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Research Article

Solving Oscillatory Delay Differential Equations Using Block Hybrid Methods

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A set of order condition for block explicit hybrid method up to order five is presented and, based on the order conditions, two-point block explicit hybrid method of order five for the approximation of special second order delay differential equations is derived. The method is then trigonometrically fitted and used to integrate second-order delay differential equations with oscillatory solutions. The efficiency curves based on the log of maximum errors versus the CPU time taken to do the integration are plotted, which clearly demonstrated the superiority of the trigonometrically fitted block hybrid method.

1. Introduction

Differential equations with a time delay are used to model the process which does not only depend on the current state of a system but also the past states. This type of equation is called delay differential equations (DDEs) in which the derivative at any time depends on the solution at prior times. The special second order DDE can be written in the form of

$$\begin{aligned} y''(t) &= f(t, y(t), y(t-\tau)), \quad a \le t \le b, \\ y(t_0) &= y_0, \\ y'(t_0) &= y'_0, \\ & \qquad t \in [-\tau, a] \end{aligned} \tag{1}$$

where τ is the delay term and the first derivative does not appear explicitly. It is a more realistic model which includes some of the past history of the system to determine the future behavior. DDEs have become an important criteria to investigate the complexities of the real-world problems concerning infectious diseases, biotic population, neuronal networks, and population dynamics.

Methods such as Runge-Kutta (RK), Runge-Kutta Nyström (RKN), hybrid, and multistep are widely used for solving DDEs. Ismail et al. [1] used RK method and Hermite interpolation to solve first-order DDEs. Taiwo and Odetunde [2] worked on decomposition method as an integrator for delay differential equations. Some authors also derived block linear multistep method (LMM) to solve DDEs; and such work can be seen in [3–6]. Hoo et al. [7] constructed Adams-Moulton Method for directly solving second-order DDEs. Mechee et al. [8] in their paper has adapted RKN for directly solving second-order DDEs.

However, all the studies previously mentioned have not been applied for solving DDE problems with oscillatory properties. Hence, in this paper we derived order condition for block explicit hybrid method up to order five using computer algebra system as proposed in Gander and Gruntz [9]. The reason to derive block hybrid method is that a faster numerical solutions can be obtained since the method approximates the solution at more than one point per step. From the order conditions we constructed a two-point three-stage fifth-order block explicit hybrid method which is then trigonometrically fitted so that it is suitable for solving second-order DDEs which are oscillatory in nature. Finally, we tested the new methods using DDEs test problems to indicate that it is superior and more efficient for solving oscillatory second order DDEs.

2. Derivation of Order Condition for Block Explicit Hybrid Method

The general formula of two-step explicit hybrid method for solving the special second-order ordinary differential equations is given as

$$Y_i = (1 + c_i)\, y_n - c_i y_{n-1} + h^2 \sum_{j=1}^{s} a_{ij} f\left(x_n + c_j h, Y_j\right), \quad (2)$$

$$y_{n+1} = 2y_n - y_{n-1} + h^2 \sum_{i=1}^{s} b_i f\left(x_n + c_i h, Y_i\right) \quad (3)$$

where $i = 1, \ldots, s$, and $i > j$. The method coefficients of b_i, c_i, and a_{ij} can be represented in Butcher tableau as follows:

The explicit hybrid methods

$$\begin{array}{c|c} \boldsymbol{c} & \boldsymbol{A} \\ \hline & \boldsymbol{b}^T \end{array} \quad (4)$$

where $\boldsymbol{A} = \begin{bmatrix} a_{11} & & \\ \vdots & \ddots & \\ a_{s1} & \cdots & a_{s,s-1} \end{bmatrix}$, $\boldsymbol{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_s \end{bmatrix}$, and $\boldsymbol{b}^T = \begin{bmatrix} b_1 \\ \vdots \\ b_s \end{bmatrix}$.

In this section, we derived the order condition for block explicit hybrid method (BEHM). The Taylor series expansion for y_{n+m} and y_{n-m} is as follows:

$$y_{n+m} = y_n + mhy_n{}' + \frac{m^2}{2!}h^2 y_n{}'' + \frac{m^3}{3!}h^3 y_n{}''' + \frac{m^4}{4!}h^4 y_n{}^{(4)} + \cdots \quad (5)$$

$$y_{n-m} = y_n - mhy_n{}' + \frac{m^2}{2!}h^2 y_n{}'' - \frac{m^3}{3!}h^3 y_n{}''' + \frac{m^4}{4!}h^4 y_n{}^{(4)} + \cdots \quad (6)$$

where h is step sizes and m is the number of step forward. Adding (5) and (6), we obtain

$$y_{n+m} - y_{n-m} = 2y_n + m^2 h^2 y_n{}'' + \frac{m^4}{12}h^4 y_n{}^{(4)} + \cdots. \quad (7)$$

or

$$y_{n+m} = 2y_n - y_{n-m} + m^2 h^2 y_n{}'' + \frac{m^4}{12}h^4 y_n{}^{(4)} + \cdots \quad (8)$$

which can be expressed as

$$y_{n+m} = 2y_n - y_{n-m} + h^2 \Delta\left(x_n, y_n, h\right) \quad (9)$$

The increment function is

$$\Delta\left(x_n, y_n, h\right) = m^2 y_n{}'' + \frac{m^4}{12}h^2 y_n{}^{(4)} + \cdots. \quad (10)$$

Then, consider the general formula for block explicit hybrid method in the form of

$$y_{n+m} = 2y_n - y_{n-m} + h^2 \sum_{i=1}^{s}{}^{(m)} b_i f\left(x_n, y_n, h\right) \quad (11)$$

Equation (11) can be simplified as

$$y_{n+m} = 2y_n - y_{n-m} + h^2 \phi\left(x_n, y_n, h\right) \quad (12)$$

such that the Taylor series increment may be written as

$$\begin{aligned} \phi\left(x_n, y_n, h\right) &= \sum_{i=1}^{s}{}^{(m)} b_i f\left(x_n + c_i h, Y_i\right) \\ &= \sum_{i=1}^{s}{}^{(m)} b_i F_1^{(2)} + h \sum_{i=1}^{s}{}^{(m)} b_i c_i F_1^{(3)} \\ &\quad + \frac{h^2}{2}\left(\sum_{i=1}^{s}{}^{(m)} b_i c_i{}^2 F_1^{(4)} + \sum_{i=1}^{s}{}^{(m)} b_i a_{ij} F_2^{(4)}\right) \end{aligned} \quad (13)$$

where the first few elementary differential are

$$F_1^{(2)} = f,$$
$$F_1^{(3)} = f_x + f_y y',$$
$$F_1^{(4)} = f_{xx} + 2y' f_{xy} + (y')^2 f_{yy} + f f_y.$$

The local truncation errors of the solution is obtain by subtracting (12) from (9). This gives us

$$t_{n+1} = h^2\left[\phi - \Delta\right] = h^2\left[\sum_{i=1}^{s}{}^{(m)} b_i f\left(x_n + c_i h, Y_i\right) - \left(m^2 y_n{}'' + \frac{m^4}{12}h^2 y_n{}^{(4)} + \cdots\right)\right] \quad (14)$$

Therefore, from (14), we obtain the order condition for block explicit hybrid method up to order 5 as follows:

$$\begin{aligned} &\text{Order 2:} && \sum_{i=1}^{s}{}^{(m)} b_i = m^2 \\ &\text{Order 3:} && \sum_{i=1}^{s}{}^{(m)} b_i c_i = 0 \\ &\text{Order 4:} && \sum_{i=1}^{s}{}^{(m)} b_i c_i{}^2 = \frac{m^4}{6} \\ & && \sum_{i=1}^{s}{}^{(m)} b_i a_{ij} = \frac{m^4}{12} \\ &\text{Order 5:} && \sum_{i=1}^{s}{}^{(m)} b_i c_i{}^3 = 0 \\ & && \sum_{i=1}^{s}{}^{(m)} b_i c_i a_{ij} = \frac{m^4}{12} \\ & && \sum_{i=1}^{s}{}^{(m)} b_i a_{ij} c_j = 0 \end{aligned} \quad (15)$$

for $m = 1, 2$.

For the first point ($m = 1$), the order condition is the same as the order conditions for explicit hybrid method given in Coleman [10], while, for the second point ($m = 2$), the order conditions are given as follows:

$$\text{Order 2:} \quad \sum_{i=1}^{s} {}^{(2)}b_i = 4 \tag{16}$$

$$\text{Order 3:} \quad \sum_{i=1}^{s} {}^{(2)}b_i c_i = 0 \tag{17}$$

$$\text{Order 4:} \quad \sum_{i=1}^{s} {}^{(2)}b_i c_i^2 = \frac{8}{3} \tag{18}$$

$$\sum_{i=1}^{s} {}^{(2)}b_i a_{ij} = \frac{4}{3} \tag{19}$$

$$\text{Order 5:} \quad \sum_{i=1}^{s} {}^{(2)}b_i c_i^3 = 0 \tag{20}$$

$$\sum_{i=1}^{s} {}^{(2)}b_i c_i a_{ij} = \frac{4}{3} \tag{21}$$

$$\sum_{i=1}^{s} {}^{(2)}b_i a_{ij} c_j = 0 \tag{22}$$

(Note: The notation ${}^{(m)}b_i$ means the coefficient of b for mh step forward.)

The general formula for block explicit hybrid method (BEHM) for $m = 1, 2$ can be written as

$$Y_i = (1 + c_i)\, y_n - c_i y_{n-1} + h^2 \sum_{j=1}^{s} a_{ij} f\left(x_n + c_j h, Y_j\right), \tag{23}$$

$$y_{n+1} = 2y_n - y_{n-1} + h^2 \sum_{i=1}^{s} {}^{(1)}b_i f\left(x_n + c_i h, Y_i\right). \tag{24}$$

$$y_{n+2} = 2y_n - y_{n-2} + h^2 \sum_{i=1}^{s} {}^{(2)}b_i f\left(x_n + c_i h, Y_i\right) \tag{25}$$

3. Construction of Trigonometrically Fitted Block Explicit Hybrid Method

In order to derive the two-point BEHM, we used the algebraic coefficients of the original three-stage explicit hybrid method in Franco [11] as the first point ($m = 1$). The block method can be written as follows:

The coefficient of the block explicit hybrid method

$$\begin{array}{c|c} \boldsymbol{c} & \boldsymbol{A} \\ \hline & {}^{(1)}\boldsymbol{b}^T \\ & {}^{(2)}\boldsymbol{b}^T \end{array} = \begin{array}{c|ccc} -1 & & & \\ 0 & & & \\ 1 & 0 & 1 & \\ \hline & \frac{1}{12} & \frac{5}{6} & \frac{1}{12} \\ & {}^{(2)}b_1 & {}^{(2)}b_2 & {}^{(2)}b_3 \end{array} \tag{26}$$

First and second point of the BEHM share the same values of $\boldsymbol{c}$ and $\boldsymbol{A}$. By solving (16)-(19) simultaneously, we obtain a unique solution of ${}^{(2)}b_i$.

$$\begin{aligned} {}^{(2)}b_1 &= \frac{4}{3}, \\ {}^{(2)}b_2 &= \frac{4}{3}, \\ {}^{(2)}b_3 &= \frac{4}{3} \end{aligned} \tag{27}$$

By checking the fifth-order conditions for the first point, we noticed that the method at the first point (${}^{(1)}b_1 = 1/12, {}^{(1)}b_2 = 5/6, {}^{(1)}b_3 = 1/12, c_1 = -1, c_2 = 0, c_3 = 1, a_{31} = 0, a_{32} = 1$) satisfies $\sum_{i=1}^{3} {}^{(1)}b_i c_i^3 = 0$, $\sum_{i=1}^{3} {}^{(1)}b_i c_i a_{ij} = 1/12$, $\sum_{i=1}^{3} {}^{(1)}b_i a_{ij} c_j = 0$.

Hence, the method in Franco [11] is order five.

And by checking fifth-order conditions for the second point, we noticed that the method at the second point satisfies

$$\begin{aligned} &\sum_{i=1}^{3} {}^{(2)}b_i c_i^3 = 0, \\ &\sum_{i=1}^{3} {}^{(2)}b_i c_i a_{ij} = \frac{4}{3}, \\ &\sum_{i=1}^{3} {}^{(2)}b_i a_{ij} c_j = 0 \end{aligned} \tag{28}$$

Hence, the block explicit hybrid method that we have derived is three-stage and order five, denoted as BEHM3(5).

To trigonometrically fit the method, we require (23), (24), and (25) to integrate exactly the linear combination of the functions $\{\sin(\omega t), \cos(\omega t)\}$ for $\in \mathscr{R}$. Hence, the following equations are obtained:

$$\cos(c_3 H) = 1 + c_3 - c_3 \cos(H) - H^2 \{a_{31}\cos(H) + a_{32}\}, \quad (29)$$

$$\sin(c_3 H) = c_3 \sin(H) + H^2 \{a_{31}\sin(H)\}, \quad (30)$$

$$2\cos(H) = 2 - H^2 \left\{ {}^{(1)}b_1 \cos(H) + {}^{(1)}b_2 + {}^{(1)}b_3 \cos(c_3 H) \right\}, \quad (31)$$

$${}^{(1)}b_1 \sin(H) = {}^{(1)}b_3 \sin(c_3 H). \quad (32)$$

$$2\cos(2H) = 2 - H^2 \left\{ {}^{(2)}b_1 \cos(H) + {}^{(2)}b_2 + {}^{(2)}b_3 \cos(c_3 H) \right\}, \quad (33)$$

$${}^{(2)}b_1 \sin(H) = {}^{(2)}b_3 \sin(c_3 H). \quad (34)$$

where $H = h\omega$, h is step size, and ω is the fitted frequency of the problem (ω depends on the problems).

By solving (29), (30) with $c_1 = -1$, $c_2 = 0$, and $c_3 = 1$, we obtain the remaining values in terms of H as follows:

$$a_{31} = 0, \qquad a_{32} = -\frac{2(\cos(H) - 1)}{H^2}. \quad (35)$$

To find ${}^{(1)}b_i$ values, we solve (31), (32) with choice of coefficients $c_1 = -1$, $c_2 = 0$, and $c_3 = 1$, we obtained the following:

$${}^{(1)}b_1 = -\frac{1}{2}\frac{2\cos(H) - 2 + H^2}{H^2(\cos(H) - 1)}, \qquad {}^{(1)}b_2 = \frac{2\cos(H) - 2 + H^2\cos(H)}{H^2(\cos(H) - 1)}, \qquad {}^{(1)}b_3 = -\frac{1}{2}\frac{2\cos(H) - 2 + H^2}{H^2(\cos(H) - 1)}. \quad (36)$$

Then, for the ${}^{(2)}b_i$ values, we solve (33), (34) with choice of coefficients $c_3 = 1$ and letting ${}^{(2)}b_1 = 4/3$ and ${}^{(2)}b_3 = 4/3$, we obtained ${}^{(2)}b_2$ in terms of H as

$${}^{(2)}\mathrm{b}_2 = \frac{-2}{3} \cdot \frac{3\cos(2H)\sin(H) - 3\sin(H) + 4H^2\cos(H)\sin(H)}{\mathrm{H}^2 \sin(H)}. \quad (37)$$

Hence, we denote the new method as three-stage fifth-order trigonometrically fitted block explicit hybrid method (TF-BEHM3(4)). The method is shown as follows:

The coefficient of the trigonometrically fitted block explicit hybrid method

(TF-BEHM3(5))

$$\begin{array}{c|ccc} -1 & & & \\ 0 & & & \\ 1 & 0 & a_{32} & \\ \hline & {}^{(1)}b_1 & {}^{(1)}b_2 & {}^{(1)}b_3 \\ & \frac{4}{3} & {}^{(2)}b_2 & \frac{4}{3} \end{array} \quad (38)$$

The coefficients can be written in Taylor expansion, to avoid heavy cancellation in the implementation of the method.

$$a_{32} = 1 - \frac{1}{2}H^2 + \frac{1}{360}H^4 - \frac{1}{20160}H^6 + \frac{1}{1814400}H^8 + O(H^{10}),$$

$${}^{(1)}b_1 = \frac{1}{12} + \frac{1}{240}H^2 + \frac{1}{6048}H^4 + \frac{1}{172800}H^6 + O(H^8),$$

$${}^{(1)}b_2 = \frac{5}{6} - \frac{1}{120}H^2 - \frac{1}{3024}H^4 - \frac{1}{86400}H^6 + O(H^8),$$

$${}^{(1)}b_3 = \frac{1}{12} + \frac{1}{240}H^2 + \frac{1}{6048}H^4 + \frac{1}{172800}H^6 + O(H^8),$$

$$^{(2)}b_2 = \frac{4}{3} + \frac{1}{15}H^4 - \frac{17}{1890}H^6 + \frac{113}{226800}H^8 - \frac{7}{427680}H^{10} + O\left(H^{11}\right). \tag{39}$$

4. Problems Tested and Numerical Results

In this section, the new methods, BEHM3(5) and TF-BEHM3(5), are used to solve oscillatory delay differential equations problems. The delay terms are evaluated using Newton divided different interpolation. A measure of the accuracy is examined using absolute error which is defined by

$$\text{Absolute error} = \max\left\{\left|y\left(t_n\right) - y_n\right|\right\}, \tag{40}$$

where $y(t_n)$ is the exact solution and y_n is the computed solution.

The test problems are listed as follows.

Problem 1 (source: Schmidt [12]).

$$y''(t) = -\frac{1}{2}y(t) + \frac{1}{2}y(t-\pi), \quad 0 \le t \le 8\pi, \; y_o = 0. \tag{41}$$

The fitted frequency is $v = 1$. Exact solution is $y(t) = \sin(t)$.

Problem 2 (source: Schmidt [12]).

$$y''(t) - y(t) + \eta(t)\, y\left(\frac{t}{2}\right) = 0, \quad 0 \le t \le 2\pi,$$
$$\text{where } \eta(t) = \frac{4\sin(t)}{(2-2\cos(t))^{1/2}}, \; \eta(0) = 4. \tag{42}$$

Fitted frequency is $v = 2$. Exact solution is $y(t) = \sin(t)$.

Problem 3 (source: Ladas and Stavroulakis [13]).

$$y''(t) = y(t-\pi), \quad 0 \le t \le 8\pi, \; y_o = 0. \tag{43}$$

Fitted frequency is $v = 1$. Exact solution is $y(t) = \sin(t)$.

Problem 4 (source: Bhagat Singh[14]).

$$y''(t) = -\frac{\sin(t)}{2-\sin(t)}y(t-\pi), \quad 0 \le t \le 8\pi, \; y_o = 2. \tag{44}$$

Fitted frequency is $v = 1$. Exact solution is $y(t) = 2 + \sin(t)$.

The following notations are used in Figures 1–4:

TF-BEHM3(5): A three-stage fifth-order trigonometrically fitted block explicit hybrid method derived in this paper.

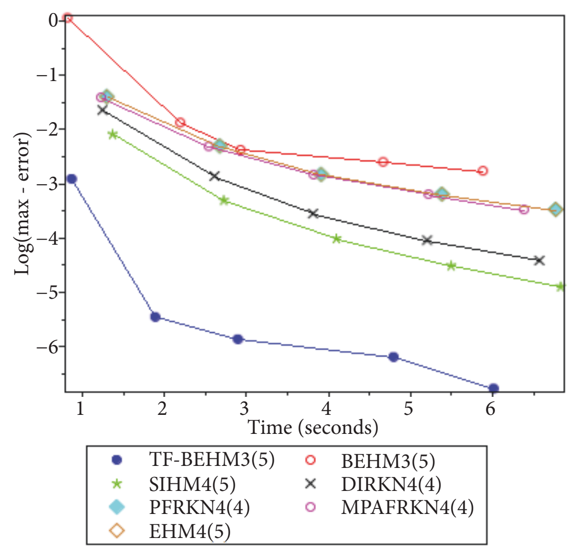

FIGURE 1: The efficiency curve of Problem 1 for $h = \pi/4^i$ ($i = 1, \ldots, 5$).

BEHM3(5): A three-stage fifth-order new block explicit hybrid method derived in this paper.

SIHM4(5): A four-stage fifth-order semi-implicit hybrid method in Ahmad et al.[15]

MPAFRKN4(4): A modified phase-fitted and amplification-fitted RKN method of four-stage fourth-order by Papadopoulos et al. [16].

PFRKN4(4): A phase-fitted RKN method of four-stage fourth-order by Papadopoulos et al. [17].

DIRKN4(4): A four-stage fourth-order diagonally implicit RKN method by Senu et al. [18].

EHM4(5): A four-stage fifth-order phase-fitted hybrid method by Franco [11].

5. Discussion and Conclusion

Figures 1–4 show the efficiency curves: the logarithm of the maximum global error versus the CPU time taken in second. From our observation, for all the problems TF-BEHM3(5) required lesser time to do the computation. In Problems 1, 3, and 4, TF-BEHM3(5) has better accuracy compared to all the methods in comparison. However, for Problem 2, efficiency of TF-BEHM3(5) is comparable to EHM4(5) and has better performance compared to the other methods. For Problems 2, 3, and 4 too BEHM3(4) is comparable to the other existing methods in comparison.

The existing methods, MPAFRKN4(4) and PFRKN4(4), were derived using specific fitting techniques. SIHM4(5), EHM4(5), and DIRKN4(4) were derived with higher order of dispersion and dissipation and all the methods purposely derived for solving oscillatory problems.

In this paper, we derived the order conditions of the block hybrid method up to order five, based on the order conditions we derived a two-point three-stage fifth-order

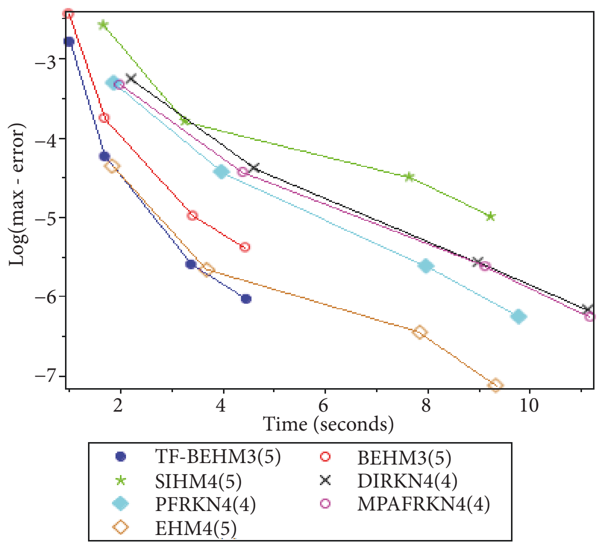

FIGURE 2: The efficiency curve of Problem 2 for $h = \pi/32^i$ ($i = 1, \ldots, 4$).

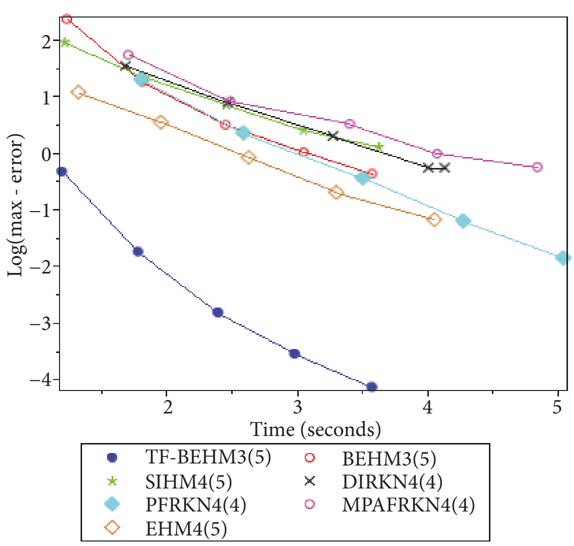

FIGURE 3: The efficiency curve of Problem 3 for $h = \pi/4^i$ ($i = 1, \ldots, 5$).

block explicit hybrid method (BEHM3(5)) and then the method is trigonometrically fitted. Both the fitted and non-fitted methods evaluate the solution of the problem at two points for each time step; hence lesser time is needed to do the computation, making them faster and cheaper compared to the other methods. From the numerical results we can conclude that BEHM3(4) is a good method for directly solving second-order DDEs; however, trigonometrically fitting the block method does improve the efficiency of the method tremendously. It can be said that TF-BEHM3(5) is a promising tool for solving special second-order DDEs with oscillatory solutions.

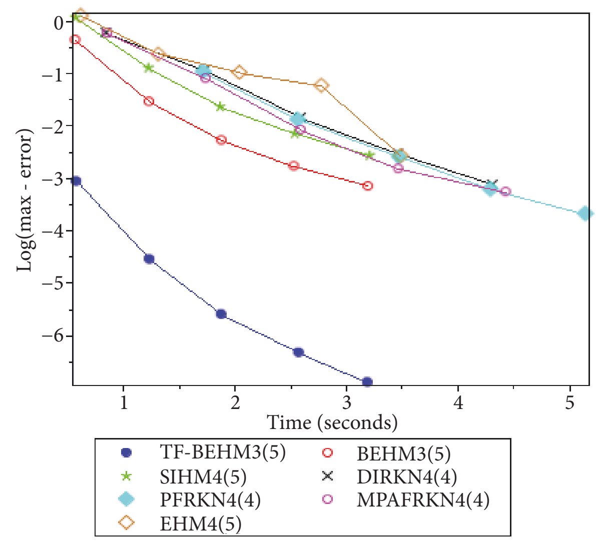

FIGURE 4: The efficiency curve of Problem 4 for $h = \pi/4^i$ ($i = 1, \ldots, 5$).

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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