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Research Article

Reduced Triangular Form of Polynomial 3-by-3 Matrices with One Characteristic Root and Its Invariants

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In this paper the semiscalar equivalence of polynomial matrices is investigated. We introduce the notion of the so-called reduced triangular form with respect to semiscalar equivalence for the 3-by-3 matrices with one characteristic root and indicate the invariants of this reduced form.

1. Introduction

We consider the following equivalence relation in the set of all polynomial matrices of fixed order over the field $\mathbf{C}$ of complex numbers: matrices $F(x), G(x)$ are called semiscalarly equivalent if there exist invertible matrices $P, Q(x)$ over $\mathbf{C}$ and $\mathbf{C}[x]$, respectively, such that $G(x) = PF(x)Q(x)$ [1] (see also [2]); notation $F(x) \approx G(x)$. Several other notions of the equivalence (so-called PS- equivalence) of the polynomial matrices are considered in [3]. Two matrices $F(x)$ and $G(x)$ are said to be PS- equivalent if there exist $P(x) \in GL(n, \mathbf{C}[x])$, $Q \in GL(n, \mathbf{C})$ with $G(x) = P(x)F(x)Q$. If $F(x), G(x)$ are semiscalarly equivalent (or PS- equivalent), then they must have the same characteristic roots and the same invariant factors. By Theorem 1 [1] (see also Theorem 1 §1, Section IV [2]) every matrix of full rank is semiscalarly equivalent to the lower triangular form with invariant factors on the main diagonal. The similar results can be found in [4]. However, the matrix of this form is not uniquely defined. Therefore, the question when two matrices are semiscalarly equivalent is open. The conditions of semiscalar equivalence of order 2 polynomial matrices in [5–7] are indicated. In this paper is determined so-called reduced form with respect to semiscalar equivalence for the 3-by-3 matrices with one characteristic root and its invariants are found. The problem of semiscalar equivalence (as of PS- equivalence) contains the classical linear algebra problem of reducing a pair of numerical matrices to a canonical form by a simultaneous similarity transformation (for the solution of this problem, see [8]).

Let $F(x) \in M(3, \mathbf{C}[x])$. We assume that characteristic polynomial $\det F(x)$ has a unique root. Without loss of generality, we assume that uniquely characteristic root is zero and the first invariant factor of the matrix $F(x)$ is unit. In accordance with [1] at this assumption we have

$$F(x) \approx G(x) = \begin{Vmatrix} 1 & 0 & 0 \\ g_1(x) & x^{k_1} & 0 \\ g_3(x) & g_2(x) & x^{k_2} \end{Vmatrix}, \tag{1}$$

where $k_1 \le k_2$, $x^{k_1} | g_2(x)$ (divides). We consider $k_1 < k_2$, since the case $k_1 = k_2$ is considered in [9].

2. Preliminary Results

Proposition 1. *In the class $\{PF(x)Q(x)\}$ of semiscalarly equivalent matrices there exists a matrix of the form (1), in which $\deg g_1 < k_1$, $\deg g_2, \deg g_3 < k_2$, $g_1(0) = g_3(0) = 0$, $g_2(x) = x^{k_1} g_2'(x)$, $g_2'(0) = 0$.*

Proof. Proof is obvious. □

Let the matrices

$$A(x) = \begin{Vmatrix} 1 & 0 & 0 \\ a_1(x) & x^{k_1} & 0 \\ a_3(x) & a_2(x) & x^{k_2} \end{Vmatrix}, \qquad B(x) = \begin{Vmatrix} 1 & 0 & 0 \\ b_1(x) & x^{k_1} & 0 \\ b_3(x) & b_2(x) & x^{k_2} \end{Vmatrix} \tag{2}$$

be given, where $k_1 < k_2$, $\deg a_1, \deg b_1 < k_1$, $\deg a_2, \deg b_2, \deg a_3, \deg b_3 < k_2$, $a_1(0) = a_3(0) = b_1(0) = b_3(0) = 0$, $a_2(x) = x^{k_1} a_2'(x)$, $a_2'(0) = 0$, $b_2(x) = x^{k_1} b_2'(x)$, $b_2'(0) = 0$.

Proposition 2. *A left reducible matrix in the passage from $A(x)$ to the semiscalarly equivalent $B(x)$ of the form (2) is an upper triangular matrix.*

Proof. Let $A(x) \approx B(x)$. Then, we have

$$\begin{aligned} \|s_{ij}\|_1^3 \begin{Vmatrix} 1 & 0 & 0 \\ a_1(x) & x^{k_1} & 0 \\ a_3(x) & a_2(x) & x^{k_2} \end{Vmatrix} \\ = \begin{Vmatrix} 1 & 0 & 0 \\ b_1(x) & x^{k_1} & 0 \\ b_3(x) & b_2(x) & x^{k_2} \end{Vmatrix} \|r_{ij}(x)\|_1^3, \end{aligned} \tag{3}$$

where $\|s_{ij}\|_1^3 \in GL(3, \mathbf{C})$, $\|r_{ij}(x)\|_1^3 \in GL(3, \mathbf{C}[x])$. From (3) it follows that

$$s_{21} + s_{22} a_1(x) + s_{23} a_3(x) = b_1(x) r_{11}(x) + x^{k_1} r_{21}(x), \tag{4}$$

$$\begin{aligned} &s_{31} + s_{32} a_1(x) + s_{33} a_3(x) \\ &\quad = b_3(x) r_{11}(x) + b_2(x) r_{21}(x) + x^{k_2} r_{31}(x), \end{aligned} \tag{5}$$

$$\begin{aligned} &s_{32} x^{k_1} + s_{33} a_2(x) \\ &\quad = b_3(x) r_{12}(x) + b_2(x) r_{22}(x) + x^{k_2} r_{32}(x). \end{aligned} \tag{6}$$

Substituting $x = 0$ in (4), (5), we find that $s_{21} = s_{31} = 0$. Since $x^{k_1} | r_{12}(x)$, the right-hand side of equality (6) and the second summand of the left-hand side of this equality are divisible by x^{k_1+1}. Therefore, $x^{k_1+1} | s_{32} x^{k_1}$. This implies that $s_{32} = 0$. Proposition is proved. □

Proposition 3. *If $A(x) \approx B(x)$ for the matrices (2), then $(a_1(x), a_3(x), x^{k_1}) = (b_1(x), b_3(x), x^{k_1})$, $(a_2(x), a_3(x), x^{k_2}) = (b_2(x), b_3(x), x^{k_2})$, where the bracket (, ,) denotes the greatest common divisor.*

Proof. Since $r_{11}(0) \neq 0$ in equality (3), it follows that, from (4) and (5), where $s_{21} = s_{31} = s_{32} = 0$, we obtain $(a_1(x), a_3(x), x^{k_1}) | b_1(x)$ and $(a_1(x), a_3(x), x^{k_1}) | b_3(x)$, respectively. Thus, $(a_1(x), a_3(x), x^{k_1}) | (b_1(x), b_3(x), x^{k_1})$. The notation of semiscalar equivalence is a symmetric relation, so that $(b_1(x), b_3(x), x^{k_1}) | (a_1(x), a_3(x), x^{k_1})$. The first part of the Proposition is thus proved. Similarly, from (5) and (6), where $s_{33} \neq 0$, we can obtain $(b_2(x), b_3(x), x^{k_2}) | a_2(x)$ and $(b_2(x), b_3(x), x^{k_2}) | a_3(x)$, respectively. Therefore, $(a_2(x), a_3(x), x^{k_2}) | (b_2(x), b_3(x), x^{k_2})$. Again by virtue of symmetrical relation of semiscalar equivalence we obtain $(a_2(x), a_3(x), x^{k_2}) | (b_2(x), b_3(x), x^{k_2})$. The Proposition is proved completely. □

Further, by using semiscalarly equivalent transformations $A(x) \longrightarrow SA(x)R(x) = B(x)$, we reduce the matrix $A(x)$ to a matrix $B(x)$ of the form (2) with the predefined properties. Furthermore, the left reducible matrix S, obviously, must be selected of the upper triangular form. We shall show how by the given matrix $A(x)$ and by the left reducible matrix S we can find the matrix $B(x)$ of the form (2) and the right reducible matrix $R(x)$ such that $A(x) \approx B(x) = SA(x)R(x)$. Then, we shall choose the matrix S of the upper unitriangular form:

$$S = \begin{Vmatrix} 1 & s_{12} & s_{13} \\ 0 & 1 & s_{23} \\ 0 & 0 & 1 \end{Vmatrix}. \tag{7}$$

By the given entries $a_1(x), a_2(x), a_3(x)$ and s_{12}, s_{13}, s_{23} of the matrices $A(x)$ and S, respectively, by means of the method of indeterminate coefficients from the congruence

$$a_1(x) + s_{23} a_3(x) \equiv b_1(x) \left(1 + s_{12} a_1(x) + s_{13} a_3(x)\right) \quad \left(\operatorname{mod} x^{k_1}\right) \tag{8}$$

we find $b_1(x) \in \mathbf{C}[x]$, $\deg b_1 < k_1$. Denote by $r_{uv}(x)$, $u, v = 1, 2$, such entries:

$$\begin{aligned} r_{11}(x) &= 1 + s_{12} a_1(x) + s_{13} a_3(x), \\ r_{12}(x) &= s_{12} x^{k_1} + s_{13} a_2(x), \\ r_{21}(x) &= \frac{a_1(x) + s_{23} a_3(x) - b_1(x) r_{11}(x)}{x^{k_1}} \in \mathbf{C}[x], \\ r_{22}(x) &= 1 + s_{23} a_2'(x) - \left(s_{12} + s_{13} a_2'(x)\right) b_1(x). \end{aligned} \tag{9}$$

Here, $a_2'(x) = a_2(x)/x^{k_1} \in \mathbf{C}[x]$. Construct the matrix $\|r_{uv}(x)\|_1^2$ and consider the congruence

$$\|b_3(x) \;\; b_2(x)\| \, \|r_{uv}(x)\|_1^2 \equiv \|a_3(x) \;\; a_2(x)\| \quad \left(\operatorname{mod} x^{k_2}\right) \tag{10}$$

in the unknowns $b_2(x), b_3(x)$. This congruence is solvable, since the free term of the matrix polynomial $\|r_{uv}(x)\|_1^2$ is a nonsingular matrix. The unknowns can be found by the method of the indefinite coefficients. It is easily verified that

$b_2'(x) = b_2(x)/x^{k_1} \in \mathbf{C}[x]$. Besides the above definition of $r_{uv}(x), u, v = 1, 2$, let us introduce the following notations:

$$\begin{aligned}
r_{13}(x) &= s_{13}x^{k_2},\\
r_{23}(x) &= s_{23}x^{k_2-k_1} - b_1(x)\,r_{13}(x),\\
r_{31}(x) &= \frac{a_3(x) - b_3(x)\,r_{11}(x) - b_2(x)\,r_{21}(x)}{x^{k_2}}\\
&\in \mathbf{C}[x],\\
r_{32}(x) &= \frac{a_2(x) - b_3(x)\,r_{12}(x) - b_2(x)\,r_{22}(x)}{x^{k_2}}\\
&\in \mathbf{C}[x],\\
r_{33}(x) &= 1 - \frac{b_3(x)\,r_{13}(x) - b_2(x)\,r_{23}(x)}{x^{k_2}} \in \mathbf{C}[x].
\end{aligned} \tag{11}$$

By the indicated above entries $r_{ij}(x), i, j = 1, 2, 3$, and by the definition from congruence (8), (10) $b_i(x)$ we construct the matrix $\|r_{ij}(x)\|_1^3$ and the matrix $B(x)$ of the form (2). Make sure that the equality $SA(x) = B(x)\|r_{ij}(x)\|_1^3$ is valid. This means that the matrix $\|r_{ij}(x)\|_1^3$ is invertible and its inverse matrix with S reduces $A(x)$ to $B(x)$. If the matrix S (7) in the passage from $A(x)$ to $B(x)$ has one of the following forms

$$\begin{Vmatrix} 1 & 0 & 0\\ 0 & 1 & s_{23}\\ 0 & 0 & 1 \end{Vmatrix}$$
$$\text{or } \begin{Vmatrix} 1 & s_{12} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{Vmatrix}, \tag{12}$$

then we shall say that to the matrix $A(x)$ is applied the transformation of type I or the transformation of type II, respectively.

3. Improvement of the Triangular Form of Matrix in the Class of Semiscalarly Equivalent Matrix: Reduced Matrix

Junior degree of polynomial $f(x) \in \mathbf{C}[x]$, $f(x) \neq 0$, is the least degree of the monomial (of nonzero coefficient) of this polynomial; notation $co\deg f$. The monomial of degree $co\deg f$ and its coefficients are called the *junior term* and *junior coefficients*, respectively. Denote by symbol $+\infty$ the junior degree of the polynomial $f(x) \equiv 0$.

Proposition 4. *If in the matrix $A(x)$ of the form (2) $co\deg a_1 = co\deg a_3 \neq +\infty$, then $A(x) \approx B(x)$, where in the matrix $B(x)$ of the form (2) $co\deg b_1 > co\deg b_3$, $co\deg b_2 = co\deg a_2$, $co\deg b_3 = co\deg a_3$.*

Proof. We will uniquely determine the value of s_{23} from condition $co\deg(a_1(x) + s_{23}a_3(x)) > co\deg a_1$ and we will apply to the matrix $A(x)$ the transformation of the type I. As a result we obtain the matrix $B(x)$ of the form (2). Its entries $b_i(x), i = 1, 2, 3$, satisfy the congruences:

$$a_1(x) + s_{23}a_3(x) - b_1(x) \equiv 0 \quad \left(\operatorname{mod} x^{k_1}\right), \tag{13}$$

$$a_2(x) - b_2(x)\left(1 + s_{23}a_2'(x)\right) \equiv 0 \quad \left(\operatorname{mod} x^{k_2}\right), \tag{14}$$

$$a_3(x) - b_2(x)\,r_{21}(x) - b_3(x) \equiv 0 \quad \left(\operatorname{mod} x^{k_2}\right), \tag{15}$$

where $r_{21}(x) = (a_1(x) + s_{23}a_3(x) - b_1(x))/x^{k_1} \in \mathbf{C}[x]$. From (14), (15), and (13), we find that $co\deg b_2 = co\deg a_2$, $co\deg b_3 = co\deg a_3$ and $co\deg b_1 > co\deg b_3$, respectively. Proposition is proved. □

Proposition 5. *Let a matrix $A(x)$ of the form (2) be given such that $co\deg a_2' = co\deg a_3 \neq +\infty$, $a_2'(x) = a_2(x)/x^{k_1} \in \mathbf{C}[x]$. Then there exists a matrix $B(x)$ of the form (2) such that $A(x) \approx B(x)$ and $co\deg b_1 = co\deg a_1$, $co\deg b_3 = co\deg a_3$, $co\deg b_2' > co\deg b_3$, where $b_2'(x) = b_2(x)/x^{k_1} \in \mathbf{C}[x]$.*

Proof. By transformation of type II we reduce the matrix $A(x)$ to the matrix $B(x)$ of the form (2). Herewith in the matrix S (see (12)) we define s_{12} such that the inequality $co\deg(s_{12}x^{k_1}a_3(x) - a_2(x)) > co\deg a_2$ is true. The entries $b_1(x), b_2(x), b_3(x)$ of the obtained matrix $B(x)$ satisfy the congruences:

$$b_1(x) - a_1(x)\left(1 - s_{12}b_1(x)\right) \equiv 0 \quad \left(\operatorname{mod} x^{k_1}\right), \tag{16}$$

$$b_2(x) + s_{12}x^{k_1}a_3(x) - a_2(x)\left(1 + s_{12}a_1(x)\right) \equiv 0 \quad \left(\operatorname{mod} x^{k_2}\right), \tag{17}$$

$$b_3(x) - a_3(x)\left(1 - s_{12}b_1(x)\right) - a_2(x)\,r_{21}(x) \equiv 0 \quad \left(\operatorname{mod} x^{k_2}\right), \tag{18}$$

where $r_{21}(x) = (b_1(x) - a_1(x)(1 - s_{12}b_1(x)))/x^{k_1} \in \mathbf{C}[x]$. From (16) and (18) we have that $co\deg b_1 = co\deg a_1$ and $co\deg b_3 = co\deg a_3$, respectively. If the principle of a choice of s_{12} is considered, then from (17) it follows that $co\deg b_2' > co\deg b_3$. Proposition is proved. □

Proposition 6. *Let the matrix $A(x)$ have the form (2) and*

$$+\infty \neq 2co\deg a_1 + co\deg a_2' = co\deg a_3 \geq co\deg a_2, \tag{19}$$

where $a_2'(x) = a_2(x)/x^{k_1} \in \mathbf{C}[x]$. Then there exists a matrix $B(x)$ of the form (2) such that $A(x) \approx B(x)$ and $b_1(x) = a_1(x)$, $co\deg b_2 = co\deg a_2$, $co\deg b_3 > co\deg a_3$.

Proof. Let us apply to the matrix $A(x)$ the transformation of type II. Moreover, in the left reducible matrix (see (12)) we can choose the value of s_{12} so that the condition $co\deg(a_3(x) + s_{12}a_1^2(x)a_2'(x)) > co\deg a_3$ is fulfilled. As a result we obtain the matrix $B(x)$ of the form (2) in which its entries $b_i(x)$, $i = 1, 2, 3$, satisfy the following congruences:

$$a_1(x) - b_1(x) - s_{12}a_1(x)b_1(x)) \equiv 0 \pmod{x^{k_1}}, \quad (20)$$

$$a_2(x) - b_2(x) + s_{12}\left(b_1(x)b_2(x) - x^{k_1}b_3(x)\right) \equiv 0 \pmod{x^{k_2}}, \quad (21)$$

$$a_3(x) + \left(b_1(x)b_2'(x) - b_3(x)\right)\left(1 + s_{12}a_1(x)\right) - b_2'(x)a_1(x) \equiv 0 \pmod{x^{k_2}}. \quad (22)$$

From (19) we have that $2co\deg a_1 \geq k_1$. Then (20) implies $b_1(x) = a_1(x)$ and from (22) we find

$$a_3(x) - b_3(x) + s_{12}a_1(x)\left(b_1(x)b_2'(x) - b_3(x)\right) \equiv 0 \pmod{x^{k_2}}. \quad (23)$$

From (21) and (23) by excluding of s_{12}, we arrive at the congruence

$$a_1(x)a_2(x) - x^{k_1}a_3(x) \equiv b_1(x)b_2(x) - x^{k_1}b_3(x) \pmod{x^{k_2+co\deg a_1}}, \quad (24)$$

or

$$a_1(x)a_2'(x) - a_3(x) \equiv b_1(x)b_2'(x) - b_3(x) \pmod{x^{k_2-k_1+co\deg a_1}}. \quad (25)$$

Since $co\deg(a_1(x)a_2(x) - x^{k_1}a_3(x)) = co\deg(a_1(x)a_2(x))$, from (21) we have $co\deg b_2 = co\deg a_2$. From the inequalities $co\deg(a_1(x)a_2'(x)) < k_2 - co\deg a_1$ and $k_2 - k_1 + co\deg a_1 \geq k_2 - co\deg a_1$ it follows that junior terms in both members (25) coincide with the junior term of the product $a_1(x)a_2'(x)$. Then from (23), taking into account the choice of s_{12}, we find $co\deg b_3 > co\deg a_3$. Proposition is proved. □

Proposition 7. *Let a matrix $A(x)$ of the form (2) be given in which $co\deg a_3 \geq co\deg a_2$ and $2co\deg a_1 < k_1$. Then $A(x) \approx B(x)$, where $B(x)$ has the form (2) in which $co\deg b_2 = co\deg a_2$, $co\deg b_3 \geq co\deg b_2$ and in $b_1(x)$ the monomial of degree $2co\deg b_1 = 2co\deg a_1$ is absent.*

Proof. Let us apply to the matrix $A(x)$ a transformation of type II. In this case, in the left reducible matrix (see (12)) we define s_{12} from the condition $s_{12}c_0^2 = c_1$, where c_0 and c_1 are junior coefficient and coefficient of the monomial of degree $2co\deg a_1$ in $a_1(x)$, respectively. Then the entries $a_i(x)$ and $b_i(x)$ of the matrix $A(x)$ and of the matrix $B(x)$, obtained as a result of transformation, satisfy the congruences (16) – (18). From (16) it follows at once that junior terms in $a_1(x)$, $b_1(x)$ coincide and in $b_1(x)$ the monomial of degree $2co\deg a_1$ is absent. From (17) and (18) we have, respectively, that $co\deg b_2 = co\deg a_2$ and $co\deg b_3 \geq co\deg b_2$. Proposition is proved. □

Taking into account Propositions 4 and 5 we shall think henceforth that $co\deg a_3 \neq co\deg a_1$ and $co\deg a_3 \neq co\deg a_2'$ in the matrix $A(x)$ of the form (2), if $co\deg a_3 < co\deg a_2$. If $co\deg a_3 \geq co\deg a_2$, then based on Propositions 6 and 7, we note that in the matrix $A(x)$ the inequality $co\deg a_3 \neq 2co\deg a_1 + co\deg a_2'$ holds true and in $a_1(x)$ the monomial of degree $2co\deg a_1$ is absent. Moreover, we may take the junior coefficients of the polynomials $a_1(x)$ and $a_2(x)$ to be unit, if $a_1(x), a_1(x) \neq 0$. If one of the polynomials $a_1(x), a_1(x)$ is identical zero, then we may take the junior coefficients of the nonzero underdiagonal entries of the matrix $A(x)$ to be unit. Such matrix $A(x)$ we shall call the *reduced matrix*. All subsequent semiscalarly equivalent transformations of the matrix $A(x)$ should not violate her property to be reduced.

4. Invariants of the Reduced Matrix

Theorem 8. *In reduced matrix $A(x)$ of the form (2) $co\deg a_1$, $co\deg a_2$, and $co\deg a_3$ are invariants with respect to semiscalarly equivalent transformations.*

Proof. Let $A(x)$ and $B(x)$ be reduced matrices of the form (2) and $A(x) \approx B(x)$. From equality (3), where matrix $\|s_{ij}\|_1^3$ by Proposition 2 is upper triangular, we get

$$s_{22}a_1(x) + s_{23}a_3(x) \equiv s_{11}b_1(x) + s_{12}a_1(x)b_1(x) + s_{13}a_3(x)b_1(x) \pmod{x^{k_1}}. \quad (26)$$

Recall that $s_{11}, s_{22} \neq 0$. If $b_1(x) \equiv 0$, then from (26) it follows that $a_1(x) \equiv 0$, i.e., $co\deg a_1 = co\deg b_1 = +\infty$. Let $a_1(x) \neq 0$ ($b_1(x) \neq 0$). If $co\deg a_1 < co\deg a_3$, then from (26) at once we have $co\deg a_1 = co\deg b_1$. If $co\deg a_1 > co\deg a_3$, then $co\deg a_3 < k_1 < co\deg a_2$. In view of Proposition 3, we get

$$k_1 > co\deg a_3 = co\deg\left(a_2(x), a_3(x), x^{k_2}\right) = co\deg\left(b_2(x), b_3(x), x^{k_2}\right) = co\deg b_3. \quad (27)$$

Also by Proposition 3 we have $co\deg(a_1(x), a_3(x), x^{k_1}) = co\deg(b_1(x), b_3(x), x^{k_1})$. For this reason $co\deg b_1 > co\deg b_3 = co\deg a_3$ and from (26) it follows that $s_{23} = 0$. Thus, $co\deg b_1 = co\deg a_1$.

From equality (37) we can write

$$s_{33}a_2(x) \equiv b_3(x)r_{12}(x) + b_2(x)r_{22}(x) \pmod{x^{k_2}}, \quad (28)$$

$$s_{33}a_3(x) \equiv b_3(x)r_{11}(x) + b_2(x)r_{21}(x) \pmod{x^{k_2}}. \quad (29)$$

We recall that $s_{33} \neq 0$. If $b_2(x) \equiv 0$, then (29) implies that $co\deg a_3 = co\deg b_3$. Then, from (28) we find $a_2(x) \equiv 0$, since $r_{12}(x) = s_{12}x^{k_1} + s_{13}a_2(x)$, $co\deg a_2 \neq co\deg a_3 + k_1$.

Let $a_2(x) \neq 0$ ($b_2(x) \neq 0$). If $co\deg b_3 < co\deg b_2$, then (29) implies that $co\deg a_3 = co\deg b_3$, and from (28), taking into account the form of $r_{12}(x)$ and $co\deg a_3 \neq co\deg a_2'$, we have $co\deg a_2 = co\deg b_2$.

If $co\deg b_3 \geq co\deg b_2$, then from (28) we get also $co\deg a_2 = co\deg b_2$. If $2co\deg b_1 < k_1$, then from (26) we obtain $s_{12} = 0$, $a_1(x) = b_1(x)$ and $r_{21}(x) = 0$. Therefore, from (29) it is clear that $co\deg a_3 = co\deg b_3$. In particular, $a_3(x) = 0 \Longleftrightarrow b_3(x) = 0$. If $2co\deg b_1 \geq k_1$, then from (26) we have $a_1(x) = b_1(x)$. In this case, rewrite (29) in the detailed form as

$$s_{33}a_3(x) - s_{11}b_3(x) + s_{12}a_1(x)\delta_B(x) + s_{13}a_3(x)\delta_B(x) - s_{23}a_3(x)b_2'(x) \equiv 0 \pmod{x^{k_2}}, \tag{30}$$

where $\delta_B(x) = b_1(x)b_2'(x) - b_3(x)$, $b_2(x) = x^{k_1}b_2'(x)$. Since $co\deg \delta_B = co\deg(b_1(x)b_2'(x)) = co\deg(a_1(x)a_2'(x))$ and $co\deg a_3, co\deg b_3 \neq 2co\deg a_1 + co\deg a_2'(x)$, as seen from the last congruence, $co\deg a_3 = co\deg b_3$. Moreover, $s_{12} = 0$, if $co\deg a_3 > 2co\deg a_1 + co\deg a_2'(x)$. Also in this case from last congruence we obtain $a_3(x) = 0 \Longleftrightarrow b_3(x) = 0$. Theorem is proved. □

Corollary 9. *Let in the reduced matrix $A(x)$ one of the following three conditions hold true:*

$$co\deg a_3 < co\deg a_1 \neq +\infty, \tag{31}$$

$$co\deg a_3 < co\deg a_2' \neq +\infty, \tag{32}$$

$$+\infty \neq co\deg a_3 > 2co\deg a_1 + co\deg a_2', \quad co\deg a_3 \geq co\deg a_2. \tag{33}$$

Then left reducible matrix in the passage from $A(x)$ to the reduced matrix $B(x)$ is of the form

$$\begin{Vmatrix} s_{11} & s_{12} & s_{13} \\ 0 & s_{22} & 0 \\ 0 & 0 & s_{33} \end{Vmatrix}, \tag{34}$$

if condition (31) is fulfilled, or

$$\begin{Vmatrix} s_{11} & 0 & s_{13} \\ 0 & s_{22} & s_{23} \\ 0 & 0 & s_{33} \end{Vmatrix}, \tag{35}$$

if one from two conditions (32), (33) is valid.

Corollary 10. *Identical equality to zero of the entry $a_1(x)$, $a_2(x)$, or $a_3(x)$ of the reduced matrix $A(x)$ is invariant with respect to semiscalarly equivalent transformations.*

Remark. If some two underdiagonal entries in the reduced semiscalarly equivalent matrices $A(x)$, $B(x)$ are nonzero, then diagonal entries of the left reducible matrix, which by Proposition 2 is upper triangular, are equal to each other. Therefore, we can choose this matrix as unitriangular.

Let reduced matrices $A(x)$, $B(x)$ of the form (2) be given. Henceforth we shall apply the following notations:

$$\begin{aligned} \Delta_A(x) &:= \begin{vmatrix} a_1(x) & x^{k_1} \\ a_3(x) & a_2(x) \end{vmatrix}, \\ \Delta_B(x) &:= \begin{vmatrix} b_1(x) & x^{k_1} \\ b_3(x) & b_2(x) \end{vmatrix}, \\ \delta_A(x) &= \frac{\Delta_A(x)}{x^{k_1}} \in \mathbf{C}[x], \\ \delta_B(x) &= \frac{\Delta_B(x)}{x^{k_1}} \in \mathbf{C}[x]. \end{aligned} \tag{36}$$

Corollary 11. *If $a_1(x), a_2(x) \neq 0$ and $co\deg a_3 \geq co\deg a_2$ in the reduced matrix $A(x)$, then $co\deg \Delta_A = co\deg(a_1(x)a_2(x))$. Therefore, $co\deg \Delta_A$ is an invariant with respect to semiscalarly equivalent transformations.*

Theorem 12. *In the reduced matrix $A(x)$ the quantity $co\deg \Delta_A$ is an invariant with respect to semiscalarly equivalent transformations.*

Proof. Let $A(x)$, $B(x)$ be reduced matrices of the form (2) and $A(x) \approx B(x)$. Then from equality (3), where matrix $\|s_{ij}\|_1^3$ is upper triangular (see Remark), we obtain

$$\begin{aligned} a_2(x) = {} & b_2(x) - s_{12}\Delta_B(x) - s_{13}a_2(x)\delta_B(x) \\ & + s_{23}a_2(x)b_2'(x) + x^{k_2}r_{32}(x), \end{aligned} \tag{37}$$

$$\begin{aligned} a_3(x) = {} & a_1(x)b_2'(x) - \delta_B(x) - s_{12}a_1(x)\delta_B(x) - \\ & - s_{13}a_3(x)\delta_B(x) + s_{23}a_3(x)b_2'(x) + x^{k_2}r_{31}(x), \end{aligned} \tag{38}$$

where $b_2'(x) = b_2(x)/x^{k_1} \in \mathbf{C}[x]$. Excluding from (37) and (38) the summand, which contains s_{12}, we define

$$\begin{aligned} \Delta_A(x) = {} & \Delta_B(x) - \Delta_A(x)\left(s_{13}\delta_B(x) - s_{23}b_2'(x)\right) \\ & + x^{k_2}\left(a_1(x)r_{32}(x) - x^{k_1}r_{31}(x)\right). \end{aligned} \tag{39}$$

Since $co\deg \Delta_A < k_2 + co\deg a_1 \leq co\deg(x^{k_2}(a_1(x)r_{32}(x) - x^{k_1}r_{31}(x))$, $co\deg \Delta_A < co\deg(\Delta_A(x)(s_{13}\delta_B(x) - s_{23}b_2'(x)))$, from (39) it follows that $co\deg \Delta_A = co\deg \Delta_B$. □

Corollary 13. *The congruence $co\deg \Delta_A \equiv 0 \pmod{x^{k_2 + co\deg a_1}}$ is an invariant of $A(x)$ with respect to semiscalarly equivalent transformations.*

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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