L\right)&=M_k\cap H_k\left(L\right)\subseteq M_k\cap H_k\left(M\right)\\ &=\mathrm{Soc}\left(M^1\right)=\mathrm{Soc}\left(L^1\right). \end{aligned} \tag{13}$$

Now Theorem 13 indicates that L is a Σ-module.

Conversely suppose L is a Σ-module. Therefore

$$\mathrm{Soc}\left(L\right)=\bigcup_{n<\omega}L_n,\quad L_n\subseteq L_{n+1},\ L_n\cap H_n\left(L\right)=\mathrm{Soc}\left(L^1\right). \tag{14}$$

Again $\mathrm{Soc}(H_m(M))=\bigcup_{n<\omega}L_n$. Now

$$\begin{aligned} L_n\cap\mathrm{Soc}\left(H_{t_n}\left(M\right)\right)&=L_n\cap\mathrm{Soc}\left(H_n\left(L\right)\right)=\mathrm{Soc}\left(L^1\right)\\ &=\mathrm{Soc}\left(M^1\right). \end{aligned} \tag{15}$$

Thus, by Theorem 13, $H_m(M)$ is a Σ-module, and so is M. □

To study the other relations between a module M and its large submodule L we need the following lemma.

Lemma 15. *Isotype submodules of countable length of summable QTAG-modules are again summable.*

Proof. Let N be an isotype submodule of countable length ρ in the summable module M. Now there is a $H_\rho(M)$-high submodule K of M such that $N \subseteq K$. Since $\mathrm{Soc}(M) = \bigoplus_{\sigma<\rho} S_\sigma$, there is $H_\rho(M)$-high submodule P of M such that $\mathrm{Soc}(P) = \bigoplus_{\sigma<\rho} S_\sigma$.

Again, for every ordinal ρ, every $H_\rho(M)$-high submodule is isotype; therefore P is isotype and it is summable. The socles of $H_\rho(M)$-high submodules have the same images under the canonical map $M \to M/H_\rho(M)$ because this maps $H_\rho(M)$-high submodules isomorphically in a height preserving manner onto submodules of $M/H_\rho(M)$.

Now N is isotype in a summable module K of countable length ρ. Therefore $\mathrm{Soc}(K)$ is the union of an ascending chain of submodules K_n, where for every n the heights of elements of K_n assume but a finite numbers of values.

Now $\mathrm{Soc}(N) = \cup K_n \cap N$, $n = 1, 2, 3, \ldots$, and the heights of the elements of $K_n \cap N$ assume a finite numbers of different values. Thus N is summable. □

The following result shows that summability is shared by large submodules.

Theorem 16. *Let L be a large submodule of a QTAG-module M. Then M is summable if and only if L is summable.*

Proof. Suppose M is summable; that is, $\mathrm{Soc}(M) = \bigoplus_{\beta<\alpha} M_\beta$, where the nonzero elements of M_β's are contained in $H_\beta(M)$ but they do not belong to $H_{\beta+1}(M)$, for every $\beta < \alpha$.

Again L is fully invariant submodule of M and $H_\rho(M) = H_\rho(L)$ for all ordinals $\rho \geq \omega$, $\mathrm{Soc}(L) = \bigoplus_{\beta<\alpha}(M_\beta \cap L)$, where the nonzero elements of $M_\beta \cap L$ are contained in $H_\beta(L)$ and not contained in $H_{\beta+1}(L)$ for every $\omega \leq \beta < \alpha$. Since $\mathrm{Soc}(H_n(L)) = \mathrm{Soc}(H_{t_n}(M))$, whenever $1 \leq n < \omega$, $n \leq t_n < \omega$, $M_\beta \cap L \subseteq L$, but $(M_\beta \cap L) \cap H_1(L) = 0$, for each $\beta < t_1$. By transfinite induction $M_\beta \cap L \subseteq H_1(L)$ and $(M_\beta \cap L) \cap H_2(L) = 0$, for $t_1 \leq \beta < t_2$ and so on; that is, $H_\beta \cap L \subseteq H_n(L)$ and $(H_\beta \cap L) \cap \mathrm{H}_{n+1}(L) = 0$, for $t_n \leq \beta < t_{n+1}$.

If we put $L_0 = \bigoplus_{0\leq\beta<t_1}(M_\beta \cap L), L_1 = \bigoplus_{t_1\leq\beta<t_2}(M_\beta \cap L), \ldots,$ and $L_n = \bigoplus_{t_n\leq\beta<t_{n+1}}(M_\beta \cap L)$, where $n < \omega$, and $M_\beta \cap L = L_\beta$ if $\beta \geq \omega$, then $\mathrm{Soc}(L) = \bigoplus_{\beta<\alpha} L_\beta$ if $L_\beta \subseteq H_\beta(L)$ and $L_\beta \cap H_{\beta+1}(L) = 0$. Therefore L is summable.

Conversely suppose L is summable. So, $L^1 = M^1$ is summable as its fully invariant submodule. Moreover, by Lemma 15, L being summable implies that L is a Σ-module. Now by Theorem 14, M is also a Σ-module. For a high submodule N of M, $\mathrm{Soc}(N) \oplus \mathrm{Soc}(M^1) = \mathrm{Soc}(M)$.

Since N is a direct sum of uniserial modules, $\mathrm{Soc}(N) = \bigoplus_{k<\omega} N_k$, where $N_k \subseteq H_k(M)$ and $N_k \cap H_{k+1}(M) = 0$ because N is h-pure in M. Again the summability of M^1 ensures that $\mathrm{Soc}(M^1) = \bigoplus_{\beta<\alpha} K_\beta$, where $K_\beta \subseteq H_\beta(M^1)$ and $K_\beta \cap H_{\beta+1}(M^1) = 0$. Therefore, $K_\beta \subseteq H_{\omega+\beta}(M)$ and $K_\beta \cap H_{\omega+\beta+1}(M) = 0$. This implies that

$$\mathrm{Soc}(M) = \bigoplus_{k<\omega} N_k \bigoplus_{\omega\leq\omega+\beta<\omega+\alpha} K_\beta. \tag{16}$$

We may infer now that M is summable. □

Theorem 17. *Let L be the large submodule of M. Then M is σ-summable if and only if L is σ-summable.*

Proof. Suppose M is unbounded. Then length of M = length of $L \geq \omega$. If M is σ-summable, then L is also σ-summable being a submodule of equal length.

If M is bounded the result holds trivially.

Conversely suppose L is σ-summable. Therefore $\mathrm{Soc}(L) = \bigcup_{n<\omega} L_n$, $L_n \subseteq L_{n+1}$ and $L_n \cap H_{\alpha_n}(L) = 0$ for all $n \geq 0$ and some $\alpha_n <$ length of M.

Now, $\mathrm{Soc}(H_m(M)) = \bigcup_{n<\omega} L_n$. Since $L^1 = M^1$ [6], $H_\alpha(M) = H_\alpha(L)$ for each ordinal $\alpha \geq \omega$ and $\mathrm{Soc}(H_{\alpha_n}(L)) = \mathrm{Soc}(H_{k_n}(M))$ for $\alpha_n < \omega$ and some $k_n \geq \max(\alpha_n, m)$ because $H_{\alpha_n}(L)$ is large in M and $H_{\alpha_n}(M)$ both. Thus $L_n \cap H_{s_n}(M) = L_n \cap H_{\alpha_n}(L) = 0$, whenever $s_n <$ length of M = length of $H_m(M) \geq \omega$, $s_n = \alpha_n \geq \omega$ or $\omega > s_n = k_n$.

We may define $M_n = \{x \mid x \in \mathrm{Soc}(M) \cap L_n$ and $x \notin H_m(M)\}$. Thus $\mathrm{Soc}(M) = \bigcup_{n<\omega} M_n$, $M_n \subseteq M_{n+1}$. By defining M_n's we observe that $M_n \cap H_{s_n}(M) = 0$. This implies that M is σ-summable. □

Theorem 18. *If M is a direct sum of σ-summable QTAG-modules, then so is L.*

Proof. Let $M = \bigoplus_{i\in I} M_i$, where each M_i is σ-summable. Now $L = \bigoplus_{i\in I}(L \cap M_i)$ because L is fully invariant in M. Since all M_i's are isotype in M, we infer that $L \cap M_i$ is large in M_i, for every i. By Theorem 17, $L \cap M_i$ are σ-summable. Thus L is also a direct sum of σ-summable modules. □

Let us recall the following.

Definition 19. A QTAG-module M is $(\omega + 1)$-projective if there exists a submodule $N \subseteq \mathrm{Soc}(M)$ such that M/N is a direct sum of uniserial modules.

Remark 20. The submodules of $(\omega + n)$-projective modules are also $(\omega + n)$-projective.

Theorem 21. *A QTAG-module M is $(\omega + 1)$-projective if and only if its large submodule L is $(\omega + 1)$-projective.*

Proof. Suppose L is $(\omega+1)$-projective. Therefore there exists a submodule $N \subseteq \mathrm{Soc}(L)$ such that $\mathrm{Soc}(L/N) = \bigcup_{n<\omega}(L_k/N)$, where $L_k \subseteq L_{k+1} \subseteq L$ and $L_k \cap H_k(L) \subseteq N$ for each $k < \omega$. Now $L_k \subseteq H^2(M)$, for every $k < \omega$. Since

$$H_k\left(H^2(L)\right) = H_{t_k}\left(H^2(M)\right) + \mathrm{Soc}\left(H_{j_k}(M)\right), \tag{17}$$

for some $k \leq j_k \leq t_k < \omega$, we have

$$\begin{aligned} &H_{t_k}\left(H^2(M)\right) \subseteq H_k\left(H^2(L)\right) \subseteq H_{j_k}\left(H^2(M)\right), \\ &L_k \cap H_{t_k}(M) = L_k \cap H_{t_k}\left(H^2(M)\right) \\ &\qquad \subseteq L_k \cap H_k\left(H^2(L)\right) = L_k \cap H_k(L) \\ &\qquad \subseteq N. \end{aligned} \tag{18}$$

Therefore the heights of the elements of L_k/N are bounded in M/N for all $k < \omega$. Now $(M/N)/(L/N) \cong M/L$ is a direct sum of uniserial modules [6]. Therefore M/N is a direct sum of uniserial modules and M is $(\omega+1)$-projective. The converse is trivial. □

The property of being h-pure complete is also shared by the large submodules of $QTAG$-modules.

First we recall the definition of h-pure completeness.

Definition 22. A $QTAG$-module M is h-pure complete if, for every subsocle $S \subseteq \mathrm{Soc}(M)$, there is a h-pure submodule N of M so that $S = \mathrm{Soc}(N)$. In other words every subsocle supports a h-pure submodule of M.

Theorem 23. *Let L be the large submodule of a $QTAG$-module M. If M is h-pure complete, so is L.*

Proof. Let S be a subsocle of L. Since $S \subseteq \mathrm{Soc}(M)$, S supports a h-pure submodule N of M. Now $N \cap L$ is also large in M and $N \cap L$ is h-pure in L. Again $S = \mathrm{Soc}(N) \cap \mathrm{Soc}(L) = \mathrm{Soc}(N \cap L)$, and therefore L is h-pure complete. □

Corollary 24. *A $QTAG$-module M is h-pure complete if and only if $H_k(M)$ is h-pure complete for some fixed but arbitrary positive integer k.*

Proof. Since $H_k(M)$ is large in M, it is h-pure complete if M is h-pure complete. Conversely suppose $H_k(M)$ is h-pure complete. We shall use transfinite induction to prove the result.

Let S be a subsocle of M such that $S \cap H_1(M) \subseteq \mathrm{Soc}(H_1(M))$ and $S \cap H_1(M) = \mathrm{Soc}(N)$ for some h-pure submodule N of $H_1(M)$. By [7] we can say that there is a h-pure submodule K of M such that $H_1(K) = N$ and $\mathrm{Soc}(K) = \mathrm{Soc}(N) = \mathrm{Soc}(H_1(K))$. Now $S \cap H_1(M) = \mathrm{Soc}(H_1(K))$.

We have to show that there exists a h-pure submodule $T \subseteq M$ such that $S = \mathrm{Soc}(T)$. We define the submodule $T = K + (S \cap H_2(M))$. Now

$$\begin{aligned} \mathrm{Soc}(T) &= \mathrm{Soc}(K + S \cap H_2(M)) \\ &= \mathrm{Soc}(K) + (S \cap H_2(M)) \end{aligned} \tag{19}$$

because $S \cap H_2(M) = \mathrm{Soc}(S \cap H_2(M))$. Again $\mathrm{Soc}(T) = \mathrm{Soc}(N) + (S \cap H_2(M)) \subseteq S$.

Now

$$\begin{aligned} S &= (S \cap H_1(M)) \cup (S \cap H_2(M)) \\ &= \mathrm{Soc}(K)\, \mathrm{Soc}(S \cap H_2(M)) \\ &= \mathrm{Soc}[K \cup (S \cap H_2(M))] \\ &\subseteq \mathrm{Soc}[K + (S \cap H_2(M))] = \mathrm{Soc}(T). \end{aligned} \tag{20}$$

Therefore $S = \mathrm{Soc}(T)$. Now,

$$\begin{aligned} \mathrm{Soc}(T) \cap H_t(M) &= S \cap H_t(M) \\ &= [\mathrm{Soc}(K) + (S \cap H_2(M))] \\ &\quad \cap H_t(M) \\ &= [\mathrm{Soc}(H_1(K)) + (S \cap H_2(M))] \\ &\quad \cap H_t(M) = \mathrm{Soc}(K) \cap H_t(M) \\ &= \mathrm{Soc}(H_t(K)) = \mathrm{Soc}(H_t(T)). \end{aligned} \tag{21}$$

This implies that T is h-pure in M. □

In the end we state the following unsolved problems.

Problem 25. Is it true that M is a HF-module if and only if its large submodule L is?

Problem 26. Is it true that M is a direct sum of closed modules if and only if its large submodule L is?

Competing Interests

The authors declare that they have no competing interests.

References

[1] A. Mehdi, M. Y. Abbasi, and F. Mehdi, "On (ω+n)-projective modules," *Ganita Sandesh*, vol. 20, no. 1, pp. 27–32, 2006.

[2] S. Singh, "Some decomposition theorems in abelian groups and their generalizations," in *Ring Theory: Proceedings of the Ohio University Conference*, vol. 25, pp. 183–189, Marcel Dekker, New York, NY, USA, 1976.

[3] L. Fuchs, *Infinite Abelian Groups. Vol. I*, Pure and Applied Mathematics, Vol. 36, Academic Press, New York, NY, USA, 1970.

[4] A. Mehdi, S. A. R. K. Naji, and A. Hasan, "Small homomorphisms and large submodules of QTAG- modules," *Scientia Series A: Mathematical Sciences*, vol. 23, pp. 19–24, 2012.

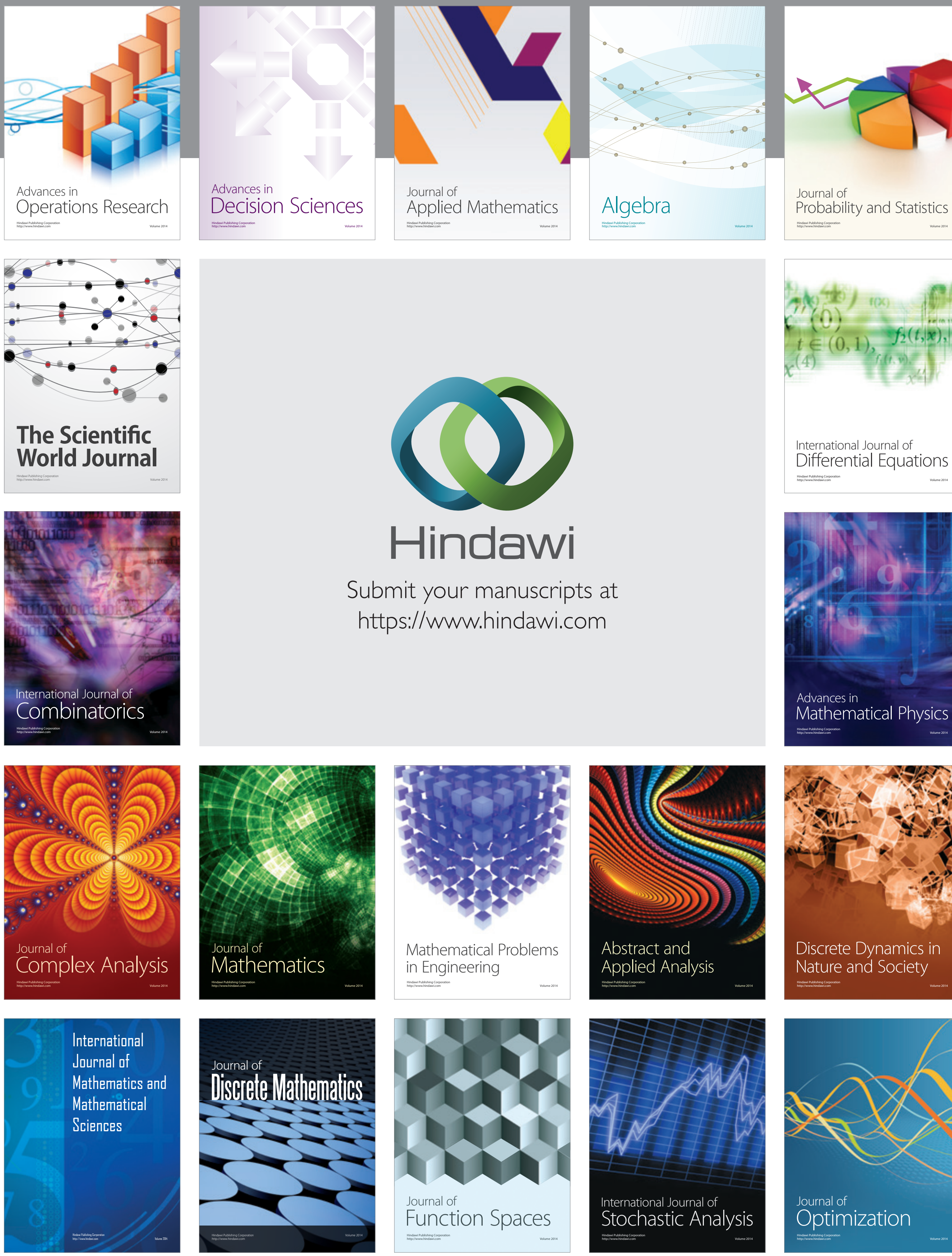
[5] M. Z. Khan, "On basic submodules," *Tamkang Journal of Mathematics*, vol. 10, no. 1, pp. 24–29, 1979.

[6] A. H. Ansari, M. Ahmad, and M. Z. Khan, "Some decomposition theorems on S2-module III," *Tamkang Journal of Mathematics*, vol. 12, no. 2, pp. 147–154, 1981.

[7] M. Z. Khan, "Modules behaving like torsion abelian groups. II," *Mathematica Japonica*, vol. 23, no. 5, pp. 509–516, 1979.

[8] S. Singh, "Some decomposition theorems on abelian groups and their generalisations. II," *Osaka Journal of Mathematics*, vol. 16, no. 1, pp. 45–55, 1979.

[9] A. Mehdi and F. Mehdi, "N-high submodules and h-topology," *The South East Asian Journal of Mathematics and Mathematical Sciences*, vol. 1, no. 1, pp. 83–88, 2002.



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