

Research Article

A Study of Chaos in Dynamical Systems

S. Effah-Poku , W. Obeng-Denteh, and I. K. Dontwi

Department of Mathematics, Kwame Nkrumah University of Science and Technology, Kumasi, Ghana

Correspondence should be addressed to S. Effah-Poku; seffahpoku@gmail.com

Received 5 July 2017; Accepted 4 February 2018; Published 8 March 2018

Academic Editor: Liwei Zhang

Copyright © 2018 S. Effah-Poku et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The behavior of systems such as periodicity, fixed points, and most importantly chaos has evolved as an integral part of mathematics, especially in dynamical system. This research presents a study on chaos as a property of nonlinear science. Systems with at least two of the following properties are considered to be chaotic in a certain sense: bifurcation and period doubling, period three, transitivity and dense orbit, sensitive dependence to initial conditions, and expansivity. These are termed as the routes to chaos.

1. Introduction

Dynamical systems are part of life. Quite often it has been studied as an abstract concept in mathematics. Chaos is one of the few concepts in mathematics which cannot usually be defined in a word or statement. Most dynamical systems are considered chaotic depending on the either the topological or metric properties of the system. Yorke et al., 1976, concluded that period three implies chaos. They discussed how a dynamical system with period three orbits gives an assurance that the system is chaotic. Several approaches and conditions are factored before the construction of any definition of chaos [1]. R. L. Devaney has one of the most popular and accepted definitions of chaos in which such systems must exhibit sensitive dependence to initial conditions, topological transitivity, and dense periodic orbits [2]. Later it was proven that if a system is transitive with dense periodic orbits then obviously sensitivity dependence to initial condition is guaranteed.

The deterministic nature of these systems does not make them predictable and this is due to chaos theory. Properties of an unknown map can be associated with that of the known via topological conjugacy; hence properties of unknown maps can always be studied in terms of the unknown. The tent map and logistic maps are two known chaotic maps. At a particular point in time, a certain type of chaos may imply or could be equivalent to another type of chaos depending on the routes of chaos the system exhibits.

The size of the scrambled set of systems is one of the most usually considered conditions in terms of defining chaos in systems [1].

2. Preliminaries

Definition 1. Given dynamical system $f : S \rightarrow S$, $S \subseteq \mathbb{R}$, the iterations of the function f are the composition of a function with itself.

If $f^1(x)$ and $f^2(x)$ represent the composition of the function with itself once and twice, respectively, the k th iteration of f at a point x represents the k times composition of f with itself. It is written as $f^k(x)$.

Definition 2. The orbit of a point x in X is the set $\text{Orb}(X, T) = \{x, Tx, T^2x, \dots\}$.

The individual elements of the set $\text{Orb}(X, T)$ represent the path of the iteration for a given function. These represent the trajectory of the function or system.

Definition 3. Assume $m \in \mathbb{Z}_+$, x is a periodic point or a period m -orbit if $f^m(x) = x$.

Under the circumstance the orbit of x is called a periodic orbit or a period- m orbit. The set of all iterations of iterations of a periodic forms a periodic orbit.

Definition 4 (Li et al., 2015). Let (X, T) be a dynamical system. A pair $(x, y) \in X \times X$ is called scrambled if

$$\liminf_{n \rightarrow \infty} d(T^n x, T^n y) = 0, \quad (1)$$
$$\limsup_{n \rightarrow \infty} d(T^n x, T^n y) > 0.$$

3. Routes to Chaos and the Types of Chaos

3.1. Sensitive Dependence to Initial Conditions and Lyapunov Exponents. Let X be a compact metric space and T a continuous map. A dynamical system (X, T) has sensitivity dependence on initial conditions if $\exists \delta > 0$ such that, for $x \in X$ and each $\epsilon > 0$, there is $y \in X$ with $d(x, y) < \epsilon$ and $n \in \mathbb{N}$ such that $d(T^n x, T^n y) > \delta$.

This idea of sensitivity dependence is otherwise called *butterfly effect*. This is perhaps due to any of these reasons or even more; lost patterns and the great effects from marginal or negligible inputs like the flap of the butterfly wings. Generally this is experienced in nonlinear science. The smallest error in change in initial condition grows to become as large as the true and actual value of the state. This makes prediction of future behavior impossible but this does not mean the system is not deterministic.

Numerically, sensitivity is measured by Lyapunov exponent such that a positive value implies the system is really sensitive to initial conditions.

This implies that Lyapunov exponents measure the rate of divergence of orbits away from each other.

Let f be a smooth map of the real line. The Lyapunov exponents are calculated using the derivatives of maps in the points of orbits [3]. The Lyapunov number $L(x)$ of the orbit $\{x_1, x_2, x_3, x_4, \dots\}$ is defined as follows:

$$L(x_1) = \lim_{n \rightarrow \infty} \left(|f'(x_1)| \cdots |f'(x_n)| \right)^{1/n}, \quad (2)$$

if this limit exists.

The Lyapunov exponent $h(x_1)$ is defined as

$$\begin{aligned} L(x_1) &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\ln |f'(x_1)| \cdots |f'(x_n)| \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \ln |f'(x_i)| \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\ln \prod_{i=1}^n |f'(x_i)| \right], \end{aligned} \quad (3)$$

if the limit exists.

Example 5 (Lyapunov exponent of the tent map). Given

$$T(x) = \begin{cases} 2x & 0 \leq x \leq \frac{1}{2} \\ 2(1-x) & \frac{1}{2} \leq x \leq 1, \end{cases} \quad (4)$$

the given map is defined on the interval $[0, 1]$.

$$h(x) = \lim_{x \rightarrow \infty} \frac{1}{k} \sum_{i=1}^k \ln |f'(x_i)|. \quad (5)$$

On the interval, the piece continuous function between 0 and $1/2$, $T'(x) = 2$.

Likewise on the interval between $1/2$ and 1, $T'(x) = -2$.

Now $|T'(x)| = 2$ in both cases. Since

$$h(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \ln |T'(x_i)| = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \ln 2 = \ln 2, \quad (6)$$

the tent map has positive Lyapunov exponent and hence is sensitive to initial conditions [4].

3.2. Topological Entropy. Topological entropy defines the rate of exponential growth of the number of distinct orbits (periodic orbits as n tends to infinity).

Entropy is one of the most important quantities in dynamical systems so far as numerical values are concerned. It basically measures the rate of complexity of the dynamical system as time varies largely and towards infinity. For autonomous systems in compact metric space, topological entropy is commutative for composite functions such that $h(f \circ g) = h(g \circ f)$ [3].

Topological entropy can be defined and determined in terms of fixed points of f^n and expressed as

$$H(f) = \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \ln (\text{number of fixed points under the map } f^n). \quad (7)$$

Li Yorke chaos can be implied from entropy if the system is autonomous and in a compact metric space [3].

3.3. Transitivity and Dense Orbits. A dynamical system (X, f) has a dense orbit if and only if

$$d(f^n(x), y) < \epsilon \quad \exists x \in X : \forall y \in X, \forall \epsilon > 0, \quad (8)$$

where $f^n(x)$ represents a specific iteration. The orbits of x , $(f^n(x))$, moves arbitrary close to another orbit at a given time, such that the metric between them is significantly small.

A dynamical system (X, F) is topologically transitive if and only if, for all nonempty subsets U and V of X , there exist n positive such that the iteration of F on U intersects V at least at a point.

Transitivity usually implies the existence of dense orbit, if $\exists x_0 \in X$ such that $\text{Orb}(x_0)$ is a dense subset of X .

Topological transitivity guarantees that there always exists a point which results from the intersection of open sets under iterative process of map.

A point x_0 is considered a transitive point in f if x_0 has a dense orbit under f .

3.4. Expansivity. Let $f : X \rightarrow X$. X is a metric space with a metric d defined on it. The map f is expansive if $\exists \epsilon > 0$ such that, $\forall x, y \in X$, $d(f^n(x), f^n(y)) \geq \epsilon$.

The constant defined by ϵ is called the expansive constant for the map. Points x and y are distinct points and $x \neq y$.

There exists an obvious and more direct link between expansivity and sensitivity dependence on initial conditions. Every expansive map exhibits sensitivity dependence to initial conditions. The converse does not hold and the two conditions are never equivalent.

Expansivity implies sensitive dependence because expansivity deals with the distance between two nearby points and how their orbits separate continuously. In expansivity, the separation is observed between two nearby points by at least the constant ϵ . In sensitive dependence the requirement is

that, at least there should be one point whose orbit moves away from the orbit of another close point after the same number of iterations.

3.5. *Period Three.* Let (X, f) be a dynamical system and be defined by the map.

The map f is said to have a periodic point if for $n > 0$, $f^n(x) = x$.

For a given map, since n is a natural number, the map is said to have periodic point of period three when $f^3(x) = x$. Period three is normally associated with chaos of dynamical systems and was first proved in [5].

Example 6 (systems with period three orbits).

$$f(x) = 2|1 - x|, \quad f: [-2, 2] \rightarrow [-2, 2]. \quad (9)$$

Let

$$\begin{aligned} x_0 &= \frac{2}{7}, \\ f(x_0) &= f\left(\frac{2}{7}\right) = \frac{10}{7}, \\ x_1 &= \frac{10}{7}, \\ f^2(x_0) &= f(x_1) = f\left(\frac{10}{7}\right) = \frac{6}{7}, \\ x_2 &= \frac{6}{7}, \\ f^3(x_0) &= f(x_2) = f\left(\frac{6}{7}\right) = \frac{2}{7}. \end{aligned} \quad (10)$$

Since

$$f^3(x_0) = (x_0) \quad (11)$$

then the map has a cycle of period or the map has period three point.

Example 7 (systems with period three orbits). Given

$$T_2(x) = \begin{cases} 2x & 0 \leq x \leq \frac{1}{2} \\ 2(1-x) & \frac{1}{2} \leq x \leq 1, \end{cases} \quad (12)$$

the given map is defined on the interval $[0, 1]$. We want to show that the tent function has period three cycles.

Let $x_0 = 2/7$

$$\begin{aligned} T_2(x_0) &= T\left(\frac{2}{7}\right) = \frac{4}{7}, \\ x_1 &= \frac{4}{7}, \end{aligned}$$

$$T_2^2(x_0) = T_2(x_1) = T\left(\frac{4}{7}\right) = \frac{6}{7},$$

$$x_2 = \frac{6}{7},$$

$$T_2^3(x_0) = T_2(x_2) = T\left(\frac{6}{7}\right) = \frac{2}{7},$$

$$x_3 = \frac{2}{7}. \quad (13)$$

Since

$$T_2^3(x_0) = (x_0) \quad (14)$$

then the tent function has a period three cycle [6].

Theorem 8 (period three theorem). *Let $f : R \rightarrow R$ be a continuous function. If f has periodic points of period three, then f has periodic point of all other periods.*

3.6. *Bifurcation and Period Doubling.* Bifurcation is defined as the structural changes of a dynamical system as a result of the alteration or changes in the parameter value. This change is usually sudden and could be topological or qualitative. In such cases, we expect the dynamical system to be a function of both the dependent variable and the parameter in context. An example is $x' = f(x, \mu)$.

The idea of bifurcation can be grouped into two, which are global bifurcation and local bifurcation. In local bifurcation, the interest is the change that happens in the system near the fixed point. It is usually analyzed through changes in stability properties and periodic orbits. Global bifurcation occurs when larger invariant sets of the system collide with each other.

Period doubling is the splitting of a trajectory into two during iteration. This unusual scene is influenced by the parameter value most of the time. Chaos is observed as the period doubling increases. This is because the path of trajectory at certain points is mixed up and inseparable making the system chaotic.

3.7. *Types of Chaos*

3.7.1. *Li Yorke Chaos.* Let $x, y \in X$. The pair $(x, y) \in (X, X)$ is a Li Yorke scrambled pair if

$$\begin{aligned} \limsup_{n \rightarrow \infty} d(f^n(x), f^n(y)) &> 0, \\ \liminf_{n \rightarrow \infty} d(f^n(x), f^n(y)) &= 0. \end{aligned} \quad (15)$$

A map is Li Yorke chaotic if it has uncountable scrambled set in X (Li et al., 2015).

(x, y) is proximal but not asymptotic. Li Yorke chaotic systems have a two-point distributional scrambled set which is equivalent to an uncountable scrambled set [1].

3.7.2. *Devaney Chaos* (see [2]). Let X be a metric space. A continuous map $f : X \rightarrow X$ is said to be chaotic if

- (1) f is transitive,
- (2) the periodic orbit of f is dense in X ,
- (3) f has sensitivity dependence on initial conditions.

3.7.3. *Wiggins Chaos (see [7]).* Let $f : X \rightarrow X$ be continuous map and X a metric space. The map is considered to be chaotic if

- (1) f is topologically transitive,
- (2) f has sensitivity dependence on initial conditions.

3.8. *Lyapunov Definition of Chaos (see [7]).* Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous differentiable map. The map f is said to be chaotic if

- (1) f is topological transitive,
- (2) f has positive Lyapunov exponent.

3.9. *Knudsen Chaotic System.* Let $F : X \rightarrow X$ be a continuous map on a metric space (x, d) , then the dynamical system $\langle X, F \rangle$ is chaotic according to Knudsen's definition iff

- (1) F has dense orbits,
- (2) F is sensitive to IC.

3.10. *Positive Expansive Chaotic System.* Let $F : X \rightarrow X$ be a continuous map on a perfect metric space (x, d) .

The dynamical system is positively expansive chaotic (E-chaotic) iff

- (1) F is topologically transitive,
- (2) F has dense periodic orbits,
- (3) F is positively expansive.

4. Interrelation of Various Types of Chaos

Let $p : X \rightarrow X$ and $q : Y \rightarrow Y$ be two mappings. p is topologically conjugated to q if there exists a homeomorphism $r : Y \rightarrow Y$ such that $r \circ p = q \circ r$.

Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = 4x(1 - x)$ and $g(x) = 2x^2 - 1$.

We verify that f and g are conjugated via $h(x) = -2x + 1$ as follows:

$$\begin{aligned} h(f(x)) &= h(4x(1 - x)) = -2(4x(1 - x)) + 1 \\ &= -2(4x - 4x^2) + 1 = -8x + 8x^2 + 1 \\ &= 8x^2 - 8x + 1. \end{aligned} \quad (16)$$

Also,

$$\begin{aligned} g(h(x)) &= g(-2x + 1) = 2((-2x + 1)(-2x + 1)) - 1 \\ &= 2(4x^2 - 2x - 2x + 1) - 1 \\ &= 8x^2 - 8x + 2 - 1 = 8x^2 - 8x + 1. \end{aligned} \quad (17)$$

Since $h \circ f = g \circ h$ and h is a homeomorphism of \mathbb{R} , f and g are conjugates.

Since the mappings in all two examples are conjugated topologically, they share the same topological properties. Hence if any of the mappings is identified with any topological property, we can as well associate the other mapping with the same property and vice versa.

At this point we take a study through the equivalent relations in various types of chaos and the implications as well.

Devaney chaos implies Wiggins chaos, Knudsen chaos, and Lyapunov chaos. Sensitivity to initial conditions is a common route in Devaney, Wiggins, and Knudsen chaos while positive Lyapunov exponents which are equivalent to sensitive dependence are used in Lyapunov chaos. Now, Devaney chaos combines both topological transitivity and the existence of dense orbits. Since Wiggins, Lyapunov, and Knudsen chaos depend on either of the two topological conditions, then Devaney chaos implies them all. A system that is Devaney chaotic has to be Wiggins chaotic, Lyapunov chaotic, and Knudsen chaotic.

Positive expansive implies Wiggins chaos, Knudsen chaos, and Lyapunov chaos. Wiggins definition, Knudsen chaos, and Lyapunov's definitions satisfies either the condition of transitivity or dense orbits.

In relation to Wiggins chaos, their distinct conditions are positive expansiveness and sensitive dependence to initial conditions. Expansivity implies sensitive dependence; hence positive expansive chaos implies Wiggins chaos.

In the case of Knudsen chaos, beyond the existence of dense orbit, the distinct condition is also sensitivity. Similarly, since expansivity implies sensitive dependence, expansive chaos implies Knudsen chaos.

Also for positive Lyapunov exponents, the distinct condition is expansivity and positive Lyapunov exponents. Every expansive map has a positive Lyapunov exponent but the converse is not true. As said earlier, if just two orbits separate apart at a point in the iteration, the Lyapunov will be positive but the map might not necessarily be expansive.

Wiggins chaos and Knudsen chaos imply each other and are quite similar or equivalent. They share a common property of transitivity. They also share an equivalent property of sensitivity and positive Lyapunov exponents. The difference in the two definitions is the space on which it is defined. Wiggins considers a continuous map on a metric while Lyapunov deals with differentiable maps.

Knudsen chaos shares an equivalent property of sensitivity dependence to initial condition and positive Lyapunov exponents with Wiggins definition and Lyapunov's definition. In some systems, transitivity is equivalent to dense orbits though not always. In such systems, all three types of chaos are the same apart from the space on which each is defined.

Devaney chaos and Li Yorke chaos are interrelated via topological entropy. Devaney chaos implies positive topological entropy and the converse is not true. Positive topological entropy implies Li Yorke chaos and here too the converse does not hold. According to the law of transitivity in analysis Devaney chaos implies Li Yorke chaos. On the interval map Devaney chaos is the strongest whereas Li Yorke's chaos is the weakest.

From above, Devaney chaos implies Wiggins chaos, Lyapunov chaos, and Knudsen chaos. Positive expansive chaos implies Wiggins chaos, Lyapunov chaos, and Knudsen chaos. Possibly then, there should be a link between the two. The two definitions share two common properties of topological transitivity and dense periodic orbit. Positive expansivity is a stronger property than sensitive dependence to initial conditions. Since positive expansivity implies sensitive dependence, the positive expansive chaos implies Devaney chaos.

Every Expanding Map Is Chaotic. Expansivity implies topological mixing which implies transitivity. When chaos of a system is defined in terms of expansivity, transitivity which is considered as one of the strongest and essential condition for chaos is captured.

There should always be a way to calculate chaos or numerically determine chaos. Entropy usually is quite difficult to obtain numerically compared to Lyapunov exponent. Since every expansive map has positive Lyapunov exponent, then by confirming the expansiveness of a map, chaos is possible if not guaranteed.

Lastly, that gives a more stronger form of chaos in that the existence of sensitive dependence alone in itself would not and cannot guarantee chaos in dynamical systems. Hence other routes become relevant in the definition of the types of chaos.

5. Conclusion

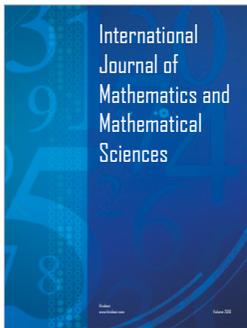
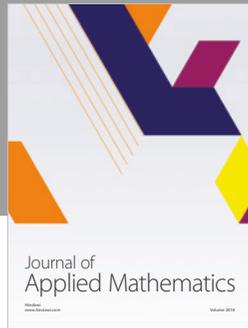
Chaos as a nonlinear science has become part of our daily lives. There is an implication or equivalence relation between the kinds of chaos based on the particular routes to chaos. Topological conjugacy is one of the means of associating topological properties of a map with known properties and a map with unknown topological properties. Expansive chaotic system implies Lyapunov chaos. Devaney chaos implies Wiggins chaos, Knudsen chaos, and Lyapunov chaos. Wiggins chaos and Knudsen chaos imply each other and are quite similar or equivalent. Devaney chaos and Li York chaos are interrelated via topological entropy.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

- [1] J. L. G. Guirao and M. Lampart, "Relations Between Distributional, Li Yorke and ω chaos," *Chaos, Solitons and Fractals*, vol. 28, no. 3, pp. 788–792, 2006.
- [2] J. Banks, J. Brooks, G. Cairns, G. Davis, and P. Stacey, "On Devaney's definition of chaos," *The American Mathematical Monthly*, vol. 99, no. 4, pp. 332–334, 1992.
- [3] F. Balibrea, "On problems of topological dynamics in non-autonomous discrete systems," *Applied Mathematics and Nonlinear Sciences*, vol. 1, no. 2, pp. 391–404, 2016.
- [4] H. R. Biswas, "One dimensional chaotic dynamical system," *Journal of Pure and Applied Mathematics: Advances and Applications*, vol. Volume 10, Number 1, pp. 69–101, 2013.
- [5] T. Y. Li and J. A. Yorke, "Period three implies chaos," *The American Mathematical Monthly*, vol. 82, no. 10, pp. 985–992, 1975.
- [6] P. R. Kulkarni and V. C. Borkar, "Period three cycle and chaos in a dynamical system," *International Journal of Multidisciplinary Research and Development*, vol. 2, no. 4, pp. 591–594, 2015, <http://www.allsubjectjournal.com>.
- [7] A. Fatiou, *Deterministic chaos*, Msc project, University of London, 2005.



Hindawi

Submit your manuscripts at
www.hindawi.com

