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*Research Article*

# An Interesting Property of a Class of Circulant Graphs

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Suppose that $\Pi = \mathrm{Cay}(\mathbb{Z}_n, \Omega)$ and $\Lambda = \mathrm{Cay}(\mathbb{Z}_n, \Psi_m)$ are two Cayley graphs on the cyclic additive group $\mathbb{Z}_n$, where $n$ is an even integer, $m = n/2 + 1$, $\Omega = \{t \in \mathbb{Z}_n \mid t \text{ is odd}\}$, and $\Psi_m = \Omega \cup \{n/2\}$ are the inverse-closed subsets of $\mathbb{Z}_n - \{0\}$. In this paper, it is shown that $\Pi$ is a distance-transitive graph, and, by this fact, we determine the adjacency matrix spectrum of $\Pi$. Finally, we show that if $n \geq 8$ and $n/2$ is an even integer, then the adjacency matrix spectrum of $\Lambda$ is $(n/2+1)^1, (1-n/2)^1, (1)^{(n-4)/2}, (-1)^{n/2}$ (we write multiplicities as exponents).

## 1. Introduction

In this paper, graph $\Gamma = (V, E)$ always means a simple connected graph with $n$ vertices (without loops, multiple edges, and isolated vertices), where $V = V(\Gamma)$ is the vertex set and $E = E(\Gamma)$ is the edge set. Graph $\Gamma$ is called a vertex-transitive graph, if, for any $x, y \in V$, there is some $\pi$ in $\mathrm{Aut}(\Gamma)$, the automorphism group of $\Gamma$, such that $\pi(x) = y$. Let $\Gamma$ be a graph, the complement $\overline{\Gamma}$ of $\Gamma$ is the graph whose vertex set is $V(\Gamma)$ and whose edges are the pairs of nonadjacent vertices of $\Gamma$. It is well known that, for any graph $\Gamma$, $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\overline{\Gamma})$ [1]. If $\Gamma$ is a connected graph and $\partial(u, v)$ denotes the distance in $\Gamma$ between the vertices $u$ and $v$, then, for any automorphism $\pi$ in $\mathrm{Aut}(\Gamma)$, we have $\partial(u, v) = \partial(\pi(u), \pi(v))$.

Let $\Upsilon = \{\gamma_1, \ldots, \gamma_{k+1}\}$ be a set and $K$ be a group; then, writing $\mathrm{Fun}(\Upsilon, K)$ to denote the set of all functions from $\Upsilon$ into $K$, we can turn $\mathrm{Fun}(\Upsilon, K)$ into a group by defining a product:

$$(fg)(\gamma) = f(\gamma)\, g(\gamma) \qquad \forall f, g \in \mathrm{Fun}(\Upsilon, K) \text{ and } \gamma \in \Upsilon, \tag{1}$$

where the product on the right is in $K$. Since $\Upsilon$ is finite then the group $\mathrm{Fun}(\Upsilon, K)$ is isomorphic to $K^{k+1}$ (a direct product of $k+1$ copies of $K$) via the isomorphism $f \to (f(\gamma_1), \ldots, f(\gamma_{k+1}))$. Let $H$ and $K$ be groups and suppose that $H$ acts on the nonempty set $\Upsilon$. Then, the wreath product of $K$ by $H$ with respect to this action is defined to be the semidirect product $\mathrm{Fun}(\Upsilon, K) \rtimes H$ where $H$ acts on the group $\mathrm{Fun}(\Upsilon, K)$ via

$$f^x(\gamma) = f\left(\gamma^{x^{-1}}\right) \qquad \forall f \in \mathrm{Fun}(\Upsilon, K),\ \gamma \in \Upsilon \text{ and } x \in H. \tag{2}$$

We denote this group by $K\mathrm{wr}_\Upsilon H$. Consider the wreath product $G = K\mathrm{wr}_\Upsilon H$. If $K$ acts on a set $\Delta$, then we can define an action of $G$ on $\Delta \times \Upsilon$ by

$$(\delta, \gamma)^{(f,h)} = \left(\delta^{f(\gamma)}, \gamma^h\right) \quad \forall (\delta, \gamma) \in \Delta \times \Upsilon, \tag{3}$$

where $(f, h) \in \mathrm{Fun}(\Upsilon, K) \rtimes H = K\mathrm{wr}_\Upsilon H$ [2].

Let $G$ be a group and $H \leq G$ a subgroup of $G$ and $S \subseteq G$. The Schreier coset graph on $G/H$ generated by $S$ is the graph $\Gamma = \Gamma(G, H, S)$ with $V(\Gamma) = G/H = \{gH \mid g \in G\}$ the set of left cosets of $H$, and there is an edge $(gH, sgH)$ for each coset $gH$ and each $s \in S$. If $S$ is inverse-closed, then $\Gamma$ is an undirected multigraph (possibly with loops). Note that if $1_G$ is the identity element of $G$, then $\Gamma(G, 1_G, S) = \Gamma(G, S)$ is the Cayley graph on $G$ generated by $S$. It is well known that every Cayley graph is vertex-transitive [3].

Let $\Gamma$ be a graph with automorphism group $\mathrm{Aut}(\Gamma)$. Say that $\Gamma$ is symmetric graph if, for all vertices $u, v, x, y$ of $\Gamma$ such that $u$ and $v$ are adjacent, also, $x$ and $y$ are adjacent, and there is an automorphism $\pi$ in $\mathrm{Aut}(\Gamma)$ such that $\pi(u) = x$

and $\pi(v) = y$. We say that $\Gamma$ is distance-transitive if, for all vertices $u, v, x, y$ of $\Gamma$ such that $\partial(u, v) = \partial(x, y)$, there is an automorphism $\pi$ in $\mathrm{Aut}(\Gamma)$ satisfying $\pi(u) = x$ and $\pi(v) = y$ [3]. It is clear that hierarchy of the conditions is

$$\begin{aligned} \text{distance-transitive} &\Longrightarrow \text{symmetric} \\ &\Longrightarrow \text{vertex-transitive.} \end{aligned} \tag{4}$$

Eigenvalues of an undirected graph $\Gamma$ are the eigenvalues of an arbitrary adjacency matrix of $\Gamma$. Harary and Schwenk [4] defined $\Gamma$ to be integral, if all of its eigenvalues are integers. For a survey of integral graphs, see [5]. In [6], the number of integral graphs on $n$ vertices is estimated. Known characterizations of integral graphs are restricted to certain graph classes; see [7].

In this paper, suppose $\Pi = \mathrm{Cay}(\mathbb{Z}_n, \Omega)$ and $\Lambda = \mathrm{Cay}(\mathbb{Z}_n, \Psi_m)$ are two Cayley graphs on the cyclic additive group $\mathbb{Z}_n$, where $n$ is an even integer, $m = n/2 + 1$, $\Omega = \{t \in \mathbb{Z}_n \mid t \text{ is odd}\}$, and $\Psi_m = \Omega \cup \{n/2\}$ are the inverse-closed subsets of $\mathbb{Z}_n - \{0\}$. One of our goals in this paper is to obtain all eigenvalues of the Cayley graph $\Lambda = \mathrm{Cay}(\mathbb{Z}_n, \Psi_m)$. First, we determine the group automorphism of $\Pi$ and we show that $\Pi$ is a distance transitive graph; also, by this fact, we determine the adjacency matrix spectrum of $\Pi$. Finally, according to these facts, we show that if $n \geq 8$ and $n/2$ is an even integer, then the adjacency matrix spectrum of $\Lambda$ is $(n/2+1)^1, (1-n/2)^1, (1)^{(n-4)/2}, (-1)^{n/2}$ (we write multiplicities as exponents).

## 2. Definitions and Preliminaries

*Definition 1* (see [3, 8]). For any vertex $v$ of a connected graph $\Gamma$, one defines

$$\Gamma_r(v) = \{u \in V(\Gamma) \mid \partial(u, v) = r\}, \tag{5}$$

where $r$ is a nonnegative integer not exceeding $d$, the diameter of $\Gamma$. It is clear that $\Gamma_0(v) = \{v\}$, and $V(\Gamma)$ is partitioned into the disjoint subsets $\Gamma_0(v), \ldots, \Gamma_d(v)$, for each $v$ in $V(\Gamma)$. The graph $\Gamma$ is called distance-regular with diameter $d$ and intersection array $\{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\}$, if it is regular of valency $k$ and, for any two vertices $u$ and $v$ in $\Gamma$ at distance $r$, one has $|\Gamma_{r+1}(v) \cap \Gamma_1(u)| = b_r$, and $|\Gamma_{r-1}(v) \cap \Gamma_1(u)| = c_r$, $(0 \leq r \leq d)$. The numbers $c_r, b_r$, and $a_r$, where

$$a_r = k - b_r - c_r \quad (0 \leq r \leq d), \tag{6}$$

is the number of neighbours of $u$ in $\Gamma_r(v)$ for $\partial(u, v) = r$, are called the intersection numbers of $\Gamma$. Clearly $b_0 = k$, $b_d = c_0 = 0$, and $c_1 = 1$.

*Remark 2* (see [3]). It is clear that if $\Gamma$ is distance-transitive graph, then $\Gamma$ is distance-regular.

**Lemma 3** (see [3]). *A connected graph $\Gamma$ with diameter $d$ and automorphism group $G = \mathrm{Aut}(\Gamma)$ is distance-transitive if and only if it is vertex-transitive and the vertex-stabilizer $G_v$ is transitive on the set $\Gamma_r(v)$, for each $r \in \{0, 1, \ldots, d\}$, and $v \in V(\Gamma)$.*

**Theorem 4** (see [8]). *Let $\Gamma$ be a distance-regular graph which the valency of each vertex as $k$, with diameter $d$, adjacency matrix $A$, and intersection array, is*

$$\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}. \tag{7}$$

*Then, the tridiagonal $(d+1) \times (d+1)$ matrix*

$$j(\Gamma) = \begin{bmatrix} a_0 & b_0 & 0 & 0 & \cdots & & & \\ c_1 & a_1 & b_1 & 0 & \cdots & & & \\ 0 & c_2 & a_2 & b_2 & & & & \\ & & & \cdots & & & & \\ & & & & c_{d-2} & a_{d-2} & b_{d-2} & 0 \\ & & & \cdots & 0 & c_{d-1} & a_{d-1} & b_{d-1} \\ & & & \cdots & 0 & 0 & c_d & a_d \end{bmatrix} \tag{8}$$

*determines all the eigenvalues of $\Gamma$.*

**Theorem 5** (see [9]). *Let $F$ be a field and let $R$ be a commutative subring of $F^{n\times n}$, the set of all $n \times n$ matrices over $F$. Let $M \in R^{m\times m}$, then $\det_F(M) = \det_F(\det_R(M))$.*

**Theorem 6** (see [10]). *Let $\Gamma$ be a graph such that contains $k+1$ components $\Gamma_1, \ldots, \Gamma_{k+1}$. If, for any $i \in I = \{1, \ldots, k+1\}$, $\Gamma_i \cong \Gamma_1$, then $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\Gamma_1) wr_I \mathrm{Sym}(k+1)$.*

### 2.1. Main Results

**Proposition 7.** *Let $\Pi = Cay(\mathbb{Z}_n, \Omega)$ be the Cayley graph on the cyclic group $\mathbb{Z}_n$ $(n \geq 4)$, where $\Omega = \{t \in \mathbb{Z}_n \mid t \text{ is odd}\}$ is the inverse-closed subset of $\mathbb{Z}_n - \{0\}$. Then $\mathrm{Aut}(\Pi) \cong \mathrm{Sym}(n/2) wr_I \mathrm{Sym}(2)$, where $I = \{1, 2\}$.*

*Proof.* Let $V(\Pi) = \{1, \ldots, n\}$ be the vertex set of $\Pi$. By assumption, the size of the every independent set of vertices in $\Pi$ is $n/2$, because $\Pi$ is a vertex-transitive graph and the size of every clique in graph $\Pi$ is 2. Therefore, for any $x \in V(\Pi)$, there is exactly $n/2$, $y \in V(\Pi)$ such that $x^{-1}y \notin \Omega$. Hence, if $x^{-1}y \notin \Omega$, then two vertices $x$ and $y$ are adjacent in the complement $\overline{\Pi}$ of $\Pi$, so $\overline{\Pi}$ contains 2 components $\Pi_1, \Pi_2$ such that $\Pi_1 \cong \Pi_2 \cong K_{n/2}$, where $K_{n/2}$ is the complete graph of $n/2$ vertices. Therefore, $\overline{\Pi} \cong 2K_{n/2}$. Hence, by Theorem 6, $\mathrm{Aut}(\overline{\Pi}) \cong \mathrm{Aut}(K_{n/2}) \mathrm{wr}_I \mathrm{Sym}(2) = \mathrm{Sym}(n/2) \mathrm{wr}_I \mathrm{Sym}(2)$. □

**Proposition 8.** *Let $\Pi = Cay(\mathbb{Z}_n, \Omega)$ be the Cayley graph on the cyclic group $\mathbb{Z}_n$ $(n \geq 4)$, where $n$ is an even integer and $\Omega = \{t \in \mathbb{Z}_n \mid t \text{ is odd}\}$ is the inverse-closed subset of $\mathbb{Z}_n - \{0\}$; then $\Pi$ is a distance-transitive graph.*

*Proof.* Suppose that $u, v, x, y$ are vertices of $\Pi$ such that $\partial(u, v) = \partial(x, y) = r$, where $r$ is a nonnegative integer not exceeding $d$, the diameter of $\Pi$. So $\partial(u, v) = \partial(x, y) = 1$ or 2, since $d = 2$.

(a) If $\partial(u, v) = \partial(x, y) = 2$, then $u^{-1}v \notin \Omega$ and $x^{-1}y \notin \Omega$. Therefore, two vertices $u$ and $v$ are adjacent in the complement $\overline{\Pi}$ of $\Pi$, also two vertices $x$ and $y$ are adjacent in the complement $\overline{\Pi}$ of $\Pi$. So $\overline{\Pi}$ contains 2 components $\Pi_1, \Pi_2$

such that $\Pi_1 \cong \Pi_2 \cong K_{n/2}$. Therefore $\overline{\Pi} \cong 2K_{n/2}$; hence we may assume $\pi = (ux)(vy) \in \mathrm{Aut}(\overline{\Pi}) = \mathrm{Aut}(\Pi)$, so $\pi(u) = x$ and $\pi(v) = y$.

(b) If $\partial(u,v) = \partial(x,y) = 1$, then, by Lemma 3, it is sufficient to show that vertex-stabilizer $G_v$ is transitive on set $\Pi_r(v)$ for every $r \in \{0,1,2\}$ and every $v \in V(\Pi)$, because $\Pi$ is a vertex-transitive graph. In this case, let $V(\Pi) = \{1,2,\ldots,n\}$ be the vertex set of $\Pi$ and $G = \mathrm{Aut}(\Pi)$. Consider the vertex $v = 1$ in $V(\Pi)$, then $\Pi_0(v) = \{1\}$, $\Pi_1(v) = \{t \in \mathbb{Z}_n \mid t \text{ is even}\}$, and $\Pi_2(v) = \{1 \neq t \in \mathbb{Z}_n \mid t \text{ is odd}\}$. Let $H$ be the group that is generated by all elements of sets $\Pi_1(v)$ and $\Pi_2(v)$, say $H = \langle (2,4,\ldots,n),(3,5,\ldots,n-1)\rangle$. It is clear that $H$ is a subgroup of $\mathrm{Aut}(\Pi)$, so the group $H$ is a subgroup of $G_v$ such that transitive on the set $\Pi_r(v)$ for each $r \in \{0,1,2\}$. Note that if $1 \neq v \in V(\Pi)$, then, we can show that vertex-stabilizer $G_v$ is transitive on the set $\Pi_r(v)$ for each $r \in \{0,1,2\}$, because $\Pi$ is a vertex-transitive graph. □

**Proposition 9.** *Let $\Pi = Cay(\mathbb{Z}_n, \Omega)$ be the Cayley graph on the cyclic group $\mathbb{Z}_n$ ($n \geq 4$), where $n$ is an even integer and $\Omega = \{t \in \mathbb{Z}_n \mid t \text{ is odd}\}$ is the inverse-closed subset of $\mathbb{Z}_n - \{0\}$; then $\Pi$ is an integral graph.*

*Proof.* By Remark 2, it is clear that $\Pi$ is distance-regular, because $\Pi$ is a distance-transitive graph. Let $V(\Pi) = \{1,2,\ldots,n\}$ be the vertex set of $\Pi$. Consider the vertex $v = 1$ in $V(\Pi)$; then $\Pi_0(v) = \{1\}$, $\Pi_1(v) = \{t \in \mathbb{Z}_n \mid t \text{ is even}\}$, and $\Pi_2(v) = \{1 \neq t \in \mathbb{Z}_n \mid t \text{ is odd}\}$. Let be $u$ in $V(\Pi)$ such that $\partial(u,v) = 0$; then $u = v = 1$ and $|\Pi_1(v) \cap \Pi_1(u)| = n/2$; hence $b_0 = n/2$ and, by Definition 1, $a_0 = n/2 - b_0 = 0$. Also, if $u$ in $V(\Pi)$ and $\partial(u,v) = 1$, then two vertices $u, v$ are adjacent in $\Pi$, so $|\Pi_0(v) \cap \Pi_1(u)| = 1$ and $|\Pi_2(v) \cap \Pi_1(u)| = n/2 - 1$; hence $c_1 = 1$, $b_1 = n/2 - 1$, and $a_1 = n/2 - b_1 - c_1 = 0$. Finally, if $u$ in $V(\Pi)$ and $\partial(u,v) = 2$, then two vertices $u, v$ are not adjacent in $\Pi$, so $|\Pi_1(v) \cap \Pi_1(u)| = n/2$; hence $c_2 = n/2$ and $a_2 = n/2 - c_2 = 0$. So the intersection array of $\Pi$ is $\{n/2, n/2-1; 1, n/2\}$. Therefore, by Theorem 4, the tridiagonal $(3) \times (3)$ matrix

$$\begin{bmatrix} a_0 & b_0 & 0 \\ c_1 & a_1 & b_1 \\ 0 & c_2 & a_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{n}{2} & 0 \\ 1 & 0 & \frac{n}{2} - 1 \\ 0 & \frac{n}{2} & 0 \end{bmatrix} \tag{9}$$

determines all the eigenvalues of $\Pi$. It is clear that all the eigenvalues of $\Pi$ are $n/2, -n/2, 0$, and their multiplicities are $1, 1, n-2$, respectively. So $\Pi$ is an integral graph. □

*Conclusion 10.* Let $\Pi = \mathrm{Cay}(\mathbb{Z}_n, \Omega)$ be the Cayley graph on the cyclic group $\mathbb{Z}_n$ as before with the adjacency matrix $M = \left[\begin{smallmatrix} A & A \\ A & A \end{smallmatrix}\right]$, and characteristic polynomial $\Phi_\Pi(x)$ then is

$$\begin{aligned} \Phi_{2A}(x) &= \det\left(xI_{n/2} - 2A\right) \\ &= \left(x - \frac{n}{2}\right)\left(x + \frac{n}{2}\right) x^{(n-4)/2}. \end{aligned} \tag{10}$$

*Proof.* It is easy to show that the adjacency matrix $M = \left[\begin{smallmatrix} A & A \\ A & A \end{smallmatrix}\right]$, where $A$ is $n/2 \times n/2$ matrix; hence, by Proposition 9 and Theorem 5,

$$\begin{aligned} \Phi_\Pi(x) &= \det\left(xI_n - M\right) = \det\begin{bmatrix} xI_{n/2} - A & -A \\ -A & xI_{n/2} - A \end{bmatrix} \\ &= \det\left[\left(xI_{n/2} - A\right)^2 - A^2\right] \\ &= \det\left[\left(xI_{n/2} - A\right) - A\right] \det\left[\left(xI_{n/2} - A\right) + A\right] \\ &= \det\left(xI_{n/2} - 2A\right) \det\left(xI_{n/2}\right) = \Phi_{2A}(x)\, x^{n/2} \\ &= \left(x - \frac{n}{2}\right)\left(x + \frac{n}{2}\right) x^{(n-4)/2} x^{n/2}. \end{aligned} \tag{11}$$

□

**Proposition 11.** *Let $\Lambda = Cay(\mathbb{Z}_n, \Psi_m)$ be the Cayley graph on the cyclic group $\mathbb{Z}_n$ as before with the adjacency matrix $\widehat{M}$ and characteristic polynomial $\Phi_\Lambda(x)$. If $n \geq 8$ and $n/2$ is an even integer, then*

$$\begin{aligned} \Phi_\Lambda(x) = &\left(x - \left(\frac{n}{2} + 1\right)\right)\left(x - \left(1 - \frac{n}{2}\right)\right) \\ &\cdot (x-1)^{(n-4)/2} (x+1)^{n/2}. \end{aligned} \tag{12}$$

*Proof.* It is easy to show that the adjacency matrix $\widehat{M} = \left[\begin{smallmatrix} A & A+I \\ A+I & A \end{smallmatrix}\right]$, where $A$ is $n/2 \times n/2$ matrix; hence, by Conclusion 10 and Theorem 5,

$$\begin{aligned} \Phi_\Lambda(x) &= \det\left(xI_n - \widehat{M}\right) \\ &= \det\begin{bmatrix} xI_{n/2} - A & -(A+I) \\ -(A+I) & xI_{n/2} - A \end{bmatrix} = \det\left[\left(xI_{n/2} - A\right)^2\right. \\ &\left. - (A+I)^2\right] = \det\left[\left(xI_{n/2} - A\right)\right. \\ &\left. - (A+I)\right] \det\left[\left(xI_{n/2} - A\right) + (A+I)\right] \\ &= \det\left[(x-1) I_{n/2} - 2A\right] \det\left[(x+1) I_{n/2}\right] \\ &= \Phi_{2A}(x-1)(x+1)^{n/2} = \left(x - \left(\frac{n}{2} + 1\right)\right)\left(x\right. \\ &\left. - \left(1 - \frac{n}{2}\right)\right)(x-1)^{(n-4)/2}(x+1)^{n/2}. \end{aligned} \tag{13}$$

□

*Conclusion 12.* Let $\Lambda = \mathrm{Cay}(\mathbb{Z}_n, \Psi_m)$ be the Cayley graph on the cyclic group $\mathbb{Z}_n$, where $n$ is an even integer, $m = n/2 + 1$, and $\Psi_m = \Omega \cup \{n/2\}$ is the inverse-closed subset of $\mathbb{Z}_n - \{0\}$. If $n \geq 8$ and $n/2$ is an even integer, then the adjacency matrix spectrum of $\Lambda$ is $(n/2+1)^1, (1-n/2)^1, (1)^{(n-4)/2}, (-1)^{n/2}$.

## Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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