Research Article

Boundary Value Problems Governed by Superdiffusion in the Right Angle: Existence and Regularity

Ramzet Dzhafarov and Nataliya Vasylyeva

Institute of Applied Mathematics and Mechanics of NAS of Ukraine, G. Batyuka St. 19, 84100 Sloviansk, Ukraine

Correspondence should be addressed to Nataliya Vasylyeva; nataliy_v@yahoo.com

Received 21 June 2018; Accepted 5 September 2018; Published 2 December 2018

Academic Editor: Reza Ezzati

Copyright © 2018 Ramzet Dzhafarov and Nataliya Vasylyeva. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

For $\alpha \in (1, 2)$, we analyze a stationary superdiffusion equation in the right angle in the unknown $u = u(x_1, x_2)$: $\mathbf{D}_{x_1}^\alpha u + \mathbf{D}_{x_2}^\alpha u = f(x_1, x_2)$, where $\mathbf{D}_x^\alpha$ is the Caputo fractional derivative. The classical solvability in the weighted fractional Hölder classes of the associated boundary problems is addressed.

1. Introduction

Fractional partial differential equations (FPDE) play a key role in the description of the so-called anomalous phenomena in nature and in the theory of complex systems (see, e.g., [1]). In particular, these equations provide a more faithful representation of the long-memory and nonlocal dependence of many anomalous processes. The signature of an anomalous diffusion species $\langle (\Delta \mathbf{x})^2 \rangle$ scales as a nonlinear power law in time; i.e., $\langle (\Delta \mathbf{x})^2 \rangle \sim t^\alpha$, $\alpha > 0$. When $\alpha > 1$, this is referred to as superdiffusion.

Superdiffusion is used in modelling turbulent flow [2, 3], chaotic dynamics of classical conservative systems [4], model solute transport in underground aquifers [5–7], and rivers [8–10], biophysics [11], and physical and chemical models described by the Lévy processes [12, 13]. In the present paper, we focus on the boundary value problems to the stationary superdiffusion equation.

Let $\Omega = \{(x_1, x_2) : x_1 \in (0, +\infty), x_2 \in (0, +\infty)\}$ be the first quarter with a boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 = \{(x_1, x_2) : x_2 = 0, x_1 \geq 0\}$, $\Gamma_2 = \{(x_1, x_2) : x_1 = 0, x_2 \geq 0\}$. For a fixed $\alpha \in (1, 2)$, we consider the linear equation in the unknown function $u = u(x_1, x_2) : \Omega \rightarrow \mathbb{R}$,

$$\mathbf{D}_{x_1}^\alpha u + \mathbf{D}_{x_2}^\alpha u = f(x_1, x_2), \quad (1)$$

subject either to the Dirichlet boundary condition (DBC),

$$\begin{aligned} u &= \psi_1(x_1) \quad \text{on } \Gamma_1 \\ \text{and } u &= \psi_2(x_2) \quad \text{on } \Gamma_2, \end{aligned} \quad (2)$$

or to the Neumann boundary condition (NBC),

$$\begin{aligned} \frac{\partial u}{\partial x_2} &= \varphi_1(x_1) \quad \text{on } \Gamma_1 \\ \text{and } \frac{\partial u}{\partial x_1} &= \varphi_2(x_2) \quad \text{on } \Gamma_2, \end{aligned} \quad (3)$$

where the functions $f, \psi_1, \psi_2, \varphi_1, \varphi_2$ are prescribed.

Here, the symbols $\mathbf{D}_{x_1}^\alpha$ and $\mathbf{D}_{x_2}^\alpha$ stand for the Caputo fractional derivative of order α with respect to x_1 and x_2 .

Introducing the function,

$$\omega_\theta(y) = \frac{y^{\theta-1}}{\Gamma(\theta)}, \quad \theta > 0, \quad y \geq 0, \quad (4)$$

we define the Caputo derivative $\mathbf{D}_y^\theta v(\cdot, y)$ as (see, e.g., (2.4.1) in [1])

$$\mathbf{D}_y^\theta v(\cdot, y) = \frac{\partial^{[\theta]}}{\partial y^{[\theta]}} \int_0^y \omega_{[\theta]-\theta}(y-z) \times \left[v(\cdot, z) - \sum_{k=0}^{[\theta]-1} \frac{\partial^k v(\cdot, 0)}{\partial z^k} \frac{z^k}{k!} \right] dz, \quad (5)$$

where $[\theta]$ is the ceiling function of θ (i.e., the smallest integer greater than or equal to θ), Γ being the Euler Gamma-function. An equivalent definition in the case of $u \in AC^{[\theta]}$ reads (see (2.4.15) in [1])

$$\mathbf{D}_y^\theta v(\cdot, y) = \int_0^y \omega_{[\theta]-\theta}(y-z) \frac{\partial^{[\theta]}}{\partial z^{[\theta]}} v(\cdot, z) dz. \quad (6)$$

In the limit case $\theta = [\theta]$, the Caputo fractional derivatives of $v(\cdot, y)$ boils down $(\partial^{[\theta]}/\partial y^{[\theta]})v(\cdot, y)$.

Elliptic boundary value problems, $\alpha = 2$, in domain with conical and dihedral singularities have been extensively studied, starting with Kondratiev's famous paper [14]. In this field of researches, it should be also noted the works of Kondratiev and Oleinik [15], Borsuk and Kondratiev [16], Grisvard [17], and Maz'ya and Plamenevskii [18]. The results of those investigations are formulated in both $W^{l,p}$ and $\mathcal{C}^{l+\beta}$ spaces and corresponding weighted classes, where the desired functions together with their derivatives are bounded with some power weights in the L^p or $\mathcal{C}^{l+\beta}$ norms.

Recently, a great attention in the literature has been devoted to the study of boundary value problems to the fractional Laplacian operator $(-\Delta)^s u$ with order $s \in (0, 1)$ (see, e.g., [19–24] and references therein). Boundary value problems for the equation with the main part,

$$\frac{\partial^2}{\partial x^2} u(x, y) + \mathbf{D}_y^{2\alpha} u(x, y), \quad \alpha \in (0, 1), \quad (7)$$

are studied with various approaches, such as spectral technique, method of potential theory, and Fourier method (see [25–28] and references therein). We also quote the works [29, 30], where the authors presented a Galerkin finite element approximation for variational solution to the steady state fractional advection dispersion equation:

$$\begin{aligned} & -D_x^1 a(p_0 D_x^{-\nu} + q_x D_1^{-\nu}) D_x^1 u + b(x) D_x^1 u + c(x) u \\ & = f, \quad \nu \in (0, 1), \end{aligned} \quad (8)$$

where $D_x^{-\nu}$, $D_1^{-\nu}$ are left and right fractional integral operators, $0 \leq p, q, \leq 1$, $p + q = 1$.

The goal of the present paper is the proof of the well-posedness and the regularity of solutions to boundary value problems (1)–(3) in weighted Hölder classes. It is worth noting that these classes allow one to control the behavior of the solution near the boundary including the corner point and at the infinity.

Outline of the Paper. In the next section, we introduce necessary functional spaces and state the main results (Theorems

3 and 4) along with the general assumptions. In Section 3, we construct the integral representation to $u(x_1, x_2)$ in the case of homogenous boundary conditions. To this end, we apply Mellin transform and reduce problem (1)–(3) to the linear nonhomogenous difference equation of the first-order with variables coefficients in the two-dimensional case. Section 4 is devoted to some auxiliary results which will play a key role in the investigation. In Section 5, we estimate the seminorms of the minor derivatives $\mathbf{D}_{x_1}^{\alpha_1} \mathbf{D}_{x_2}^{\alpha_2} u$, $0 \leq \alpha_1 + \alpha_2 \leq 1$, and in Section 6 we evaluate the Hölder coefficients of the major derivatives $\mathbf{D}_{x_1}^{\alpha_1} \mathbf{D}_{x_2}^{\alpha_2} u$, $1 < \alpha_1 + \alpha_2 \leq \alpha$. In Section 7, using these estimates, we provide the proofs of Theorems 3 and 4. Moreover, in Remark 27 we show how results of Theorems 3 and 4 can be extended to the more general equation compared to (1).

2. Functional Setting and Main Results

Throughout this work, the symbol C will denote a generic positive constant, depending only on the structural quantities of the problem. We will carry out our analysis in the framework of the weighted Hölder spaces. Let $\beta \in (0, 1)$ be arbitrary fixed. We denote by $r(x)$ and $d(x)$ the distance from a point $x(x_1, x_2) \in \Omega$ to the origin $(0, 0)$ and to the boundary $\partial\Omega$, correspondingly. Then for every x and $\bar{x}$ from Ω we define $r(x, \bar{x}) = \min\{r(x), r(\bar{x})\}$ and $d(x, \bar{x}) = \min\{d(x), d(\bar{x})\}$. Note that if $x, \bar{x} \in \partial\Omega$, then $r(x) = d(x)$ and $r(x, \bar{x}) = d(x, \bar{x})$.

For fixed $s \in \mathbb{R}$, we introduce the Banach spaces $E_s^\beta(\bar{\Omega})$ and $\mathcal{C}_s^\beta(\bar{\Omega})$ of the functions v with the norms

$$\begin{aligned} \|v\|_{E_s^\beta(\bar{\Omega})} &= \sup_{\bar{\Omega}} r^{-s}(x) |v| + \langle v \rangle_{x_1, s-\beta, \Omega}^{(\beta)} + \langle v \rangle_{x_2, s-\beta, \Omega}^{(\beta)}, \\ \|v\|_{\mathcal{C}_s^\beta(\bar{\Omega})} &= \sup_{\bar{\Omega}} d^{-s}(x) |v| + \langle \langle v \rangle \rangle_{x_1, s-\beta, \Omega}^{(\beta)} \\ &\quad + \langle \langle v \rangle \rangle_{x_2, s-\beta, \Omega}^{(\beta)}, \end{aligned} \quad (9)$$

where

$$\begin{aligned} \langle v \rangle_{y, s-\beta, \Omega}^{(\beta)} &= \sup_{y, \bar{y} \in \bar{\Omega}} r^{-s+\beta}(y, \bar{y}) \frac{|v(y) - v(\bar{y})|}{|y - \bar{y}|^\beta}, \\ \langle \langle v \rangle \rangle_{y, s-\beta, \Omega}^{(\beta)} &= \sup_{y, \bar{y} \in \bar{\Omega}} d^{-s+\beta}(y, \bar{y}) \frac{|v(y) - v(\bar{y})|}{|y - \bar{y}|^\beta}, \end{aligned} \quad (10)$$

for $y \neq \bar{y}$.

Definition 1. A function $v = v(x_1, x_2)$ belongs to the class $\mathcal{M}_s^{l+\beta}(\bar{\Omega})$, for $l > 0$, if $v \in E_s^\beta(\bar{\Omega})$, $\mathbf{D}_{x_i}^l v \in E_{s-l}^\beta(\bar{\Omega})$, $i = 1, 2$, and the norms below are finite,

$$\begin{aligned} \|v\|_{\mathcal{M}_s^{l+\beta}(\bar{\Omega})} &= \|v\|_{E_s^\beta(\bar{\Omega})} + \sum_{|j|=1}^{[l]-1} \|\mathbf{D}_x^j v\|_{E_{s-|j|}^\beta(\bar{\Omega})} \\ &\quad + \sum_{i=1}^2 \|\mathbf{D}_{x_i}^l v\|_{E_{s-l}^\beta(\bar{\Omega})}, \end{aligned} \quad (11)$$

if $l \geq 1$, and

$$\|v\|_{\mathcal{M}_s^{l+\beta}(\bar{\Omega})} = \|v\|_{E_s^\beta(\bar{\Omega})} + \sum_{i=1}^2 \|\mathbf{D}_{x_i}^l v\|_{E_{s-i}^\beta(\bar{\Omega})} \quad (12)$$

in the case of $l \in (0, 1)$.

Definition 2. A function $v = v(x_1, x_2)$ belongs to the class $\mathcal{N}_s^{l+\beta}(\bar{\Omega})$, for $l > 0$, if $v \in \mathcal{C}_s^\beta(\bar{\Omega})$, $\mathbf{D}_{x_i}^l v \in \mathcal{C}_{s-l}^\beta(\bar{\Omega})$, $i = 1, 2$, and the norms below are finite,

$$\begin{aligned} \|v\|_{\mathcal{N}_s^{l+\beta}(\bar{\Omega})} &= \|v\|_{\mathcal{C}_s^\beta(\bar{\Omega})} + \sum_{|j|=1}^{\lceil l \rceil - 1} \|\mathbf{D}_x^j v\|_{\mathcal{C}_{s-|j|}^\beta(\bar{\Omega})} \\ &\quad + \sum_{i=1}^2 \|\mathbf{D}_{x_i}^l v\|_{\mathcal{C}_{s-i}^\beta(\bar{\Omega})}, \end{aligned} \quad (13)$$

if $l \geq 1$, and

$$\|v\|_{\mathcal{N}_s^{l+\beta}(\bar{\Omega})} = \|v\|_{\mathcal{C}_s^\beta(\bar{\Omega})} + \sum_{i=1}^2 \|\mathbf{D}_{x_i}^l v\|_{C_{s-i}^\beta(\bar{\Omega})} \quad (14)$$

in the case of $l \in (0, 1)$.

In a similar way we introduce the spaces $E_s^\beta(\partial\Omega)$ and $\mathcal{M}_s^{l+\beta}(\partial\Omega)$. As for $\mathcal{C}_s^\beta(\partial\Omega)$ and $\mathcal{N}_s^{l+\beta}(\partial\Omega)$, these spaces concave with $E_s^\beta(\partial\Omega)$ and $\mathcal{M}_s^{l+\beta}(\partial\Omega)$.

$\mathcal{M}_s^{l+\beta}$ and $\mathcal{N}_s^{l+\beta}$ defined above are Banach spaces. Indeed, the fact that they are normed spaces is easily seen, whereas the completeness follows from Theorem 2.7 [31] together with standard arguments (see, e.g., Remark 3.1.3 in [32]).

We begin to stipulate the general assumptions.

h1 (condition on the parameters): we require

$$\begin{aligned} \alpha - 1 &< \beta < 1, \\ s &= \alpha + \beta, \\ 0 &< s_1 < 2 - \alpha. \end{aligned} \quad (15)$$

h2 (conditions on the right-hand sides of boundary conditions): we demand

$$\begin{aligned} \psi_1 &\in \mathcal{M}_s^{\alpha+\beta}(\Gamma_1) \cap \mathcal{M}_{-s_1}^{\alpha+\beta}(\Gamma_1), \\ \psi_2 &\in \mathcal{M}_s^{\alpha+\beta}(\Gamma_2) \cap \mathcal{M}_{-s_1}^{\alpha+\beta}(\Gamma_2), \\ \varphi_1 &\in \mathcal{M}_{s-1}^{1+\beta}(\Gamma_1) \cap \mathcal{M}_{-s_1}^{1+\beta}(\Gamma_1), \\ \varphi_2 &\in \mathcal{M}_{s-1}^{1+\beta}(\Gamma_2) \cap \mathcal{M}_{-s_1-1}^{1+\beta}(\Gamma_2). \end{aligned} \quad (16)$$

h3 (conditions on the right-hand side of the equation): we suppose

$$\text{either } f \in E_{s-\alpha}^\beta(\bar{\Omega}) \cap E_{-s_1-\alpha}^\beta(\bar{\Omega}) \quad (17)$$

$$\text{or } f \in \mathcal{C}_{s-\alpha}^\beta(\bar{\Omega}) \cap \mathcal{C}_{-s_1-\alpha}^\beta(\bar{\Omega}). \quad (18)$$

We are now in the position to state our main results.

Theorem 3. Let assumptions (h1), (h2), and (17) hold and moreover for every points $x_1 \in \Gamma_1$ and $x_2 \in \Gamma_2$

$$f(x_1, 0) - \mathbf{D}_{x_1}^\alpha \psi_1(x_1) = 0,$$

$$f(0, x_2) - \mathbf{D}_{x_2}^\alpha \psi_2(x_2) = 0, \quad (19)$$

if the DBC (2) holds;

and

$$f(x_1, 0) - \mathbf{D}_{x_1}^{\alpha-1} \varphi_1(x_1) = 0,$$

$$f(0, x_2) - \mathbf{D}_{x_2}^{\alpha-1} \varphi_2(x_2) = 0, \quad (20)$$

if the NBC (3) holds.

Then, (1) subject either to the DBC (2) or to the NBC (3), admits a unique classical solution $u = u(x_1, x_2)$ on Ω , satisfying the regularity

$$u \in \mathcal{M}_s^{\alpha+\beta}(\bar{\Omega}) \cap \mathcal{M}_{-s_1}^{\alpha+\beta}(\bar{\Omega}) \quad (21)$$

$$\text{and } \mathbf{D}_{x_1}^{\alpha_1} \mathbf{D}_{x_2}^{\alpha_2} u = \mathbf{D}_{x_2}^{\alpha_2} \mathbf{D}_{x_1}^{\alpha_1} u$$

for $0 < \alpha_1 + \alpha_2 < \alpha$. Besides, the following estimates hold:

$$\begin{aligned} &\|u\|_{\mathcal{M}_s^{\alpha+\beta}(\bar{\Omega})} + \|u\|_{\mathcal{M}_{-s_1}^{\alpha+\beta}(\bar{\Omega})} + \|u\|_{\mathcal{N}_s^{1+\beta}(\bar{\Omega})} \\ &\leq C \left\{ \|f\|_{E_{s-\alpha}^\beta(\bar{\Omega})} + \|f\|_{E_{-s_1-\alpha}^\beta(\bar{\Omega})} \right. \\ &\quad \left. + \sum_{i=1}^2 \left[\|\psi_i\|_{\mathcal{M}_s^{\alpha+\beta}(\Gamma_i)} + \|\psi_i\|_{\mathcal{M}_{-s_1}^{\alpha+\beta}(\Gamma_i)} \right] \right\} \\ &\equiv \mathcal{F}_{\mathcal{M}}(f, \psi_1, \psi_2), \end{aligned} \quad (22)$$

$$\begin{aligned} &\|\mathbf{D}_{x_1}^{\alpha_1} \mathbf{D}_{x_2}^{\alpha_2} u\|_{E_{s-\alpha_1-\alpha_2}^\beta(\bar{\Omega})} + \|\mathbf{D}_{x_1}^{\alpha_1} \mathbf{D}_{x_2}^{\alpha_2} u\|_{E_{-s_1-\alpha_1-\alpha_2}^\beta(\bar{\Omega})} \\ &\leq \mathcal{F}_{\mathcal{M}}(f, \psi_1, \psi_2) \end{aligned} \quad (23)$$

in the case of DBC (2), and

$$\begin{aligned} &\|u\|_{\mathcal{M}_s^{\alpha+\beta}(\bar{\Omega})} + \|u\|_{\mathcal{M}_{-s_1}^{\alpha+\beta}(\bar{\Omega})} + \|u\|_{\mathcal{C}_s^\beta(\bar{\Omega})} \\ &\quad + \|\mathbf{D}_{x_1}^{\alpha_1} \mathbf{D}_{x_2}^{\alpha_2} u\|_{E_{s-\alpha_1-\alpha_2}^\beta(\bar{\Omega})} + \|\mathbf{D}_{x_1}^{\alpha_1} \mathbf{D}_{x_2}^{\alpha_2} u\|_{E_{-s_1-\alpha_1-\alpha_2}^\beta(\bar{\Omega})} \\ &\leq C \left\{ \|f\|_{E_{s-\alpha}^\beta(\bar{\Omega})} + \|f\|_{E_{-s_1-\alpha}^\beta(\bar{\Omega})} \right. \\ &\quad \left. + \sum_{i=1}^2 \left[\|\varphi_i\|_{\mathcal{M}_{s-1}^{(\alpha-1)+\beta}(\Gamma_i)} + \|\varphi_i\|_{\mathcal{M}_{-s_1-1}^{(\alpha-1)+\beta}(\Gamma_i)} \right] \right\} \end{aligned} \quad (24)$$

if NBC (3) hold.

Theorem 4. Let assumptions (h1), (h2), and (18) hold. Then, boundary value problems (1)-(3) admit a unique classical solution $u = u(x_1, x_2)$ on Ω , satisfying the regularity

$$u \in \mathcal{N}_s^{\alpha+\beta}(\bar{\Omega}) \cap \mathcal{N}_{-s_1}^{\alpha+\beta}(\bar{\Omega}) \quad (25)$$

$$\text{and } \mathbf{D}_{x_1}^{\alpha_1} \mathbf{D}_{x_2}^{\alpha_2} u = \mathbf{D}_{x_2}^{\alpha_2} \mathbf{D}_{x_1}^{\alpha_1} u$$

for $0 < \alpha_1 + \alpha_2 < \alpha$. Besides, the following estimates hold:

$$\begin{aligned} \|u\|_{\mathcal{M}_s^{\alpha+\beta}(\bar{\Omega})} + \|u\|_{\mathcal{M}_{-s_1}^{\alpha+\beta}(\bar{\Omega})} &\leq C \left\{ \|f\|_{\mathcal{E}_{s-\alpha}^\beta(\bar{\Omega})} \right. \\ &\quad \left. + \|f\|_{\mathcal{E}_{-s_1-\alpha}^\beta(\bar{\Omega})} + \sum_{i=1}^2 \left[\|\psi_i\|_{\mathcal{M}_s^{\alpha+\beta}(\Gamma_i)} + \|\psi_i\|_{\mathcal{M}_{-s_1}^{\alpha+\beta}(\Gamma_i)} \right] \right\} \quad (26) \end{aligned}$$

$$\equiv \mathcal{F}_{\mathcal{N}}(f, \psi_1, \psi_2),$$

$$\begin{aligned} \|\mathbf{D}_{x_1}^{\alpha_1} \mathbf{D}_{x_2}^{\alpha_2} u\|_{\mathcal{E}_{s-\alpha_1-\alpha_2}^\beta(\bar{\Omega})} + \|\mathbf{D}_{x_1}^{\alpha_1} \mathbf{D}_{x_2}^{\alpha_2} u\|_{\mathcal{E}_{-s_1-\alpha_1-\alpha_2}^\beta(\bar{\Omega})} \\ \leq \mathcal{F}_{\mathcal{N}}(f, \psi_1, \psi_2), \quad (27) \end{aligned}$$

in the case of DBC (2), and

$$\begin{aligned} \|u\|_{\mathcal{M}_s^{\alpha+\beta}(\bar{\Omega})} + \|u\|_{\mathcal{M}_{-s_1}^{\alpha+\beta}(\bar{\Omega})} + \|\mathbf{D}_{x_1}^{\alpha_1} \mathbf{D}_{x_2}^{\alpha_2} u\|_{\mathcal{E}_{s-\alpha_1-\alpha_2}^\beta(\bar{\Omega})} \\ + \|\mathbf{D}_{x_1}^{\alpha_1} \mathbf{D}_{x_2}^{\alpha_2} u\|_{\mathcal{E}_{-s_1-\alpha_1-\alpha_2}^\beta(\bar{\Omega})} \leq C \left\{ \|f\|_{\mathcal{E}_{s-\alpha}^\beta(\bar{\Omega})} \right. \\ + \|f\|_{\mathcal{E}_{-s_1-\alpha}^\beta(\bar{\Omega})} \\ \left. + \sum_{i=1}^2 \left[\|\varphi_i\|_{\mathcal{M}_{s-1}^{(\alpha-1)+\beta}(\Gamma_i)} + \|\varphi_i\|_{\mathcal{M}_{-s_1-1}^{(\alpha-1)+\beta}(\Gamma_i)} \right] \right\} \quad (28) \end{aligned}$$

if NBC (3) hold.

Remark 5. It is easily apparent that the functions,

$$f = \frac{(x_1^2 + x_2^2)^{\beta/2} x_1 x_2}{(1 + x_1^2 + x_2^2)^{(s_1+\alpha)/2} (x_1^2 + x_2^2)}, \quad (29)$$

$$\psi_i, \varphi_i \equiv 0, \quad i = 1, 2,$$

satisfy conditions (h2) and (17), (19), and (20).

The remainder of the paper is devoted to the proof of Theorems 3 and 4 in the DBC case. The proof of these theorems for NBC is almost identical and is left to the interested reader.

3. Integral Representation for $u(x_1, x_2)$ in the Special Case

We first dwell on the special case where $\psi_i \equiv 0$ and f is a finite function. Namely (2), (17), and (19) are replaced by the simpler conditions

$$u|_{\partial\Omega} = 0, \quad (30)$$

$$f \equiv 0 \quad \text{if} \quad (x_1^2 + x_2^2)^{1/2} > R_0 \quad (31)$$

for some given positive R_0 and

$$\begin{aligned} \text{either} \quad f &\in E_{s-\alpha}^\beta(\bar{\Omega}), \\ f(x_1, 0) &= f(0, x_2) = 0, \quad (32) \end{aligned}$$

$$\text{or} \quad f \in E_{s-\alpha}^\beta(\bar{\Omega}).$$

We denote by $u^*(p_1, p_2)$ the Mellin transform of the function $u(x_1, x_2)$. Due to conditions (30)-(32) and assumptions of Theorem 3, we can apply, at least formally, the Mellin transformation to problem (1) and (2) (see for details § 1.4 and § 2.5 in [1]). Then simple calculations lead to the equation

$$\begin{aligned} u^*(p_1 - \alpha, p_2 + \alpha) \\ + \frac{\Gamma(1 - p_1)}{\Gamma(1 + \alpha - p_1)} \frac{\Gamma(1 - p_2)}{\Gamma(1 - p_2 - \alpha)} u^*(p_1, p_2) \\ = \frac{\Gamma(1 - p_1)}{\Gamma(1 + \alpha - p_1)} f^*(p_1, p_2 + \alpha). \quad (33) \end{aligned}$$

Introducing new variables

$$p_1 = -\alpha q_1 \quad (34)$$

$$\text{and } p_2 = \alpha q_2,$$

and new functions

$$\begin{aligned} V(q_1, q_2) &= u^*(-\alpha q_1, \alpha q_2), \\ F(q_1, q_2) &= \frac{\Gamma(1 + \alpha q_1)}{\Gamma(1 + \alpha + \alpha q_1)} f^*(-\alpha q_1, \alpha q_2 + \alpha), \\ \mathcal{P}(q_1, q_2) &= \frac{\Gamma(1 + \alpha q_1)}{\Gamma(1 + \alpha + \alpha q_1)} \frac{\Gamma(1 - \alpha q_2)}{\Gamma(1 - \alpha q_2 - \alpha)}, \quad (35) \end{aligned}$$

we rewrite the equation in the more compact form

$$\begin{aligned} V(q_1 + 1, q_2 + 1) + \mathcal{P}(q_1, q_2) V(q_1, q_2) \\ = F(q_1, q_2). \quad (36) \end{aligned}$$

Thus, we transform problem (1) and (2) to the linear non-homogeneous difference equation of the first-order with variable coefficients. In order to solve this equation, we adapt the technique from Section 3 in [33] to our case.

Proposition 6. Let $\bar{\gamma}$ denote Euler-Mascheroni constant (see, e.g., Definition in [34]),

$$\begin{aligned} \operatorname{Re} q_1 &\neq -\frac{m+1}{\alpha}, \\ \operatorname{Re} q_2 &\neq \frac{m+1}{\alpha}, \quad (37) \end{aligned}$$

$$m = 0, 1, 2, \dots,$$

and let $\mathcal{E}_1(q_1)$ and $\mathcal{E}_2(q_2)$ be arbitrary analytic functions such that

$$\begin{aligned} \mathcal{E}_1(q_1) &= \mathcal{E}_1(q_1 + 1), \\ \mathcal{E}_2(q_2) &= \mathcal{E}_2(q_2 + 1). \quad (38) \end{aligned}$$

Then the function

$$V_0(q_1, q_2) = \frac{\pi e^{\bar{\gamma}\alpha} e^{i\pi q_1}}{\Gamma(1 + \alpha q_1) \Gamma(1 - \alpha q_2)} \mathcal{E}_1(q_1) \mathcal{E}_2(q_2) \quad (39)$$

solves homogenous equation (36) (i.e., $F \equiv 0$) and does not have any poles.

If, in addition, the functions $\mathcal{E}_1(q_1)$ and $\mathcal{E}_2(q_2)$ have no zeros, then the function $1/V_0(q_1, q_2)$ does not have any poles and the following estimate holds:

$$\left| \frac{\mathcal{E}_1(q_1) \mathcal{E}_2(q_2)}{V_0(q_1, q_2)} \right| \leq C |q_1|^{\alpha \operatorname{Re} q_1 + 1/2} |q_2|^{1/2 - \alpha \operatorname{Re} q_2} \times \exp \left\{ -\pi \operatorname{Im} q_1 - \frac{\pi \alpha}{2} |\operatorname{Im} q_1| - \frac{\pi \alpha}{2} |\operatorname{Im} q_2| \right\} \quad (40)$$

for $|\operatorname{Im} q_1|, |\operatorname{Im} q_2| \rightarrow +\infty$ and for every fixed $\operatorname{Re} q_1$ and $\operatorname{Re} q_2$ satisfying inequalities above.

Proof. In order to verify that the function $V_0(q_1, q_2)$ solves homogenous equation (36), it is enough to substitute V_0 in (36) and take into account the properties of the Gamma-function: $\Gamma(q+1) = q\Gamma(q)$. Besides, the properties of the function V_0 are simple consequences of the well-known properties of Gamma function:

(i) $\Gamma(z)$ has simple poles in the points

$$\operatorname{Re} z = -m, \quad m = 0, 1, 2, \dots \quad (41)$$

(ii) The Stirling asymptotic formula holds:

$$\Gamma(z) = \exp \left\{ \left(z - \frac{1}{2} \right) \ln z - z + \frac{\ln 2\pi}{2} + \frac{1}{12z} + O(z^{-3}) \right\} \quad (42)$$

as $|\operatorname{Im} z| \rightarrow +\infty$ while $|\operatorname{Re} z|$ remains bounded

□

We are now in the position to construct the solution of nonhomogeneous equation (36).

Till the end of the paper, we assume that $\gamma \in [0, 1]$ is an arbitrary fixed quantity and define the contour L_γ in the complex plane ζ as

(i)

$$L_\gamma = \{\operatorname{Re} \zeta = -\gamma, \operatorname{Im} \zeta \in (-\infty, +\infty)\}, \quad \text{if } \gamma \in (0, 1). \quad (43)$$

(ii) If $\gamma = 1$, then L_1 consists of three parts:

the half-circle $|\zeta + 1| = \gamma_0$, $\operatorname{Re} \zeta > -1$,

the intervals $\operatorname{Re} \zeta = -1$, $\operatorname{Im} \zeta \in (-\infty, -\gamma_0)$ and $\operatorname{Re} \zeta = -1$, $\operatorname{Im} \zeta \in (\gamma_0, +\infty)$

with the small positive number γ_0 , $\gamma_0 < \gamma/2$.

(iii) The contour L_0 ($\gamma = 0$) is obtained from L_1 after its shifting to the right on $\operatorname{Re} \zeta = 1$.

Introducing the periodic function $K(\zeta)$ with period 1,

$$K(\zeta) = \frac{1}{2i} [\cot \pi \zeta + i] \frac{\sin^2 \pi a}{\sin^2 \pi (\zeta + a)}, \quad (44)$$

$$0 < a - \gamma < 1,$$

we assert the following results.

Proposition 7. Let $\zeta = -\gamma + iz$, $z \in \mathbb{R}$; conditions (32) and (31) hold, and let

$$\begin{aligned} \operatorname{Re} q_1 &\neq -\frac{m+1}{\alpha} + \gamma, \\ \operatorname{Re} q_2 &\neq \frac{m+1}{\alpha} + \gamma - 1, \end{aligned} \quad (45)$$

$$m = 0, 1, 2, \dots$$

Then the solution of inhomogeneous equation (36) is given by

$$\begin{aligned} V(q_1, q_2) &= \frac{V_0(q_1, q_2)}{2i} \int_{L_\gamma} K(\zeta) \frac{F(q_1 + \zeta, q_2 + \zeta)}{V_0(q_1 + \zeta + 1, q_2 + \zeta + 1)} d\zeta \end{aligned} \quad (46)$$

where $V_0(q_1, q_2)$ is defined in (39) with $\mathcal{E}_1(q_1)$, $\mathcal{E}_2(q_2) \equiv 1$.

Proof. First, we prove this statement in the case $\gamma = 1$; i.e., $L_\gamma = L_1$. To this end, we will seek the solution of (36) as a product

$$V(q_1, q_2) = V_0(q_1, q_2) Y(q_1, q_2), \quad (47)$$

where the unknown function $Y(q_1, q_2)$ solves the first-order difference inhomogeneous equation with the constant coefficients

$$Y(q_1 + 1, q_2 + 1) - Y(q_1, q_2) = \frac{F(q_1, q_2)}{V_0(q_1 + 1, q_2 + 1)}. \quad (48)$$

Then adapting the technique from [35] to our case, we deduce that

$$\begin{aligned} Y(q_1, q_2) &= \frac{1}{2i} \int_{L_{-1}} \frac{F(q_1 + \zeta, q_2 + \zeta)}{V_0(q_1 + 1 + \zeta, q_2 + 1 + \zeta)} K(\zeta) d\zeta, \end{aligned} \quad (49)$$

if condition (45) holds.

Indeed, substituting $Y(q_1, q_2)$ to (48) and applying Cauchy's residue theorem arrive at

$$\begin{aligned} Y(q_1 + 1, q_2 + 1) - Y(q_1, q_2) &= \frac{1}{2i} \left[\int_{L_{-1}} \frac{F(q_1 + 1 + \zeta, q_2 + 1 + \zeta)}{V_0(q_1 + 2 + \zeta, q_2 + 2 + \zeta)} K(\zeta) d\zeta \right. \\ &\quad \left. - \int_{L_{-1}} \frac{F(q_1 + \zeta, q_2 + \zeta)}{V_0(q_1 + 1 + \zeta, q_2 + 1 + \zeta)} K(\zeta) d\zeta \right] \\ &= \pi \operatorname{res}_{\zeta=0} \frac{F(q_1 + \zeta, q_2 + \zeta)}{V_0(q_1 + 1 + \zeta, q_2 + 1 + \zeta)} K(\zeta) \\ &= \frac{F(q_1, q_2)}{V_0(q_1 + 1, q_2 + 1)}. \end{aligned} \quad (50)$$

To reach these equalities, we essentially used the following properties of the functions F , V_0 , and K :

- (i) $F(q_1 + \zeta, q_2 + \zeta)/V_0(q_1 + 1 + \zeta, q_2 + 1 + \zeta)$ has no poles if $\operatorname{Re} \zeta \in (-1, \gamma_0)$ and $\operatorname{Re} q_1, \operatorname{Re} q_2$ satisfy (45).
- (ii) $K(\zeta)$ is periodic of period 1 with simple poles at $\zeta = 0, \pm 1$ and multiple poles at $\zeta = -a$.

Besides, the following inequality holds:

$$|K(-\gamma + iz)| \leq C \begin{cases} e^{-4\pi z}, & z \rightarrow +\infty; \\ e^{2\pi z}, & z \rightarrow -\infty, \end{cases} \quad (51)$$

for $\gamma \in [0, 1]$

- (iii) There is the asymptotic representation

$$K(\zeta) \frac{F(q_1 + \zeta, q_2 + \zeta)}{V_0(q_1 + 1 + \zeta, q_2 + 1 + \zeta)} \rightarrow 0 \quad (52)$$

as $|\operatorname{Im} \zeta| \rightarrow 0$,

for the bounded $\operatorname{Im} q_1$ and $\operatorname{Im} q_2$ and if $\operatorname{Re} q_1, \operatorname{Re} q_2$ meet requirement (45).

Note that statements (i) and (ii) are simple consequences of Proposition 6 and (32). As for assessment (iii), it can be easily drawn from (i) and (ii).

After that, we return to the representation of $V(q_1, q_2)$ and obtain solution (36) as

$$V(q_1, q_2) = \frac{V_0(q_1, q_2)}{2i} \cdot \int_{L_{-1}} K(\zeta) \frac{F(q_1 + \zeta, q_2 + \zeta)}{V_0(q_1 + 1 + \zeta, q_2 + 1 + \zeta)} d\zeta. \quad (53)$$

This proves Proposition 7 in the case $L_\gamma := L_{-1}$.

Recasting the arguments above in the case $\gamma \in [0, 1)$ and applying Cauchy theorem allow us to prove Proposition 7 in the case of arbitrary contour L_γ , $\gamma \in [0, 1)$. It completes the proof of Proposition 7. $\square$

We are now in the position to obtain integral representation of the solution $u(x_1, x_2)$. Indeed, Propositions 6 and 7 provide

$$\begin{aligned} u^*(p_1, p_2) &= \frac{V_0(-p_1/\alpha, p_2/\alpha)}{2i} \int_{L_{-\gamma}} K(\zeta) \\ &\times \frac{F(-p_1/\alpha + \zeta, p_2/\alpha + \zeta)}{V_0(-p_1/\alpha + 1 + \zeta, p_2/\alpha + 1 + \zeta)} d\zeta = -\frac{1}{2i} \\ &\times \int_{L_{-\gamma}} K(\zeta) e^{-i\pi\zeta} \\ &\times \frac{\Gamma(1-p_1+\alpha\zeta)\Gamma(1-p_2-\alpha-\alpha\zeta)}{\Gamma(1-p_1)\Gamma(1-p_2)} \\ &\times f^*(p_1-\alpha\zeta, p_2+\alpha\zeta+\alpha) d\zeta, \end{aligned} \quad (54)$$

where

$$\begin{aligned} \operatorname{Re} p_1 &\neq m+1-\gamma\alpha, \\ \operatorname{Re} p_2 &\neq m+1-\alpha+\gamma\alpha, \end{aligned} \quad (55)$$

$m = 0, 1, 2, \dots, \gamma \in (0, 1).$

Note that, conditions on $\operatorname{Re} p_i$ hold, for example, in the domain

$$\begin{aligned} \operatorname{Re} p_1 &< 1-\gamma\alpha \\ \text{and } \operatorname{Re} p_2 &< 1-\alpha+\gamma\alpha. \end{aligned} \quad (56)$$

At last, we carry out the inverse Mellin transform to derive an explicit integral representation for $u(x_1, x_2)$,

$$\begin{aligned} u(x_1, x_2) &= \frac{1}{8\pi i} \int_0^{+\infty} \frac{dt_1}{t_1} \\ &\times \int_0^{+\infty} \frac{dt_2}{t_2} f\left(\frac{x_1}{t_1}, \frac{x_2}{t_2}\right) \mathcal{L}_\gamma(t_1, t_2, x_1, x_2), \end{aligned} \quad (57)$$

with

$$\begin{aligned} \mathcal{L}_\gamma(t_1, t_2, x_1, x_2) &= \int_{L_{-\gamma}} K(\zeta) \left(\frac{x_1}{t_1}\right)^{-\alpha\zeta} \left(\frac{x_2}{t_2}\right)^{\alpha+\alpha\zeta} \\ &\times e^{-i\pi\zeta} \int_{\gamma_1-i\infty}^{\gamma_1+i\infty} t_1^{-p_1} \frac{\Gamma(1-p_1+\alpha\zeta)}{\Gamma(1-p_1)} dp_1 \\ &\times \int_{\gamma_2-i\infty}^{\gamma_2+i\infty} t_2^{-p_2} \frac{\Gamma(1-p_2-\alpha-\alpha\zeta)}{\Gamma(1-p_2)} dp_2 d\zeta, \end{aligned} \quad (58)$$

for $\operatorname{Re} \gamma_1 \in (\mathcal{A}_1, \mathcal{B}_1)$ and $\operatorname{Re} \gamma_2 \in (\mathcal{A}_2, \mathcal{B}_2)$, where $\mathcal{A}_i$ and $\mathcal{B}_i$, $i = 1, 2$, are chosen that

$$\begin{aligned} m+1-\gamma\alpha &\notin (\mathcal{A}_1, \mathcal{B}_1) \\ \text{and } m+1-\alpha+\gamma\alpha &\notin (\mathcal{A}_2, \mathcal{B}_2), \end{aligned} \quad (59)$$

$m = 0, 1, 2, \dots$

In the light of (56), we can pick up $(\mathcal{A}_1, \mathcal{B}_1)$ and $(\mathcal{A}_2, \mathcal{B}_2)$ as

$$\begin{aligned} (\mathcal{A}_1, \mathcal{B}_1) &\subseteq \{\operatorname{Re} \gamma_1 < 1-\gamma\alpha\} \\ \text{and } (\mathcal{A}_2, \mathcal{B}_2) &\subseteq \{\operatorname{Re} \gamma_2 < 1-\alpha+\gamma\alpha\}. \end{aligned} \quad (60)$$

4. Some Technical Results

First we introduce some equivalent norms.

Proposition 8. Let s and s_1 be any nonnegative numbers. Then for any functions v and w , $v \in E_s^\beta(\bar{\Omega}) \cap E_{-s_1}^\beta(\bar{\Omega})$, and $w \in \mathcal{E}_s^\beta(\bar{\Omega}) \cap \mathcal{E}_{-s_1}^\beta(\bar{\Omega})$, we have the following norm equivalence:

$$\begin{aligned}
 & C_1 \left(\|v\|_{E_s^\beta(\bar{\Omega})} + \|v\|_{E_{-s_1}^\beta(\bar{\Omega})} \right) \leq \sup_{\bar{\Omega}} r^{-s}(x) \\
 & \quad \times \left(1 + r^2(x) \right)^{(s_1+s)/2} |v(x)| + \sup_{x, \bar{x} \in \bar{\Omega}} r^{-s+\beta}(x, \bar{x}) \\
 & \quad \times \left(1 + r^2(x, \bar{x}) \right)^{(s_1+s)/2} \frac{|v(x) - v(\bar{x})|}{|x - \bar{x}|^\beta} \\
 & \leq C_2 \left(\|v\|_{E_s^\beta(\bar{\Omega})} + \|v\|_{E_{-s_1}^\beta(\bar{\Omega})} \right), \\
 & C_1 \left(\|w\|_{\mathcal{E}_s^\beta(\bar{\Omega})} + \|w\|_{\mathcal{E}_{-s_1}^\beta(\bar{\Omega})} \right) \leq \sup_{\bar{\Omega}} d^{-s}(x) \\
 & \quad \times \left(1 + d(x) \right)^{s_1+s} |w(x)| + \sup_{x, \bar{x} \in \bar{\Omega}} d^{-s+\beta}(x, \bar{x}) \\
 & \quad \times \left(1 + d(x, \bar{x}) \right)^{s_1+s} \frac{|w(x) - w(\bar{x})|}{|x - \bar{x}|^\beta} \\
 & \leq C_2 \left(\|w\|_{\mathcal{E}_s^\beta(\bar{\Omega})} + \|w\|_{\mathcal{E}_{-s_1}^\beta(\bar{\Omega})} \right).
 \end{aligned} \tag{61}$$

The proof of this statement follows with direct calculations. Let

$$a \geq 1, \quad b \in [0, 2], \quad z, r \in \mathbb{R}. \tag{62}$$

Next we represent certain estimates for the functions

$$I_\pm^1(b, z, r) = \int_{-\infty}^{+\infty} e^{\pm i p r} \frac{\Gamma(a \pm i(p - z))}{\Gamma(a + b \pm i p)} dp, \tag{63}$$

which will be frequently used to evaluate the functions $u(x_1, x_2)$ and $\mathbf{D}_{x_i}^\alpha u$, $i = 1, 2$.

Proposition 9. Let ε be positive number, $\varepsilon \in (0, 1)$. Then there are estimates.

$$\begin{aligned}
 |I_\pm^1(b, z, r)| & \leq C \left(1 + z^2 \right) \cosh \left(\frac{\pi z}{2} \right) \\
 & \quad \times \begin{cases} e^{-\varepsilon|r|}, & \text{if } b \in (1, 2]; \\ e^{-\varepsilon|r|} (1 + |r|^{-1}), & \text{if } b \in (0, 1]; \\ (1 + |r|^{-1}), & \text{if } b = 0. \end{cases}
 \end{aligned} \tag{64}$$

Proof. For simplicity consideration, we put $a = 1$. Here we prove this proposition for the function $I_+^1(b, z, r)$. The case of $I_-^1(b, z, r)$ is considered in the similar way. We start our consideration with the case of positive b (i.e., $b \neq 0$).

FIGURE 1

The Stirling asymptotic formula for the Gamma-function,

$$\begin{aligned}
 \Gamma(im + d) & \approx C e^{-(\pi/2)|m|} |im + d|^{d-1/2} \\
 & \quad \times \exp \{im \ln |im + d| - im - d\} \\
 & \quad \times [1 + O(|im + d|^{-3})],
 \end{aligned} \tag{65}$$

as $|m| \rightarrow +\infty$,

provides that the integrand $\Gamma(1 + i(p - z))/\Gamma(1 + b + ip)$, $b > 0$, has the order $|p|^{-b}$ at the infinity. Moreover, the function $\Gamma(1 + \bar{p} - iz)/\Gamma(1 + b + \bar{p})$ with $\bar{p} = \text{Re } \bar{p} + ip$ has no poles in the domain $\text{Re } \bar{p} > -1$. Thus, by the residue theorem, we get

$$\begin{aligned}
 I_+^1(b, z, r) & = \begin{cases} e^{-\varepsilon r} \int_{-\infty}^{+\infty} \frac{\Gamma(1 - \varepsilon + i(p - z))}{\Gamma(1 + b - \varepsilon + ip)} e^{ipr} dp, & r \geq 0, \\ e^{\varepsilon r} \int_{-\infty}^{+\infty} \frac{\Gamma(1 + \varepsilon + i(p - z))}{\Gamma(1 + b + \varepsilon + ip)} e^{ipr} dp, & r < 0 \end{cases} \\
 & = e^{-\varepsilon|r|} \int_{-\infty}^{+\infty} \frac{\Gamma(1 - \varepsilon \text{sgn } r + i(p - z))}{\Gamma(1 + b - \varepsilon \text{sgn } r + ip)} e^{ipr} dp.
 \end{aligned} \tag{66}$$

Then, following [33], we decompose the plane (p, z) into 15 subdomains, as shown in Figure 1

$$(p, z) = \Omega_0 \cup \left(\bigcup_{i=1}^7 \Omega_i^+ \right) \cup \left(\bigcup_{i=1}^7 \Omega_i^- \right). \tag{67}$$

In this decomposition M is a sufficiently large number such that the terms $O((im + d)^{-3})$ in (65) can be neglected in all regions $\Omega_i^\pm$.

Taking into account this decomposition, we represent the function $I_+^1(b, z, r)$ as

$$I_+^1(b, z, r) = e^{-\varepsilon|r|} \times \begin{cases} \sum_{i=2}^7 \int_{\Omega_i^+} \frac{\Gamma(1 - \varepsilon \operatorname{sgn} r + i(p - z))}{\Gamma(1 + b - \varepsilon \operatorname{sgn} r + ip)} e^{ipr} dp, & z > 2M, \\ \sum_{i=2}^7 \int_{\Omega_i^-} \frac{\Gamma(1 - \varepsilon \operatorname{sgn} r + i(p - z))}{\Gamma(1 + b - \varepsilon \operatorname{sgn} r + ip)} e^{ipr} dp, & z < -2M, \\ \int_{\Omega_1^+ \cup \Omega_1^-} \frac{\Gamma(1 - \varepsilon \operatorname{sgn} r + i(p - z))}{\Gamma(1 + b - \varepsilon \operatorname{sgn} r + ip)} e^{ipr} dp \\ + \int_{\Omega_0} \frac{\Gamma(1 - \varepsilon \operatorname{sgn} r + i(p - z))}{\Gamma(1 + b - \varepsilon \operatorname{sgn} r + ip)} e^{ipr} dp, & |z| < 2M. \end{cases} \quad (68)$$

It is apparent that, if $z < -2M$ the function $I_+^1(b, z, r)$ is estimated analogous to the function $I_+^1(b, z, r)$ if $z > 2M$.

Thus, we consider here just case $z > -2M$. To this end, we apply Stirling formula (65) and the well-known properties of the Gamma-function. Introducing the functions

$$\begin{aligned} d_1(p, z) &= \left| \frac{1 - \varepsilon \operatorname{sgn} r + i(p - z)}{1 + b - \varepsilon \operatorname{sgn} r + ip} \right|^{-\varepsilon \operatorname{sgn} r + 1/2}, \\ d_2(p, z) &= \exp \{ i(p - z) \ln |1 - \varepsilon \operatorname{sgn} r + i(p - z)| \\ &\quad - ip \ln |1 + b - \varepsilon \operatorname{sgn} r + ip| \}, \end{aligned} \quad (69)$$

we arrive at the representation to the function $\Gamma(1 - \varepsilon \operatorname{sgn} r + i(p - z))/\Gamma(1 + b - \varepsilon \operatorname{sgn} r + ip)$ in every domain Ω_i^+ , Ω_1^- , and Ω_0 ,

$$\frac{\Gamma(1 - \varepsilon \operatorname{sgn} r + i(p - z))}{\Gamma(1 + b - \varepsilon \operatorname{sgn} r + ip)} = \begin{cases} \Phi_0(p, z, \operatorname{sgn} r), & \text{in } \Omega_0, \\ \frac{d_1(p, z) d_2(p, z)}{|1 + b - \varepsilon \operatorname{sgn} r + ip|^b} \Phi_1(p, z, \operatorname{sgn} r), & \text{in } \Omega_1^+, \\ \frac{e^{\pi z/2} d_1(p, z) d_2(p, z)}{|1 + b - \varepsilon \operatorname{sgn} r + ip|^b} \Phi_2(p, z, \operatorname{sgn} r), & \text{in } \Omega_2^+, \\ \frac{e^{\pi p/2}}{|1 + b - \varepsilon \operatorname{sgn} r + ip|^b} \Phi_3(p, z, \operatorname{sgn} r), & \text{in } \Omega_3^+, \\ \frac{e^{\pi(p-z)+\pi z/2} d_1(p, z) d_2(p, z)}{|1 + b - \varepsilon \operatorname{sgn} r + ip|^b} \Phi_4(p, z, \operatorname{sgn} r), & \text{in } \Omega_4^+, \\ e^{(\pi/2)(p-z)} |1 - \varepsilon \operatorname{sgn} r + ip|^{-\varepsilon \operatorname{sgn} r + 1/2} \Phi_5(p, z, \operatorname{sgn} r), & \text{in } \Omega_5^+, \\ \frac{e^{-\pi z/2} d_1(p, z) d_2(p, z)}{|1 + b - \varepsilon \operatorname{sgn} r + ip|^b} \Phi_6(p, z, \operatorname{sgn} r), & \text{in } \Omega_6^+ \cup \Omega_7^+, \\ \frac{d_1(p, z) d_2(p, z)}{|1 + b - \varepsilon \operatorname{sgn} r + ip|^b} \Phi_7(p, z, \operatorname{sgn} r), & \text{in } \Omega_1^-, \end{cases} \quad (70)$$

where the functions $\Phi_i(p, z, \operatorname{sgn} r)$, $i = 0, 1, 2, \dots, 7$, are uniformly bounded with respect to p and z together with $(\partial \Phi_i / \partial z)(p, z, \operatorname{sgn} r)$.

Moreover,

$$\left| \frac{\partial \Phi_1}{\partial p} \right|, \left| \frac{\partial \Phi_2}{\partial p} \right|, \left| \frac{\partial \Phi_6}{\partial p} \right|, \left| \frac{\partial \Phi_7}{\partial p} \right| \leq \frac{C}{|p|^3}, \quad (71)$$

for (p, z) belongs to one of the corresponding domains Ω_1^+ , Ω_2^+ , Ω_7^+ , Ω_1^- . It is worth mentioning that representation (70) holds for each $b \in [0, 2]$.

If $b \in (1, 2]$, representation (70) with aid (65) and (71) arrives at the inequality

$$\begin{aligned} I_+^1(b, z, r) &\leq C \begin{cases} e^{-\varepsilon|r|} (1 + z) \cosh\left(\frac{\pi z}{2}\right), & \text{if } z > 2M, \\ e^{-\varepsilon|r|} (1 + |z|), & \text{if } |z| \leq 2M. \end{cases} \end{aligned} \quad (72)$$

These relations guarantee the first estimate in Proposition 9 for $b \in (1, 2]$.

Further, we verify the estimate $I_+^1(b, z, r)$ if $b \in (0, 1]$. Representation (70) and straightforward calculations lead to the inequality

$$\begin{aligned} &\left| \int_{\Omega_0} \frac{\Gamma(1 - \varepsilon \operatorname{sgn} r + i(p - z))}{\Gamma(1 + b - \varepsilon \operatorname{sgn} r + ip)} e^{ipr} dp \right| \\ &\quad + \sum_{i=3}^6 \left| \int_{\Omega_i^+} \frac{\Gamma(1 - \varepsilon \operatorname{sgn} r + i(p - z))}{\Gamma(1 + b - \varepsilon \operatorname{sgn} r + ip)} e^{ipr} dp \right| \\ &\leq C(1 + |z|) \cosh\left(\frac{\pi z}{2}\right). \end{aligned} \quad (73)$$

To get the same estimate in the domains Ω_i^+ , $i = 1, 2, 7$, and Ω_1^- , it is enough to integrate by parts. Namely, let us consider the case Ω_1^+ . Based on representation (70), we can write

$$\begin{aligned}
& \int_{3M}^{+\infty} \frac{\Gamma(1 - \varepsilon \operatorname{sgn} r + i(p - z))}{\Gamma(1 + b - \varepsilon \operatorname{sgn} r + ip)} e^{ipr} dp \\
&= \int_{3M}^{+\infty} \frac{d_1(p, z) d_2(p, z)}{|1 + b - \varepsilon \operatorname{sgn} r + ip|^b} \Phi_1(p, z, \operatorname{sgn} r) \\
&\quad \times e^{ipr} dp = \frac{ie^{i3Mr}}{r} \frac{d_1(3M, z) d_2(3M, z)}{|1 + b - \varepsilon \operatorname{sgn} r + i3M|^b} \\
&\quad \times \Phi_1(3M, z, \operatorname{sgn} r) + \frac{i}{r} \\
&\quad \times \int_{3M}^{+\infty} e^{ipr} \frac{\partial}{\partial p} \left[\frac{d_1(p, z) d_2(p, z)}{|1 + b - \varepsilon \operatorname{sgn} r + ip|^b} \right. \\
&\quad \left. \times \Phi_1(p, z, \operatorname{sgn} r) \right] dp.
\end{aligned} \tag{74}$$

It is apparent that

$$\begin{aligned}
& |d_1(3M, z)| + |d_2(3M, z)| + |\Phi_1(3M, z, \operatorname{sgn} r)| \leq C \\
& \left| \frac{\partial d_1(p, z)}{\partial p} \right| + \left| \frac{\partial d_2(p, z)}{\partial p} \right| \leq C \frac{1 + z^2}{|p|}.
\end{aligned} \tag{75}$$

Thus, we can enhance estimate (74), taking into account (71), so as to get

$$\left| \int_{3M}^{+\infty} e^{ipr} \frac{\Gamma(1 - \varepsilon \operatorname{sgn} r + i(p - z))}{\Gamma(1 + b - \varepsilon \operatorname{sgn} r + ip)} dp \right| \leq C \frac{1 + z^2}{|r|}. \tag{76}$$

Recasting the arguments above in the case of domains Ω_2^+ , Ω_6^+ , and Ω_7^+ yields the same estimate to $|\int_{\Omega_2^+ \cup \Omega_6^+ \cup \Omega_7^+} e^{ipr} (\Gamma(1 - \varepsilon \operatorname{sgn} r + i(p - z))/\Gamma(1 + b - \varepsilon \operatorname{sgn} r + ip)) dp|$.

After that, we combine inequalities (73) and (76) and obtain the required estimate to $|I_+^1(b, z, r)|$ if $b \in (0, 1]$.

At last, we are left to produce suitable estimate of $|I_+^1(0, z, r)|$. Standard direct calculations lead to

$$\begin{aligned}
& |d_1(p, z)| + |d_2(p, z)| \leq C(1 + |z|) \\
& \text{for } (p, z) \in \left(\bigcup_{i=2}^6 \Omega_i^+ \right) \cup \Omega_0 \text{ and } b \in [0, 2].
\end{aligned} \tag{77}$$

In conclusion, we derive that

$$\begin{aligned}
& \left| \int_{\bigcup_{i=3}^6 \Omega_i^+ \cup \Omega_0} \frac{\Gamma(1 + i(p - z))}{\Gamma(1 + ip)} e^{ipr} dp \right| \\
& \leq C \cosh\left(\frac{\pi z}{2}\right) (1 + |z|).
\end{aligned} \tag{78}$$

After that, we remark that the terms $d_1(p, z)$, $d_2(p, z)$, and Φ_i share the same properties in Ω_1^- and Ω_i^+ , $i = 1, 2, 6, 7$. Thus, the estimate of $\int_{\Omega_1^+} (\Gamma(1 + i(p - z))/\Gamma(1 + ip)) e^{ipr} dp$ is very similar and we will confine ourselves the consideration of the case Ω_1^+ .

Simple conclusions draw to representations in Ω_1^+

$$\begin{aligned}
& d_1(p, z) = 1 + g_1(z, p), \\
& (p - z) \ln[1 + (p - z)^2] - p \ln[1 + p^2] \\
& = -z \ln p - z + g_2(p, z),
\end{aligned} \tag{79}$$

where

$$\begin{aligned}
& |g_1| \leq \frac{C(1 + z^2)}{p^2} \\
& \text{and } |g_2| \leq \frac{C}{|p|}.
\end{aligned} \tag{80}$$

Therefore, the main term in the integral $\int_{\Omega_1^+} (\Gamma(1 + i(p - z))/\Gamma(1 + ip)) e^{ipr} dp$ is

$$\begin{aligned}
& \int_{3M}^{+\infty} e^{irp} e^{-iz - iz \ln p} dp \\
&= e^{-iz} \left[\int_0^{+\infty} e^{irp} e^{-iz \ln p} dp - \int_0^{3M} e^{irp} e^{-iz \ln p} dp \right].
\end{aligned} \tag{81}$$

Note that the second term here is a regular function of r and z . In order to evaluate the first term in this representation, we apply Proposition 7.1 from [33] and obtain

$$\begin{aligned}
& \int_0^{+\infty} e^{irp} e^{-iz \ln p} dp \\
&= e^{iz \ln |r|} [z |r|^{-1} g_3(z) + (1 + z) r^{-1} g_4(z)],
\end{aligned} \tag{82}$$

where g_3 and g_4 are twice continuously differentiated and bounded functions for $|z| < 2M$.

In summary, we obtain the estimate

$$\left| \int_{\Omega_1^+} \frac{\Gamma(1 + i(p - z))}{\Gamma(1 + ip)} e^{ipr} dp \right| \leq C(1 + |r|^{-1}). \tag{83}$$

Finally, this inequality together with (78) completes the proof of Proposition 9. $\square$

Recasting the proof of the previous proposition, we state the results, which will be used to evaluate $|\mathbf{D}_{x_i}^{\alpha_i} u|$, $i = 1, 2$, $0 < \alpha_i < \alpha$, in Sections 5 and 6.

Proposition 10. *Let $z \in (-2M, 2M)$, $a \geq 1$, $b \in [0, 2]$, $r \in \mathbb{R}$; then the inequalities are fulfilled*

$$\begin{aligned}
& \left| \frac{\partial}{\partial z} I_{\pm}^1(b, z, r) \right| \leq C \cosh\left(\frac{\pi z}{2}\right) \\
& \times e^{-\varepsilon|r|} \begin{cases} 1, & \text{if } b \in (1, 2], \\ (1 + |r|^{-1}), & \text{if } b \in (0, 1], \end{cases}
\end{aligned}$$

$$\begin{aligned}
& \left| \frac{\partial}{\partial z} \int_{\Omega_0} \frac{\Gamma(a \pm i(p-z))}{\Gamma(a \pm ip)} e^{\pm ipr} dp \right| \\
& + \left| \frac{\partial}{\partial z} e^{\mp iz \ln |r|} \int_{\Omega_1^+ \cup \Omega_1^-} \frac{\Gamma(a \pm i(p-z))}{\Gamma(a \pm ip)} e^{\pm ipr} dp \right| \\
& \leq C(1 + |r|^{-1}).
\end{aligned} \tag{84}$$

The following results are related to the properties of the function

$$\begin{aligned}
I_{\pm}^2(\varepsilon_1, \varepsilon_2, \varepsilon, b, z, \rho) &:= \int_{-\varepsilon_1 \rho}^{\varepsilon_2 \rho} dr e^{-\varepsilon |r|} \\
&\times \int_{-\infty}^{+\infty} \frac{\Gamma(1 \pm \varepsilon \operatorname{sgn} r \pm i(p-z))}{\Gamma(1 + b \pm \varepsilon \operatorname{sgn} r \pm ip)} e^{\pm ir(p-z)} dp,
\end{aligned} \tag{85}$$

where $\varepsilon_1, \varepsilon_2$ are some positive constant, $\varepsilon \in [0, 1]$, $b \in [0, 2]$.

Proposition 11. *Let $z \in \mathbb{R}$, $\rho \in (0, +\infty)$. Then there are the following estimates for every fixed ρ :*

(i)

$$\begin{aligned}
I_{\pm}^2(\varepsilon_1, \varepsilon_2, \varepsilon, b, z, \rho) \\
\leq C[1 + |z|] \cosh\left(\frac{\pi z}{2}\right) [1 + \rho(\varepsilon_1 + \varepsilon_2)]
\end{aligned} \tag{86}$$

if $b \in (0, 2]$.

(ii)

$$\begin{aligned}
I_{\pm}^2(\varepsilon_1, \varepsilon_2, \varepsilon, 0, z, \rho) &\leq C[1 + |z|] \cosh\left(\frac{\pi z}{2}\right) \\
&\times \left[1 + \rho(\varepsilon_1 + \varepsilon_2) + \frac{1}{\rho \varepsilon_1} + \frac{1}{\rho \varepsilon_2}\right] \\
&\leq C[1 + |z|] \\
&\times \cosh\left(\frac{\pi z}{2}\right) \left[1 + \rho(\varepsilon_1 + \varepsilon_2) + \frac{\varepsilon_1}{\varepsilon_2} + \frac{\varepsilon_2}{\varepsilon_1}\right].
\end{aligned} \tag{87}$$

(iii) Besides, if $\rho = +\infty$, then

$$\begin{aligned}
I_{\pm}^2(\varepsilon_1, \varepsilon_2, 0, b, z, +\infty) \\
&= \int_{-\infty}^{+\infty} dr \int_{-\infty}^{+\infty} \frac{\Gamma(1 \pm i(p-z))}{\Gamma(1 + b \pm ip)} e^{\pm ir(p-z)} dp \\
&= \frac{\pi}{\Gamma(1 + b \pm iz)}, \quad \text{if } b \in [0, 2], \\
&|I_{\pm}^2(\varepsilon_1, \varepsilon_2, \varepsilon, 0, z, +\infty)| = |I_{\pm}^2(\varepsilon, z)| \\
&\leq C(1 + |z|) \cosh \frac{\pi z}{2}.
\end{aligned} \tag{88}$$

Proof. We will carry out the detailed proof in $I_{+}^2(\varepsilon_1, \varepsilon_2, \varepsilon, b, z, \rho)$ case. The arguments for $I_{-}^2(\varepsilon_1, \varepsilon_2, b, z, \rho)$ are almost identical and left to the interested reader.

Straightforward calculations lead to the representation

$$\begin{aligned}
I_{+}^2(\varepsilon_1, \varepsilon_2, \varepsilon, b, z, \rho) &= 2 \int_{-\infty}^{+\infty} \frac{\Gamma(1 + i(p-z))}{\Gamma(1 + b + ip)} \\
&\times \frac{\sin[(\varepsilon_1 + \varepsilon_2)(p-z)\rho]}{(p-z)} e^{i(\varepsilon_2 - \varepsilon_1)(p-z)\rho/2} dp \\
&\equiv 2 \int_{-\infty}^{+\infty} i_2 dp.
\end{aligned} \tag{89}$$

First, we consider case $b \in (0, 2]$. Asymptotic representation (70) with the decomposition in Figure 1 provides estimate in $\mathbb{R}^2 \setminus (\Omega_0 \cup \Omega_3^+ \cup \Omega_3^- \cup \Omega_5^+ \cup \Omega_5^-)$

$$|i_2| \leq C \frac{[1 + |z|] \cosh((\pi z)/2)}{|p-z| [p^2 + (1 + b - \varepsilon)^2]^{b/2}} \tag{90}$$

with $\varepsilon \in (0, 1)$ and

$$\begin{aligned}
|i_2| &\leq C \left| \frac{\sin(\varepsilon_1 + \varepsilon_2)(p-z)\rho}{(\varepsilon_1 + \varepsilon_2)(p-z)\rho} \right| \left[1 + \cosh\left(\frac{\pi z}{2}\right) \right] \\
&\cdot (\varepsilon_1 + \varepsilon_2)\rho
\end{aligned} \tag{91}$$

in $\Omega_0 \cup \Omega_3^+ \cup \Omega_3^- \cup \Omega_5^+ \cup \Omega_5^-$. Then, this inequality together with (89) leads to statement (ii).

Let us verify point (ii). One can easily check that

$$\begin{aligned}
&\left| \frac{\sin(\varepsilon_1 + \varepsilon_2)(p-z)\rho}{p-z} \right| \\
&\leq C \begin{cases} (\varepsilon_1 + \varepsilon_2)\rho, & \text{if } |p-z| \leq M, \\ |p-z|^{-1}, & \text{if } |p-z| > M. \end{cases}
\end{aligned} \tag{92}$$

After that, recasting the proof of Proposition 9 with aid the last inequality arrives at the estimate

$$\begin{aligned}
&\sum_{j=3}^6 \left| \int_{\Omega_j^+} i_2 dp \right| + \left| \int_{\Omega_0} i_2 dp \right| \\
&\leq C(1 + |z|) \cosh\left(\frac{\pi z}{2}\right) [1 + \rho(\varepsilon_1 + \varepsilon_2)].
\end{aligned} \tag{93}$$

Then we are left to evaluate the terms $\int_{\Omega_j^+} i_2 dp$, $j = 1, 2, 7$. We restrict ourselves the estimate of $\int_{\Omega_2^+} i_2 dp$, the remaining terms are evaluated the same way. Asymptotic (70) with

$b = 0$ and the change of variables, $p - z = k$, provide the representation

$$\begin{aligned} e^{-\pi z/2} \int_{\Omega_2^+} i_2 dp &= \int_{2M}^{+\infty} \frac{d_1(k+z, z) d_2(k+z, z)}{2ik} \Phi_2(k+z, z) \\ &\times [e^{i\varepsilon_2 \rho k} - e^{-i\varepsilon_1 \rho k}] dk. \end{aligned} \quad (94)$$

After that, following the proof of Proposition 9, we integrate by parts and deduce

$$\begin{aligned} e^{-\pi z/2} \left| \int_{\Omega_2^+} i_2 dp \right| &\leq C \left[\frac{1}{\rho \varepsilon_1} + \frac{1}{\rho \varepsilon_2} \right] \\ &\times \left\{ \frac{|d_1(2M+z, z)| |d_2(2M+z, z)|}{2M} \right. \\ &\times |\Phi_2(2M+z, z)| + \left| \int_{2M}^{+\infty} [e^{i\varepsilon_2 \rho k} + e^{-i\varepsilon_1 \rho k}] \right. \\ &\times \left. \frac{\partial}{\partial k} \frac{d_1(k+z, z) d_2(k+z, z) \Phi_2(k+z, z)}{k} dk \right\}. \end{aligned} \quad (95)$$

The standard calculations lead to

$$\left| \frac{\partial}{\partial k} \frac{d_1(k+z, z) d_2(k+z, z) \Phi_2(k+z, z)}{k} \right| \leq \frac{1+|z|}{k^2}. \quad (96)$$

In conclusion, we reach the estimate

$$\left| \int_{\Omega_2^+} i_2 dp \right| \leq C e^{-\pi z/2} [1+|z|] \left[\frac{1}{\rho \varepsilon_1} + \frac{1}{\rho \varepsilon_2} \right]. \quad (97)$$

Collecting this inequality with (93), we deduce the first estimate in statement (ii) of Proposition 11.

Further, to improve the estimate for $I_+^2(\varepsilon_1, \varepsilon_2, \varepsilon, 0, z, \rho)$, we consider two different cases:

(i)

$$2M\rho(\varepsilon_1 + \varepsilon_2) > \frac{\pi}{2}, \quad (98)$$

(ii)

$$2M\rho(\varepsilon_1 + \varepsilon_2) \leq \frac{\pi}{2}. \quad (99)$$

It is apparent that, in the first case there is

$$\frac{1}{\varepsilon_1 \rho} + \frac{1}{\varepsilon_2 \rho} < \frac{4M}{\pi} [\varepsilon_1^{-1} + \varepsilon_2^{-1}] [\varepsilon_1 + \varepsilon_2], \quad (100)$$

and, therefore, keeping in mind the previous estimate of $I_{\pm}^2(\varepsilon_1, \varepsilon_2, \varepsilon, 0, z, \rho)$, we deduce that

$$\begin{aligned} I_+^2(\varepsilon_1, \varepsilon_2, \varepsilon, 0, z, \rho) &\leq C [1+|z|] \cosh\left(\frac{\pi z}{2}\right) \\ &\times \left[1 + \rho(\varepsilon_1 + \varepsilon_2) + \frac{\varepsilon_1}{\varepsilon_2} + \frac{\varepsilon_2}{\varepsilon_1} \right]. \end{aligned} \quad (101)$$

Coming to the second case (i.e., $\rho^{-1} > 4M(\varepsilon_1^{-1} + \varepsilon_2^{-1})\pi^{-1}$), the change of the variable, $k(\varepsilon_1 + \varepsilon_2)\rho = v$, leads to

$$\begin{aligned} e^{-\pi z/2} \int_{\Omega_2^+} i_2 dp &= \int_{2M\rho(\varepsilon_1 + \varepsilon_2)}^{+\infty} d_1\left(z + \frac{v}{\rho(\varepsilon_1 + \varepsilon_2)}, z\right) \\ &\times d_2\left(z + \frac{v}{\rho(\varepsilon_1 + \varepsilon_2)}, z\right) \frac{\sin v}{v} e^{iv(2\varepsilon_1 + 2\varepsilon_2)^{-1}(\varepsilon_2 - \varepsilon_1)} dv \\ &= \int_{2M\rho(\varepsilon_1 + \varepsilon_2)}^{\pi/2} d_1\left(z + \frac{v}{\rho(\varepsilon_1 + \varepsilon_2)}, z\right) \\ &\times d_2\left(z + \frac{v}{\rho(\varepsilon_1 + \varepsilon_2)}, z\right) \frac{\sin v}{v} e^{iv(2\varepsilon_1 + 2\varepsilon_2)^{-1}(\varepsilon_2 - \varepsilon_1)} dv \\ &+ \int_{\pi/2}^{+\infty} d_1\left(z + \frac{v}{\rho(\varepsilon_1 + \varepsilon_2)}, z\right) \\ &\times d_2\left(z + \frac{v}{\rho(\varepsilon_1 + \varepsilon_2)}, z\right) \frac{\sin v}{v} e^{iv(2\varepsilon_1 + 2\varepsilon_2)^{-1}(\varepsilon_2 - \varepsilon_1)} dv. \end{aligned} \quad (102)$$

The straightforward calculations ensure the uniformly boundedness of the first term in (102); moreover, this estimate is independent of ε_1 , ε_2 , and ρ .

Further, we treat the second term in (102). Integrating by parts leads to

$$\begin{aligned} &\int_{\pi/2}^{+\infty} d_1\left(z + \frac{v}{\rho(\varepsilon_1 + \varepsilon_2)}, z\right) d_2\left(z + \frac{v}{\rho(\varepsilon_1 + \varepsilon_2)}, z\right) \\ &\times \Phi_2\left(z + \frac{v}{\rho(\varepsilon_1 + \varepsilon_2)}, z\right) \frac{\sin v}{v} e^{iv(2\varepsilon_1 + 2\varepsilon_2)^{-1}(\varepsilon_2 - \varepsilon_1)} dv \\ &= -i \frac{8e^{i\pi(\varepsilon_2 - \varepsilon_1)(\varepsilon_1 + \varepsilon_2)^{-1}/4}}{4\pi - \pi(\varepsilon_2 - \varepsilon_1)^2(\varepsilon_2 + \varepsilon_1)^{-2}} \\ &\times d_1\left(\frac{\pi}{2\rho(\varepsilon_1 + \varepsilon_2)} + z, z\right) d_2\left(\frac{\pi}{2\rho(\varepsilon_1 + \varepsilon_2)} + z, z\right) \\ &\times \Phi_2\left(\frac{\pi}{2\rho(\varepsilon_1 + \varepsilon_2)} + z, z\right) - \frac{4}{4 - (\varepsilon_2 - \varepsilon_1)^2(\varepsilon_2 + \varepsilon_1)^{-2}} \\ &\times \int_{\pi/2}^{+\infty} e^{-iv + iv(\varepsilon_2 - \varepsilon_1)/2(\varepsilon_2 + \varepsilon_1)} \\ &\times \frac{\partial}{\partial v} [d_1(v\rho^{-1}(\varepsilon_1 + \varepsilon_2)^{-1} + z, z) d_2(v\rho^{-1}(\varepsilon_1 + \varepsilon_2)^{-1} + z, z) \\ &\times \Phi_2(v\rho^{-1}(\varepsilon_1 + \varepsilon_2)^{-1} + z, z) v^{-1}] dv. \end{aligned} \quad (103)$$

After that, the properties of functions d_1 , d_2 , and Φ_2 allow us to extend this estimate and conclude

$$\begin{aligned} & \left| \int_{\pi/2}^{+\infty} d_1 \left(z + \frac{v}{\rho(\varepsilon_1 + \varepsilon_2)}, z \right) d_2 \left(z + \frac{v}{\rho(\varepsilon_1 + \varepsilon_2)}, z \right) \right. \\ & \quad \times \frac{\sin v}{v} e^{iv(2\varepsilon_1 + 2\varepsilon_2)^{-1}(\varepsilon_2 - \varepsilon_1)} \Phi_2 \left(z + \frac{v}{\rho(\varepsilon_1 + \varepsilon_2)}, z \right) dv \Big| \\ & \leq C \left| \frac{4}{4 - (\varepsilon_2 - \varepsilon_1)^2 (\varepsilon_2 + \varepsilon_1)^{-2}} \right| \\ & \leq C \left[1 + \frac{\varepsilon_2}{\varepsilon_1} + \frac{\varepsilon_1}{\varepsilon_2} \right]. \end{aligned} \quad (104)$$

Hence, this estimate completes the proof of statement (ii).

Finally, we are left to verify statement (iii). As for the first equality in (iii), it is a simple consequence of Lemma 5.3 [36]. Let us check the second inequality in (iii). To this end, we rewrite $I_+^2(\varepsilon, z)$ as

$$\begin{aligned} I_+^2(\varepsilon, z) &= \int_{-\infty}^{+\infty} \frac{\Gamma(1 + \varepsilon + i(p - z))}{\Gamma(1 + \varepsilon + ip)} \frac{dp}{\varepsilon - i(p - z)} \\ & \quad + \int_{-\infty}^{+\infty} \frac{\Gamma(1 - \varepsilon + i(p - z))}{\Gamma(1 - \varepsilon + ip)} \frac{dp}{\varepsilon + i(p - z)} \\ & \equiv i_1^+(z) + i_2^+(z), \end{aligned} \quad (105)$$

where we set

$$\begin{aligned} i_1^+(z) &= \int_{-\infty}^{+\infty} \frac{\Gamma(1 + \varepsilon + i(p - z))}{\Gamma(1 + \varepsilon + ip)} \frac{2\varepsilon dp}{\varepsilon^2 + (p - z)^2}, \\ i_2^+(z) &= - \int_{-1}^1 dq \\ & \quad \times \int_{-\infty}^{+\infty} \frac{\varepsilon^2}{\varepsilon + i(p - z)} \frac{\partial}{\partial(1 - q\varepsilon)} \frac{\Gamma(1 + q\varepsilon + i(p - z))}{\Gamma(1 + q\varepsilon + ip)} dp. \end{aligned} \quad (106)$$

Then representation (70) with $b = 0$ and direct calculations provide

$$|i_1^+(z)| + |i_2^+(z)| \leq C [1 + |z|] \cosh \frac{\pi z}{2}, \quad (107)$$

which implies

$$|I_+^2(\varepsilon, z)| \leq C [1 + |z|] \cosh \frac{\pi z}{2}. \quad (108)$$

Thus, the claim is proven. $\square$

Finally, we complete this preliminary section with two estimates that will be frequently used in the following sections.

Proposition 12. *Let $z \in [-2M, 2M]$, $x_1, x_2 \in [0, +\infty)$, $r, \rho \in \mathbb{R}$, $\varepsilon \in (0, 1)$. Then the following inequalities are fulfilled:*

(i)

$$\begin{aligned} & \left| \frac{\partial}{\partial z} I_{\pm}^2(\varepsilon_1, \varepsilon_2, \varepsilon, b, z, \rho) \right| \\ & \leq C \begin{cases} 1 + \rho(\varepsilon_1 + \varepsilon_2), & b \in (0, 2], \\ 1 + \rho(\varepsilon_1 + \varepsilon_2) + \frac{\varepsilon_1}{\varepsilon_2} + \frac{\varepsilon_2}{\varepsilon_1}, & b = 0, \end{cases} \end{aligned} \quad (109)$$

for every fixed ρ . Besides,

$$\left| \frac{\partial}{\partial z} I_{\pm}^2(\varepsilon_1, \varepsilon_2, \varepsilon, 0, z, +\infty) \right| \leq C. \quad (110)$$

(ii)

$$\begin{aligned} & J_1(x_1, x_2, t, \rho, M) \\ & := \left| \int_0^1 \frac{dq}{q} \int_{-2M}^{2M} \frac{\partial}{\partial z} \exp \left\{ izq \ln \frac{x_2 |r|}{x_1 |\rho|} \right\} dz \right| \leq C, \end{aligned} \quad (111)$$

where the positive constant C is independent of ρ , ε_1 , ε_2 , ε , x_1 , and x_2 .

Proof. To show the validity of statement (i), it is enough to recast the arguments in Propositions 11 and 10 if $z \in [-2M, 2M]$.

Next, we obtain the second inequality from this proposition. For the sake of clarity, we consider case $x_2 |r|/x_1 |\rho| \geq 1$. In the opposite case, we exchange the function $e^{izq \ln(x_2 |r|/x_1 |\rho|)}$ by $e^{-izq \ln(x_1 |\rho|/x_2 |r|)}$ and repeat the arguments below.

The direct calculations together with the change of variables, $v = 2Mq \ln(x_2 |r|/x_1 |\rho|)$, reduce the function J_1 to the form

$$J_1 = \left| \int_0^{2M \ln(x_2 |r|/x_1 |\rho|)} \frac{\sin v}{v} dv \right|. \quad (112)$$

At this point, we consider the two different cases for J_1 :

(i)

$$2M \ln \frac{x_2 |r|}{x_1 |\rho|} > \frac{\pi}{2}, \quad (113)$$

(ii)

$$0 \leq b_2 M \ln \frac{x_2 |r|}{x_1 |\rho|} \leq \frac{\pi}{2}. \quad (114)$$

In the first case, we have

$$\begin{aligned} J_1 & \leq \left| \int_0^{\pi/2} \frac{\sin v}{v} dv \right| + \left| \int_{\pi/2}^{2M \ln(x_2 |r|/x_1 |\rho|)} \frac{\sin v}{v} dv \right| \leq \frac{\pi}{2} \\ & \quad + \left| -\frac{\cos(2M \ln(x_2 |r|/x_1 |\rho|))}{2M \ln(x_2 |r|/x_1 |\rho|)} \right| \\ & \quad + \left| \int_{\pi/2}^{2M \ln(x_2 |r|/x_1 |\rho|)} \frac{\cos v}{v^2} dv \right| \leq \frac{\pi}{2} + \frac{2}{\pi} + \int_{\pi/2}^{+\infty} \frac{dv}{v^2} \\ & \leq C. \end{aligned} \quad (115)$$

Note that we apply integrating by parts in order to reach the last estimate.

Further, we obtain the same estimate in the second case. To this end, we use the easily verified inequality,

$$\left| \frac{\sin v}{v} \right| \leq 1 \quad 0 < v < \frac{\pi}{2}, \quad (116)$$

and deduce the bound

$$J_1 \leq \int_0^{2M \ln(x_2|r|/x_1|\rho|)} dv \leq \int_0^{\pi/2} dv \leq C. \quad (117)$$

In summary, we complete the proof of Proposition 12. $\square$

5. Estimates on Minor Derivatives of $u(x_1, x_2)$

To evaluate the function $u(x_1, x_2)$, we use representation (57) with $\gamma \in (0, 1)$ and $\gamma_1 = -\alpha\gamma$, $\gamma_2 = -\alpha(1 - \gamma)$. The change of variables

$$\begin{aligned} t_1 &= e^{r_1} \\ \text{and } t_2 &= e^{r_2}, \end{aligned} \quad (118)$$

in the integrals in (57) yields

$$\begin{aligned} u(x_1, x_2) &= \frac{e^{i\pi\gamma} x_1^{\alpha\gamma} x_2^{\alpha(1-\gamma)}}{i8\pi\alpha} \int_{-\infty}^{+\infty} dr_1 \\ &\quad \times \int_{-\infty}^{+\infty} dr_2 f(x_1 e^{-r_1}, x_2 e^{-r_2}) \mathcal{L}_\gamma(r_1, r_2, x_1, x_2), \end{aligned} \quad (119)$$

where

$$\begin{aligned} \mathcal{L}_\gamma(r_1, r_2, x_1, x_2) &= \int_{-\infty}^{+\infty} K\left(-\gamma - \frac{iz}{\alpha}\right) \\ &\quad \times e^{\pi z/\alpha} x_1^{-iz} x_2^{iz} e^{iz(r_1 - r_2)} \int_{-\infty}^{+\infty} e^{-ip_1 r_1} \\ &\quad \times \frac{\Gamma(1 - i(p_1 - z))}{\Gamma(1 + \alpha\gamma - ip_1)} dp_1 \int_{-\infty}^{+\infty} e^{ip_2 r_2} \\ &\quad \times \frac{\Gamma(1 + i(p_2 - z))}{\Gamma(1 + \alpha - \alpha\gamma - ip_2)} dp_2 dz, \quad \gamma \in (0, 1). \end{aligned} \quad (120)$$

5.1. Estimate on Maximum of $|u(x_1, x_2)|$. Before evaluating the functions u and $\mathbf{D}_{x_i}^{\alpha_i} u$, $i = 1, 2$, $0 < \alpha_i \leq 1$, we describe the suitable properties of the kernel $\mathcal{L}_\gamma$.

Lemma 13. Let $\varepsilon \in (0, 1)$ and let $r_1, r_2 \in \mathbb{R}$. Then there are estimates.

(i)

$$\begin{aligned} |\mathcal{L}_\gamma(r_1, r_2, x_1, x_2)| &\leq C \exp\{-\varepsilon|r_1| - \varepsilon|r_2|\} \\ &\quad \times \begin{cases} 1 + |r_2|^{-1} & \text{if } \gamma \in \left(\frac{1}{\alpha}, 1\right); \\ 1 + |r_1|^{-1} & \text{if } \gamma \in \left(0, 1 - \frac{1}{\alpha}\right); \\ (1 + |r_2|^{-1})(1 + |r_1|^{-1}) & \text{if } \gamma \in \left(1 - \frac{1}{\alpha}, \frac{1}{\alpha}\right). \end{cases} \end{aligned} \quad (121)$$

(ii) If $\gamma = 0$ or $\gamma = 1$, the following representations hold:

$$\begin{aligned} \mathcal{L}_0(r_1, r_2, x_1, x_2) &= \exp\{-\varepsilon|r_1| - \varepsilon|r_2|\} \\ &\quad \times ([\delta(r_1) + \delta(-r_1)] \mathcal{L}_0^1(r_1, r_2, x_1, x_2) \\ &\quad + \mathcal{L}_0^2(r_1, r_2, x_1, x_2)), \\ \mathcal{L}_1(r_1, r_2, x_1, x_2) &= \exp\{-\varepsilon|r_1| - \varepsilon|r_2|\} \\ &\quad \times ([\delta(r_2) + \delta(-r_2)] \mathcal{L}_1^1(r_1, r_2, x_1, x_2) \\ &\quad + \mathcal{L}_1^2(r_1, r_2, x_1, x_2)), \end{aligned} \quad (122)$$

where $\delta(r)$ is a Dirac delta function and

$$\begin{aligned} |\mathcal{L}_0^1(r_1, r_2, x_1, x_2)| + |\mathcal{L}_1^1(r_1, r_2, x_1, x_2)| &\leq C, \\ |\mathcal{L}_0^2(r_1, r_2, x_1, x_2)| &\leq C[1 + |r_1|^{-1}(1 + |r_2|)], \\ |\mathcal{L}_1^2(r_1, r_2, x_1, x_2)| &\leq C[1 + |r_2|^{-1}(1 + |r_1|)]. \end{aligned} \quad (123)$$

(iii)

$$\begin{aligned} \int_0^{+\infty} dr_1 \int_0^{+\infty} r_2^\beta |\mathcal{L}_\gamma(r_1, r_2, x_1, x_2)| dr_2 &\leq C \\ \text{if } \gamma &\in \left(\frac{1}{\alpha}, 1\right); \end{aligned}$$

$$\begin{aligned} \int_0^{+\infty} dr_1 r_1^\beta \left| \int_0^{+\infty} \mathcal{L}_\gamma(r_1, r_2, x_1, x_2) dr_2 \right| &\leq C \\ \text{if } \gamma &\in \left(0, 1 - \frac{1}{\alpha}\right); \end{aligned}$$

$$\begin{aligned} \int_0^{+\infty} dr_1 \int_0^{+\infty} r_2^\beta e^{-\varepsilon r_1 - \varepsilon r_2} |\mathcal{L}_1^2(r_1, r_2, x_1, x_2)| dr_2 \\ + \int_0^{+\infty} dr_1 r_1^\beta \left| \int_0^{+\infty} e^{-\varepsilon r_1 - \varepsilon r_2} \mathcal{L}_1^2(r_1, r_2, x_1, x_2) dr_2 \right| \\ \leq C \quad \text{if } \gamma = 1; \end{aligned} \quad (124)$$

$$\begin{aligned} \int_0^{+\infty} dr_1 \int_0^{+\infty} r_2^\beta e^{-\varepsilon r_1 - \varepsilon r_2} |\mathcal{L}_0^2(r_1, r_2, x_1, x_2)| dr_2 \\ + \int_0^{+\infty} dr_1 r_1^\beta \left| \int_0^{+\infty} e^{-\varepsilon r_1 - \varepsilon r_2} \mathcal{L}_0^2(r_1, r_2, x_1, x_2) dr_2 \right| \\ \leq C \quad \text{if } \gamma = 0. \end{aligned}$$

Here the constant C is independent on x_1, x_2 and γ .

Proof. First of all, we verify statement (i). It is easy to see that inequality (51) guarantees for $\gamma \in [0, 1]$

$$\begin{aligned}
\left| K\left(-\gamma - \frac{iz}{\alpha}\right) e^{\pi z/\alpha} \right| &\leq C e^{-3\pi|z|/\alpha}, \quad |z| \rightarrow +\infty, \\
\left| zK\left(1 - \frac{iz}{\alpha}\right) \right| &= \left| zK\left(-\frac{iz}{\alpha}\right) \right| \leq C, \quad |z| \leq 2M, \\
\left| \frac{\partial}{\partial z} zK\left(1 - \frac{iz}{\alpha}\right) \right| &= \left| \frac{\partial}{\partial z} zK\left(-\frac{iz}{\alpha}\right) \right| \leq C, \\
|z| &\leq 2M.
\end{aligned} \tag{125}$$

Then the results of Proposition 9 with aid of (125) provide statement (i).

As for statement (ii), we confirm ourselves the case of $\gamma = 0$, due to the arguments in the case $\delta = 1$ are similar.

First, we represent $\mathcal{L}_\gamma$, $\gamma \in (0, 1)$, as

$$\begin{aligned}
\mathcal{L}_\gamma(r_1, r_2, x_1, x_2) &= e^{-\varepsilon|r_1| - \varepsilon|r_2|} \int_{-\infty}^{+\infty} dz K\left(-\gamma - \frac{iz}{\alpha}\right) \\
&\times e^{\pi z/\alpha} e^{iz \ln(x_2/x_1)} e^{iz(r_1-r_2)} \int_{-\infty}^{+\infty} e^{-ip_1 r_1} \\
&\times \frac{\Gamma(1 - \varepsilon \operatorname{sgn} r_1 - i(p_1 - z))}{\Gamma(1 + \alpha\gamma - \varepsilon \operatorname{sgn} r_1 - ip_1)} dp_1 \int_{-\infty}^{+\infty} e^{ip_2 r_2} \\
&\times \frac{\Gamma(1 + \varepsilon \operatorname{sgn} r_2 + i(p_2 - z))}{\Gamma(1 + \alpha - \alpha\gamma + \varepsilon \operatorname{sgn} r_2 + ip_2)} dp_2.
\end{aligned} \tag{126}$$

Here we use the same reasons as in the proof of Proposition 9.

Then, as mentioned in the proof of Proposition 7 (see (ii) there) the function $K(-\gamma - iz/\alpha)$ has the simple poles if $\gamma = 0, 1$. Thus, following Cauchy's residue theorem, we rewrite $\mathcal{L}_\gamma$ as

$$\begin{aligned}
\mathcal{L}_\gamma(r_1, r_2, x_1, x_2) \\
= e^{-\varepsilon|r_1| - \varepsilon|r_2|} \left[\overline{\mathcal{L}}_0^1(r_1, r_2) + \overline{\mathcal{L}}_0^2(r_1, r_2, x_1, x_2) \right], \tag{127}
\end{aligned}$$

$\gamma = 0,$

where we set

$$\begin{aligned}
\overline{\mathcal{L}}_0^1(r_1, r_2) &= C \int_{-\infty}^{+\infty} e^{-ip_1 r_1} \int_{-\infty}^{+\infty} e^{ip_2 r_2} \\
&\times \frac{\Gamma(1 + \varepsilon \operatorname{sgn} r_2 + ip_2)}{\Gamma(1 + \alpha + \varepsilon \operatorname{sgn} r_2 + ip_2)} dp_2, \\
\overline{\mathcal{L}}_0^2(r_1, r_2, x_1, x_2) &= \int_{-\infty}^{+\infty} K\left(-\frac{iz}{\alpha}\right) \\
&\times e^{\pi z/\alpha} e^{iz \ln(x_2|x_1|/|r_2|x_1)} e^{iz(r_1-r_2)} \\
&\times \int_{-\infty}^{+\infty} e^{-ip_1 r_1} e^{-iz \ln|r_1|} \\
&\times \frac{\Gamma(1 - \varepsilon \operatorname{sgn} r_1 - i(p_1 - z))}{\Gamma(1 - \varepsilon \operatorname{sgn} r_1 - ip_1)} dp_1 \\
&\times \int_{-\infty}^{+\infty} e^{ip_2 r_2} e^{iz \ln|r_2|} \\
&\times \frac{\Gamma(1 + \varepsilon \operatorname{sgn} r_2 + i(p_2 - z))}{\Gamma(1 + \alpha + \varepsilon \operatorname{sgn} r_2 + ip_2)} dp_2 dz.
\end{aligned} \tag{128}$$

Concerning the estimate of $\overline{\mathcal{L}}_0^1(r_1, r_2)$, we apply Lemma 5.3 from [36] and deduce

$$\overline{\mathcal{L}}_0^1(r_1, r_2) = C [\delta(-r_1) + \delta(r_1)] \tag{129}$$

$$\times \int_{-\infty}^{+\infty} e^{ip_2 r_2} \frac{\Gamma(1 + \varepsilon \operatorname{sgn} r_2 + i(p_2 - z))}{\Gamma(1 + \alpha + \varepsilon \operatorname{sgn} r_2 + ip_2)} dp_2.$$

Then the asymptotic representation of Gamma-function (see, e.g., (1.5.15) in [1]),

$$\frac{\Gamma(p+a)}{\Gamma(p+b)} = p^{a-b} [1 + O(p^{-1})], \tag{130}$$

$$|\arg(p+a)| < \pi, \quad |p| \rightarrow +\infty,$$

provides the estimate

$$\left| \int_{-\infty}^{+\infty} e^{ip_2 r_2} \frac{\Gamma(1 + \varepsilon \operatorname{sgn} r_2 + i(p_2 - z))}{\Gamma(1 + \alpha + \varepsilon \operatorname{sgn} r_2 + ip_2)} dp_2 \right| \leq C. \tag{131}$$

Hence, we are left to evaluate the function $\overline{\mathcal{L}}_0^2$. To this end, we rewrite the term $\mathcal{L}_0^2$ in the form

$$\overline{\mathcal{L}}_0^2 = \sum_{i=1}^3 l_i, \tag{132}$$

where we put

$$\begin{aligned}
B(z, r_1, r_2) &= zK\left(-\frac{iz}{\alpha}\right) e^{iz(r_2-r_1)} \int_{-\infty}^{+\infty} e^{-ip_1 r_1} \\
&\times \frac{\Gamma(1 - \varepsilon \operatorname{sgn} r_1 - i(p_1 - z))}{\Gamma(1 - \varepsilon \operatorname{sgn} r_1 - ip_1)} e^{iz \ln|r_1|} dp_1 \int_{-\infty}^{+\infty} e^{ip_2 r_2} \\
&\times \frac{\Gamma(1 + \varepsilon \operatorname{sgn} r_2 + i(p_2 - z))}{\Gamma(1 + \alpha + \varepsilon \operatorname{sgn} r_2 + ip_2)} e^{-iz \ln|r_2|} dp_2, \\
l_1 &= \int_{|z| > 2M} K\left(-\frac{iz}{\alpha}\right) e^{iz \ln(x_2/x_1)} e^{iz(r_2-r_1)} \int_{-\infty}^{+\infty} e^{-ip_1 r_1} \\
&\times \frac{\Gamma(1 - \varepsilon \operatorname{sgn} r_1 - i(p_1 - z))}{\Gamma(1 - \varepsilon \operatorname{sgn} r_1 - ip_1)} dp_1 \int_{-\infty}^{+\infty} e^{ip_2 r_2} \\
&\times \frac{\Gamma(1 + \varepsilon \operatorname{sgn} r_2 + i(p_2 - z))}{\Gamma(1 + \alpha + \varepsilon \operatorname{sgn} r_2 + ip_2)} dp_2, \\
l_2 &= B(0, r_1, r_2) \int_{-2M}^{2M} \frac{e^{iz \ln(x_2|x_1|/|r_2|x_1)} - 1}{z} dz, \\
l_3 &= \int_{-2M}^{2M} \frac{e^{iz \ln(x_2|x_1|/|r_2|x_1)} [B(z, r_1, r_2) - B(0, r_1, r_2)]}{z} dz.
\end{aligned} \tag{133}$$

Treating the first term in (132) via Proposition 9 and inequalities (125), we arrive at

$$|l_1| \leq C (1 + |r_1|^{-1}). \tag{134}$$

Concerning l_2 , we get

$$B(0, r_1, r_2) = C \overline{\mathcal{L}}_0^1(r_1, r_2), \tag{135}$$

and therefore

$$l_2 = C \overline{\mathcal{L}}_0^1 \int_0^1 \frac{dq}{q} \int_{-2M}^{2M} \frac{\partial}{\partial z} e^{izq \ln(x_2 |r_1|/x_1 |r_2|)} dz. \quad (136)$$

After that, statement (ii) of Proposition 12 (with $r = r_1, \rho = r_2$) provides

$$\left| \int_0^1 \frac{dq}{q} \int_{-2M}^{2M} \frac{\partial}{\partial z} e^{izq \ln(x_2 |r_1|/x_1 |r_2|)} dz \right| \leq C. \quad (137)$$

In summary, we can conclude that

$$\begin{aligned} l_2 + \overline{\mathcal{L}}_0^1 &= [\delta(-r_1) + \delta(r_1)] \\ &\times C \left[1 + \int_0^1 \frac{dq}{q} \int_{-2M}^{2M} \frac{\partial}{\partial z} e^{izq \ln(x_2 |r_1|/x_1 |r_2|)} dz \right] \\ &\times \int_{-\infty}^{+\infty} e^{ip_2 r_2} \frac{\Gamma(1 + \varepsilon \operatorname{sgn} r_2 + ip_2)}{\Gamma(1 + \varepsilon \operatorname{sgn} r_2 + \alpha + ip_2)} dp_2 \\ &\equiv [\delta(-r_1) + \delta(r_1)] \mathcal{L}_0^1(r_1, r_2, x_1, x_2), \\ &\left| \mathcal{L}_0^1(r_1, r_2, x_1, x_2) \right| < C. \end{aligned} \quad (138)$$

Finally, coming to l_3 , we first rewrite it as

$$l_3 = \int_0^1 dq \int_{-2M}^{2M} e^{iz \ln(x_2 |r_1|/x_1 |r_2|)} \frac{\partial}{\partial qz} B(qz, r_1, r_2) dz. \quad (139)$$

Then, the tedious calculations with Propositions 9 and 10 and estimates (125) entail

$$|l_3| \leq C \left(1 + \frac{1 + |r_2|}{|r_1|} \right). \quad (140)$$

Hence, representations (127) and (132) together with estimates (134), (138), and (140) provide statement (ii) for the function $\mathcal{L}_0$ where

$$\mathcal{L}_0^2(r_1, r_2, x_1, x_2) = l_1 + l_3. \quad (141)$$

At last, the proof of statement (iii) is simple consequences of statements (i) and (ii) from this proposition. Thus, the claim is proven. $\square$

We are ready now to state estimates of the function $u(x_1, x_2)$.

Lemma 14. *Let assumptions (h1), (31), and (32) hold. Then the function $u(x_1, x_2)$ represented with (119) satisfies inequalities*

$$\begin{aligned} &\sup_{\overline{\Omega}} r^{-\alpha-\beta}(x) \left(1 + r^2(x) \right)^{(s_1+\alpha)/2} |u| \\ &+ \sup_{\overline{\Omega}} (x_1^{-\alpha} + x_2^{-\alpha}) r^{-\beta}(x) |u| \leq C(R_0) \|f\|_{E_{\beta}^{\beta}(\overline{\Omega})}, \quad (142) \\ &\sup_{\overline{\Omega}} d^{-\alpha-\beta}(x) (1 + d(x))^{s_1+\alpha} |u| \leq C(R_0) \|f\|_{E_{\beta}^{\beta}(\overline{\Omega})}. \end{aligned}$$

Proof. At the beginning, we verify the first inequality in this lemma. To this end, we evaluate the term $r^{-\alpha-\beta}(x)|u|$. Putting $\gamma \in (1/\alpha, 1)$ in (119), we conclude

$$\begin{aligned} r^{-\alpha-\beta}(x) |u| &\leq r^{-\beta}(x) \left| \int_{-\infty}^{+\infty} dr_1 \right. \\ &\times \int_{-\infty}^{+\infty} dr_2 [f(x_1 e^{-r_1}, x_2 e^{-r_2}) - f(x_1 e^{-r_1}, x_2)] \\ &\times \mathcal{L}_{\gamma}(r_1, r_2, x_1, x_2) \Big| + r^{-\beta}(x) \\ &\times \left| \int_{-\infty}^{+\infty} dr_1 f(x_1 e^{-r_1}, x_2) \right. \\ &\times \left. \int_{-\infty}^{+\infty} dr_2 \mathcal{L}_{\gamma}(r_1, r_2, x_1, x_2) \right| \equiv r^{-\beta}(x) [J_1 + J_2]; \end{aligned} \quad (143)$$

here we use the simple inequality

$$x_1^{\alpha\gamma} x_2^{\alpha(1-\gamma)} < r^{\alpha\gamma}(x) r^{\alpha(1-\gamma)}(x) = r^{\alpha}(x). \quad (144)$$

Concerning J_1 , we apply statement (i) from Lemma 13 with $\varepsilon > \beta$ to deduce

$$\begin{aligned} J_1 &\leq C \langle f \rangle_{x_2, 0, \Omega}^{(\beta)} \int_0^{+\infty} dr_1 e^{-\varepsilon r_1} \\ &\times \int_0^{+\infty} x_2^{\beta} |e^{-r_2} - 1|^{\beta} e^{-\varepsilon |r_2|} (1 + |r_2|^{-1}) dr_2 \\ &\leq C x_2^{\beta} \langle f \rangle_{x_2, 0, \Omega}^{(\beta)} \leq C r^{\beta}(x) \langle f \rangle_{x_2, 0, \Omega}^{(\beta)} \end{aligned} \quad (145)$$

with the constant is independent of x_1, x_2 , and γ .

Coming to the term J_2 , applying Proposition 11 to the function

$$\int_{-\infty}^{+\infty} dr_2 \int_{-\infty}^{+\infty} e^{-izr_2} e^{ip_2 r_2} \frac{\Gamma(1 + i(p_2 - z))}{\Gamma(1 + \alpha - \alpha\gamma + ip_2)} dp_2, \quad (146)$$

and keeping in mind Proposition 9 with inequality (125), we get

$$\begin{aligned} &\left| \int_{-\infty}^{+\infty} \mathcal{L}_{\gamma}(r_1, r_2, x_1, x_2) dr_2 \right| \\ &\leq C e^{-\varepsilon |r_1|} \int_{-\infty}^{+\infty} \left| K \left(-\gamma - \frac{iz}{\alpha} \right) \right| e^{\pi z/\alpha} (1 + |z|) \\ &\times \frac{\cosh(\pi z/2)}{|\Gamma(1 + \alpha - \alpha\gamma + iz)|} dz \leq C e^{-\varepsilon |r_1|}. \end{aligned} \quad (147)$$

In conclusion, we have

$$\begin{aligned} r^{-\beta}(x) J_2 &\leq r^{-\beta}(x) \|f\|_{E_{\beta}^{\beta}(\overline{\Omega})} \int_0^{+\infty} (x_2^2 e^{2r_1} + x_1^2)^{\beta/2} e^{-\varepsilon r_1} dr_1 \\ &\leq C \|f\|_{E_{\beta}^{\beta}(\overline{\Omega})}, \end{aligned} \quad (148)$$

if $\varepsilon > \beta$.

Then, collecting this inequality with (143) and (145) yields

$$\sup_{\overline{\Omega}} r^{-\alpha-\beta}(x) |u| \leq C \|f\|_{E_{\beta}^{\beta}(\overline{\Omega})}. \quad (149)$$

Further, we estimate $r^{-\alpha-\beta}(x)(1 + r^2(x))^{(s_1+\alpha)/2}|u|$. Assumption (31) provides that

$$f(x_1, x_2) = \frac{F_0(x_1, x_2)}{(1 + x_1^2 + x_2^2)^{(s_1+\alpha)/2}} \quad (150)$$

where the function F_0 satisfies conditions (31) and (32). Thus, substituting this representation of f in (143) and recasting arguments above, we deduce

$$\begin{aligned} \sup_{\overline{\Omega}} r^{-\alpha-\beta}(x) (1 + r^2(x))^{(s_1+\alpha)/2} |u| \\ \leq C(R_0) \|f\|_{E_{\beta}^{\beta}(\overline{\Omega})}. \end{aligned} \quad (151)$$

It is easy to see that the second estimate of Lemma 14 is proved with the similar arguments.

Finally, we are left to estimate $x_i^{-\alpha} r^{-\beta}(x) |u|$, $i = 1, 2$. For simplicity consideration, we evaluate $x_1^{-\alpha} r^{-\beta}(x) |u|$. Another case is studied with the same arguments. Using statement (ii) of Lemma 13 with $\gamma = 1$, we rewrite $u(x_1, x_2)$ as

$$\begin{aligned} x_1^{-\alpha} u(x_1, x_2) &= C \int_{-\infty}^{+\infty} dr_1 \int_{-\infty}^{+\infty} dr_2 f(x_1 e^{-r_1}, x_2 e^{-r_2}) \\ &\quad \times e^{-\epsilon(|r_1|+|r_2|)} \mathcal{L}_1^1(r_1, r_2, x_1, x_2) [\delta(r_2) + \delta(-r_2)] \\ &\quad + C \int_{-\infty}^{+\infty} dr_1 \\ &\quad \times \int_{-\infty}^{+\infty} dr_2 [f(x_1 e^{-r_1}, x_2 e^{-r_2}) - f(x_1 e^{-r_1}, x_2)] \\ &\quad \times \mathcal{L}_1^2(r_1, r_2, x_1, x_2) e^{-\epsilon(|r_1|+|r_2|)} \\ &\quad + C \int_{-\infty}^{+\infty} dr_1 f(x_1 e^{-r_1}, x_2) \int_{-\infty}^{+\infty} \mathcal{L}_1^2(r_1, r_2, x_1, x_2) \\ &\quad \times e^{-\epsilon(|r_1|+|r_2|)} dr_2. \end{aligned} \quad (152)$$

After that, statements (ii) and (iii) of Lemma 13 and Proposition 10 yield inequality

$$\sup_{\overline{\Omega}} x_1^{-\alpha} r^{-\beta}(x) |u| \leq C \|f\|_{E_{\beta}^{\beta}(\overline{\Omega})}. \quad (153)$$

This completes the proof of Lemma 14. $\square$

5.2. Estimate on $|\mathbf{D}_{x_1}^{\alpha_1} \mathbf{D}_{x_2}^{\alpha_2} u(x_1, x_2)|$, $0 < \alpha_1 + \alpha_2 \leq 1$. First, based on representation (57) and formula (2.1.17) [1], we obtain

$$\begin{aligned} \mathbf{D}_{x_1}^{\alpha_1} \mathbf{D}_{x_2}^{\alpha_2} u &= \mathbf{D}_{x_2}^{\alpha_2} \mathbf{D}_{x_1}^{\alpha_1} u = \frac{e^{i\pi\gamma}}{8\pi\alpha} \\ &\quad \times x_1^{\alpha\gamma-\alpha_1} x_2^{\alpha(1-\gamma)-\alpha_2} \int_{-\infty}^{+\infty} dr_1 \\ &\quad \times \int_{-\infty}^{+\infty} dr_2 [f(x_1 e^{-r_1}, x_2 e^{-r_2}) - f(x_1, x_2)] \\ &\quad \times \mathcal{M}_{\gamma}(r_1, r_2, x_1, x_2) + \frac{e^{i\pi\gamma} f(x_1, x_2)}{8\pi\alpha} \\ &\quad \times x_1^{\alpha\gamma-\alpha_1} x_2^{\alpha(1-\gamma)-\alpha_2} \int_{-\infty}^{+\infty} dr_1 \\ &\quad \times \int_{-\infty}^{+\infty} dr_2 \mathcal{M}_{\gamma}(r_1, r_2, x_1, x_2), \end{aligned} \quad (154)$$

where we set

$$\begin{aligned} \mathcal{M}_{\gamma}(r_1, r_2, x_1, x_2) &= \int_{-\infty}^{+\infty} dz K\left(-\gamma + \frac{iz}{\alpha}\right) e^{\frac{\pi z}{\alpha}} e^{iz(r_2-r_1)} e^{-iz \ln\left(\frac{x_1}{x_2}\right)} \\ &\quad \times \int_{-\infty}^{+\infty} e^{-ip_1 r_1} \frac{\Gamma(1-i(p_1-z))}{\Gamma(1+\alpha\gamma-\alpha_1-ip_1)} dp_1 \\ &\quad \times \int_{-\infty}^{+\infty} e^{ip_2 r_2} \frac{\Gamma(1+i(p_2-z))}{\Gamma(1+\alpha(1-\gamma)-\alpha_2+ip_2)} dp_2, \\ &\quad \gamma \in \left[\frac{\alpha_1}{\alpha}, 1 - \frac{\alpha_2}{\alpha}\right]. \end{aligned} \quad (155)$$

After that, recasting the arguments of Lemma 14 with aid of Propositions 9–11, we can state the following results.

Lemma 15. Let $\epsilon \in (\beta, 1)$ and $\gamma \in [\alpha_1/\alpha, 1 - \alpha_2/\alpha]$, and let the nonnegative numbers α_1 and α_2 satisfy $0 < \alpha_1 + \alpha_2 \leq 1$. Then the kernel $\mathcal{M}_{\gamma}(r_1, r_2, x_1, x_2)$ possesses the following properties:

(i)

$$\left| \int_{-\infty}^{+\infty} dr_1 \int_{-\infty}^{+\infty} dr_2 \mathcal{M}_{\gamma}(r_1, r_2, x_1, x_2) \right| \leq C; \quad (156)$$

(ii)

$$\begin{aligned} \int_0^{+\infty} dr_1 \int_0^{r_1} |e^{\pm r_2} - 1|^{\beta} |\mathcal{M}_{\gamma}(r_1, r_2, x_1, x_2)| dr_2 \\ + \int_0^{+\infty} dr_2 \int_0^{r_2} |e^{\pm r_1} - 1|^{\beta} |\mathcal{M}_{\delta}(r_1, r_2, x_1, x_2)| dr_1 \\ \leq C; \end{aligned} \quad (157)$$

(iii)

$$\left| \int_0^{+\infty} dr_1 |e^{\pm r_1} - 1|^\beta \left| \int_{-r_1}^{r_1} \mathcal{M}_\gamma(r_1, r_2, x_1, x_2) dr_2 \right| \right. \\ \left. + \int_0^{+\infty} dr_2 |e^{\pm r_2} - 1|^\beta \left| \int_{-r_2}^{r_2} \mathcal{M}_\gamma(r_1, r_2, x_1, x_2) dr_1 \right| \right| \quad (158)$$

 $\leq C$;(iv) If either $\gamma = 1$ and $\alpha_1 = 1, \alpha_2 = 0$ or $\gamma = 0$ and $\alpha_2 = 1, \alpha_1 = 0$, then

$$\begin{aligned} \mathcal{M}_1(r_1, r_2, x_1, x_2) &= [\delta(r_2) + \delta(-r_2)] \mathcal{M}_1^1(r_1) \\ &\quad + \mathcal{M}_1^2(r_1, r_2, x_1, x_2), \\ \mathcal{M}_0(r_1, r_2, x_1, x_2) &= [\delta(r_1) + \delta(-r_1)] \mathcal{M}_0^1(r_2) \\ &\quad + \mathcal{M}_0^2(r_1, r_2, x_1, x_2), \end{aligned} \quad (159)$$

where the functions $\mathcal{M}_1^1$ and $\mathcal{M}_0^1$ satisfy statements (i)-(iii) of this lemma and

$$\begin{aligned} |\mathcal{M}_1^1(r_1)| &\leq C \frac{e^{-\varepsilon|r_1|}}{|r_1|}, \\ |\mathcal{M}_0^1(r_1)| &\leq C \frac{e^{-\varepsilon|r_2|}}{|r_2|}, \end{aligned} \quad (160)$$

$$\left| \int_{-\infty}^{+\infty} \mathcal{M}_1^1(r_1) dr_1 \right| + \left| \int_{-\infty}^{+\infty} \mathcal{M}_0^1(r_2) dr_2 \right| \leq C.$$

Now we are ready to describe properties of low derivatives to the function u .

Lemma 16. Let assumptions of Lemma 14 hold, and $0 < \alpha_1 + \alpha_2 \leq 1$. Then there are the estimates.

(i)

$$\begin{aligned} \sup_{\bar{\Omega}} r^{-\alpha-\beta+\alpha_1+\alpha_2}(x) (1+r^2(x))^{(s_1+\alpha)/2} &|\mathbf{D}_{x_1}^{\alpha_1} \mathbf{D}_{x_2}^{\alpha_2} u| \\ &+ \sup_{\bar{\Omega}} r^{-\alpha-\beta+\alpha_1+\alpha_2}(x) (1+r^2(x))^{(s_1+\alpha)/2} |\mathbf{D}_{x_2}^{\alpha_2} \mathbf{D}_{x_1}^{\alpha_1} u| \\ &+ \sup_{\bar{\Omega}} x_1^{-\alpha+1} r^{-\beta}(x) |\mathbf{D}_{x_1} u| \\ &+ \sup_{\bar{\Omega}} x_2^{-\alpha+1} r^{-\beta}(x) |\mathbf{D}_{x_2} u| \leq C(R_0) \|f\|_{E_\beta^\beta(\bar{\Omega})}; \end{aligned} \quad (161)$$

(ii)

$$\begin{aligned} \sup_{\bar{\Omega}} d^{-\alpha-\beta+\alpha_1+\alpha_2}(x) (1+d(x))^{s_1+\alpha} &|\mathbf{D}_{x_1}^{\alpha_1} \mathbf{D}_{x_2}^{\alpha_2} u| \\ &+ \sup_{\bar{\Omega}} d^{-\alpha-\beta+\alpha_1+\alpha_2}(x) (1+d(x))^{s_1+\alpha} |\mathbf{D}_{x_2}^{\alpha_2} \mathbf{D}_{x_1}^{\alpha_1} u| \\ &\leq C(R_0) \|f\|_{E_\beta^\beta(\bar{\Omega})}. \end{aligned} \quad (162)$$

Proof. Taking into account (154), we will carry out the detailed proof of the inequality

$$\begin{aligned} \sup_{\bar{\Omega}} r^{-\alpha-\beta+\alpha_1+\alpha_2}(x) (1+r^2(x))^{(s_1+\alpha)/2} &|\mathbf{D}_{x_1}^{\alpha_1} \mathbf{D}_{x_2}^{\alpha_2} u| \\ &+ \sup_{\bar{\Omega}} x_1^{-\alpha+1} |\mathbf{D}_{x_1} u| \leq C(R_0) \|f\|_{E_\beta^\beta(\bar{\Omega})}. \end{aligned} \quad (163)$$

The remaining terms in (i) and (ii) are estimated in the same way and with the recasting of the corresponding arguments of Lemma 14.

We begin to evaluate the first term in (163). To this end, we represent $\mathbf{D}_{x_1}^{\alpha_1} \mathbf{D}_{x_2}^{\alpha_2} u$ as

$$\mathbf{D}_{x_1}^{\alpha_1} \mathbf{D}_{x_2}^{\alpha_2} u = \frac{e^{i\pi\gamma}}{8\pi\alpha} x_1^{\alpha\gamma-\alpha_1} x_2^{\alpha(1-\gamma)-\alpha_2} \sum_{j=1}^9 Q_j, \quad (164)$$

where we put

$$\begin{aligned} Q_1 &= \int_0^{+\infty} dr_1 \int_{-r_1}^{r_1} [f(x_1 e^{r_1}, x_2 e^{r_2}) - f(x_1 e^{r_1}, x_2)] \\ &\quad \times \mathcal{M}_\gamma(r_1, r_2, x_1, x_2) dr_2, \\ Q_2 &= \int_0^{+\infty} dr_1 [f(x_1 e^{r_1}, x_2) - f(x_1, x_2)] \\ &\quad \times \int_{-r_1}^{r_1} \mathcal{M}_\gamma(r_1, r_2, x_1, x_2) dr_2, \\ Q_3 &= \int_{-\infty}^0 dr_1 \int_{r_1}^{-r_1} [f(x_1 e^{r_1}, x_2 e^{r_2}) - f(x_1 e^{r_1}, x_2)] \\ &\quad \times \mathcal{M}_\gamma(r_1, r_2, x_1, x_2) dr_2, \\ Q_4 &= \int_{-\infty}^0 dr_1 [f(x_1 e^{r_1}, x_2) - f(x_1, x_2)] \\ &\quad \times \int_{r_1}^{-r_1} \mathcal{M}_\gamma(r_1, r_2, x_1, x_2) dr_2, \\ Q_5 &= \int_0^{+\infty} dr_2 \int_{-r_2}^{r_2} [f(x_1 e^{r_1}, x_2 e^{r_2}) - f(x_1, x_2 e^{r_2})] \\ &\quad \times \mathcal{M}_\gamma(r_1, r_2, x_1, x_2) dr_1, \\ Q_6 &= \int_0^{+\infty} dr_2 [f(x_1, x_2 e^{r_2}) - f(x_1, x_2)] \\ &\quad \times \int_{-r_2}^{r_2} \mathcal{M}_\gamma(r_1, r_2, x_1, x_2) dr_1, \\ Q_7 &= \int_{-\infty}^0 dr_2 \int_{r_2}^{-r_2} [f(x_1 e^{r_1}, x_2 e^{r_2}) - f(x_1, x_2 e^{r_2})] \\ &\quad \times \mathcal{M}_\gamma(r_1, r_2, x_1, x_2) dr_1, \\ Q_8 &= \int_{-\infty}^0 dr_2 [f(x_1, x_2 e^{r_2}) - f(x_1, x_2)] \\ &\quad \times \int_{r_2}^{-r_2} \mathcal{M}_\gamma(r_1, r_2, x_1, x_2) dr_1, \\ Q_9 &= f(x_1, x_2) \int_{-\infty}^{+\infty} dr_1 \int_{-\infty}^{+\infty} \mathcal{M}_\gamma(r_1, r_2, x_1, x_2) dr_2. \end{aligned} \quad (165)$$

Statement (ii) of Lemma 15 provides

$$\begin{aligned} |Q_1| + |Q_3| + |Q_5| + |Q_7| &\leq C \left[x_2^\beta \langle f \rangle_{x_2,0,\Omega}^{(\beta)} \int_0^{+\infty} dr_1 \right. \\ &\quad \times \int_0^{r_1} |e^{r_2} - 1|^\beta |\mathcal{M}_\gamma(r_1, r_2, x_1, x_2)| dr_2 \\ &\quad + x_1^\beta \langle f \rangle_{x_1,0,\Omega}^{(\beta)} \int_0^{+\infty} dr_2 \\ &\quad \times \int_0^{r_2} |e^{r_1} - 1|^\beta |\mathcal{M}_\gamma(r_1, r_2, x_1, x_2)| dr_1 \Big] \leq C \left[x_1^\beta \right. \\ &\quad \left. + x_2^\beta \right] \left[\langle f \rangle_{x_1,0,\Omega}^{(\beta)} + \langle f \rangle_{x_2,0,\Omega}^{(\beta)} \right]. \end{aligned} \quad (166)$$

As for terms Q_2 , Q_4 , Q_6 , and Q_8 , we apply statement (iii) of Lemma 15 and deduce

$$\begin{aligned} |Q_2| + |Q_4| + |Q_6| + |Q_8| \\ \leq C \left[x_1^\beta + x_2^\beta \right] \left[\langle f \rangle_{x_1,0,\Omega}^{(\beta)} + \langle f \rangle_{x_2,0,\Omega}^{(\beta)} \right]. \end{aligned} \quad (167)$$

Finally, statement (i) of Lemma 16 guarantees the estimate to Q_9

$$|Q_9| \leq C r^\beta(x) \sup_{\bar{\Omega}} r^{-\beta}(x) |f|. \quad (168)$$

Hence, representation (164) and inequalities for Q_j entail

$$\sup_{\bar{\Omega}} r^{-\alpha-\beta+\alpha_1+\alpha_2}(x) |\mathbf{D}_{x_1}^{\alpha_1} \mathbf{D}_{x_2}^{\alpha_2}| \leq C \|f\|_{E_\beta^\beta(\bar{\Omega})}. \quad (169)$$

In order to complete the proof, we need a similar estimate to the term $x_1^{-\alpha+1} r^{-\beta}(x) |\mathbf{D}_{x_1} u|$. We exchange $\mathcal{M}_\gamma$ in (154) by the function $\mathcal{M}_1$ from (iv) in Lemma 15. After that, recasting the arguments above together with statement (iv) in Lemma 15 derives

$$\sup_{\bar{\Omega}} x_1^{-\alpha+1} r^{-\beta}(x) |\mathbf{D}_{x_1} u| \leq C \|f\|_{E_\beta^\beta(\bar{\Omega})}. \quad (170)$$

This finishes the proof of the lemma. $\square$

The following result is a simple consequence of Lemmas 14 and 16, interpolation inequalities, and Proposition 8.

Proposition 17. *Let assumptions of Lemma 14 hold. Then*

$$\begin{aligned} \|u\|_{E_s^\beta(\bar{\Omega})} + \|u\|_{E_{-s_1}^\beta(\bar{\Omega})} &\leq C(R_0) \|f\|_{E_\beta^\beta(\bar{\Omega})}, \\ \|u\|_{\mathcal{E}_s^\beta(\bar{\Omega})} + \|u\|_{\mathcal{E}_{-s_1}^\beta(\bar{\Omega})} &\leq C(R_0) \|f\|_{\mathcal{E}_\beta^\beta(\bar{\Omega})}. \end{aligned} \quad (171)$$

6. Estimates on $\mathbf{D}_{x_1}^{\alpha_1} \mathbf{D}_{x_2}^{\alpha_2} u(x_1, x_2)$, $1 < \alpha_1 + \alpha_2 \leq \alpha$

We begin our consideration with the estimates of the function $\mathcal{D}_{x_1}^\alpha u$. Here, we use representation (119) to the function $u(x_1, x_2)$ with $\gamma = 1$; i.e., $\mathcal{L}_\gamma = \mathcal{L}_1$.

Recasting the arguments leading to (127) arrives at

$$\begin{aligned} u(x_1, x_2) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(e^{y_1}, e^{y_2}) \\ &\quad \times x_1^\alpha \overline{\mathcal{L}}_1^1(y_1 - \ln x_1, y_2 - \ln x_2) e^{-\varepsilon|y_1 - \ln x_1|} dy_1 dy_2 \\ &\quad + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(e^{y_1}, e^{y_2}) \\ &\quad \times x_1^\alpha \overline{\mathcal{L}}_1^2(y_1 - \ln x_1, y_2 - \ln x_2, x_1, x_2) \\ &\quad \times e^{-\varepsilon|y_1 - \ln x_1| - \varepsilon|y_2 - \ln x_2|} dy_1 dy_2 \end{aligned} \quad (172)$$

where we put

$$\begin{aligned} \overline{\mathcal{L}}_1^1(y_1 - \ln x_1, y_2 - \ln x_2) &= -\frac{1}{4\alpha} \\ &\quad \times \text{res}_{z=0} \left[K\left(-1 + \frac{iz}{\alpha}\right) \int_{-\infty}^{+\infty} \frac{\Gamma(1+i(p_2-z))}{\Gamma(1+ip_2)} e^{-ip_2(y_2 - \ln x_2)} dp_2 \right. \\ &\quad \times \left. \int_{-\infty}^{+\infty} \frac{\Gamma(1-i(p_1-z))}{\Gamma(1+\alpha-ip_1)} e^{ip_1(y_1 - \ln x_1)} dp_1 e^{\pi z/\alpha} e^{iz(y_2 - y_1)} \right], \\ \overline{\mathcal{L}}_1^2(y_1 - \ln x_1, y_2 - \ln x_2, x_1, x_2) &= -\frac{1}{8\pi i} \\ &\quad \times \int_{-\infty}^{+\infty} dz e^{\pi z/\alpha} K\left(-1 + \frac{iz}{\alpha}\right) e^{iz(y_2 - y_1)} \\ &\quad \times \int_{-\infty}^{+\infty} \frac{\Gamma(1+\varepsilon \operatorname{sgn}(y_1 - \ln x_1) - i(p_1 - z))}{\Gamma(1+\varepsilon \operatorname{sgn}(y_1 - \ln x_1) - ip_1)} e^{ip_1(y_1 - \ln x_1)} dp_1 \\ &\quad \times \int_{-\infty}^{+\infty} \frac{\Gamma(1-\varepsilon \operatorname{sgn}(y_2 - \ln x_2) + i(p_2 - z))}{\Gamma(1-\varepsilon \operatorname{sgn}(y_2 - \ln x_2) + ip_2)} e^{-ip_2(y_2 - \ln x_2)} dp_2. \end{aligned} \quad (173)$$

After that, straightforward calculations arrive at

$$\begin{aligned} \overline{\mathcal{L}}_1^1(y_1 - \ln x_1, y_2 - \ln x_2) \\ = C [\delta(y_2 - \ln x_2) + \delta(-y_2 + \ln x_2)] \\ \times \int_{-\infty}^{+\infty} \frac{\Gamma(1-ip_1) e^{ip_1(y_1 - \ln x_1)}}{\Gamma(1+\alpha-ip_1)} dp_1. \end{aligned} \quad (174)$$

Thus, we can rewrite $u(x_1, x_2)$ in the form

$$\begin{aligned} u(x_1, x_2) &= C \int_{-\infty}^{+\infty} dy_1 f(e^{y_1}, x_2) \\ &\quad \times \int_{-\infty}^{+\infty} \frac{\Gamma(1-ip_1)}{\Gamma(1+\alpha-ip_1)} x_1^\alpha e^{ip_1(y_1 - \ln x_1)} dp_1 \\ &\quad + C \int_{-\infty}^{+\infty} f(e^{y_1}, e^{y_2}) \\ &\quad \times x_1^\alpha \overline{\mathcal{L}}_1^2(y_1 - \ln x_1, y_2 - \ln x_2, x_1, x_2) \\ &\quad \times e^{-\{\varepsilon|y_1 - \ln x_1| - \varepsilon|y_2 - \ln x_2|\}} dy_1 dy_2. \end{aligned} \quad (175)$$

We are now in the position to calculate $\mathbf{D}_{x_1}^\alpha u(x_1, x_2)$. Indeed, formula (2.1.17) [1] and direct calculations provide the representation

$$\mathbf{D}_{x_1}^\alpha u(x_1, x_2) = \sum_{j=1}^3 \mathcal{U}_j(x_1, x_2) \quad (176)$$

where we put

$$\begin{aligned} \mathcal{U}_1 &= \int_{-\infty}^{+\infty} f(e^{y_1}, x_2) \mathcal{R}_1(x_1, y_1 - \ln x_1) dy_1, \\ \mathcal{U}_2 &= f(x_1, x_2) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathcal{R}_2(x_1, x_2, y_1 - \ln x_1, y_2 - \ln x_2) dy_1 dy_2, \\ \mathcal{U}_3 &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [f(e^{y_1}, e^{y_2}) - f(x_1, x_2)] \\ &\quad \times \mathcal{R}_2(x_1, x_2, y_1 - \ln x_1, y_2 - \ln x_2) dy_1 dy_2, \end{aligned} \quad (177)$$

with

$$\begin{aligned} \mathcal{R}_1(x_1, y_1 - \ln x_1) &= C \int_{-\infty}^{+\infty} e^{ip_1(y_1 - \ln x_1)} dp_1, \\ \mathcal{R}_2(x_1, x_2, y_1 - \ln x_1, y_2 - \ln x_2) &= e^{-\varepsilon|y_2 - \ln x_2|} \mathbf{D}_{x_1}^\alpha \left[x_1^\alpha e^{-\varepsilon|y_1 - \ln x_1|} \overline{\mathcal{L}}_1^2(y_1 - \ln x_1, y_2 - \ln x_2, x_1, x_2) \right]. \end{aligned} \quad (178)$$

After that, introducing new variables

$$\begin{aligned} r_1 &= \ln x_1 - y_1, \\ r_2 &= \ln x_2 - y_2, \end{aligned} \quad (179)$$

and new functions

$$\begin{aligned} B_1(z, r_1, r_2) &= e^{\pi z/\alpha} z K \left(-1 + \frac{iz}{\alpha} \right) \\ &\quad \times e^{iz(r_2 - r_1)} e^{iz \ln |r_2/r_1|} \int_{-\infty}^{+\infty} e^{-ip_1 r_1} \\ &\quad \times \frac{\Gamma(1 - \varepsilon \operatorname{sgn} r_1 - i(p_1 - z))}{\Gamma(1 - \varepsilon \operatorname{sgn} r_1 - ip_1)} dp_1 \int_{-\infty}^{+\infty} e^{ip_2 r_2} \\ &\quad \times \frac{\Gamma(1 + \varepsilon \operatorname{sgn} r_2 + i(p_2 - z))}{\Gamma(1 + \varepsilon \operatorname{sgn} r_2 + ip_2)} dp_2, \\ \mathcal{R}_3(x_1, x_2, r_1, r_2) &= \mathcal{R}_2(x_1, x_2, -r_1, -r_2) \\ &\quad - e^{-\varepsilon|r_1| - \varepsilon|r_2|} B_1(0, r_1, r_2) \int_{|z| < 2M} \frac{dz}{z} \\ &\quad \times \int_0^1 \frac{\partial}{\partial q} e^{-iqz \ln(x_1|r_2|/x_2|r_1|)} dq, \end{aligned} \quad (180)$$

we rewrite the functions $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{U}_j$ in the more comfortable forms

$$\begin{aligned} \mathcal{R}_1(x_1, -r_1) &= C [\delta(r_1) + \delta(-r_1)], \\ \mathcal{R}_2(x_1, x_2, -r_1, -r_2) &= e^{-\varepsilon|r_1| - \varepsilon|r_2|} \int_{-\infty}^{+\infty} \frac{B_1(z, r_1, r_2) e^{-iz \ln(x_1|r_2|/x_2|r_1|)}}{z} dz, \\ \mathcal{U}_1(x_1, x_2) &= C f(x_1, x_2), \\ \mathcal{U}_2(x_1, x_2) &= f(x_1, x_2) \\ &\quad \times \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathcal{R}_2(x_1, x_2, -r_1, -r_2) dr_1 dr_2, \\ \mathcal{U}_3(x_1, x_2) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [f(x_1 e^{-r_1}, x_2 e^{-r_2}) - f(x_1, x_2)] \\ &\quad \times \mathcal{R}_3(x_1, x_2, r_1, r_2) dr_1 dr_2. \end{aligned} \quad (181)$$

In order to reach the last equality in (181), we recast the arguments of Lemma 13 which lead to representations (132), (134), and (138). Indeed,

$$\begin{aligned} B_1(0, r_1, r_2) &= C [\delta(-r_1) + \delta(r_1)] [\delta(-r_2) + \delta(r_2)], \end{aligned} \quad (182)$$

and

$$\begin{aligned} \left| \int_{|z| < 2M} \frac{dz}{z} \int_0^1 \frac{\partial}{\partial q} e^{-iqz \ln(x_1|r_2|/x_2|r_1|)} dq \right| \\ = \left| \int_0^1 \frac{dq}{q} \int_{-2M}^{2M} \frac{\partial}{\partial z} e^{-iqz \ln(x_1|r_2|/x_2|r_1|)} dz \right| \leq C, \end{aligned} \quad (183)$$

where the positive constant C is independent of x_i and r_i .

Hence, we obtain

$$\begin{aligned} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [f(x_1 e^{-r_1}, x_2 e^{-r_2}) - f(x_1, x_2)] \\ \times B_1(0, r_1, r_2) \int_{|z| < 2M} \frac{dz}{z} \\ \times \int_0^1 \frac{\partial}{\partial q} e^{-iqz \ln(x_1|r_2|/x_2|r_1|)} dq dr_1 dr_2 = 0. \end{aligned} \quad (184)$$

6.1. Estimates on Maximum $|\mathbf{D}_{x_1}^\alpha u(x_1, x_2)|$. Further, in virtue of representations (176) and (181), estimates of $|\mathbf{D}_{x_1}^\alpha u(x_1, x_2)|$ and $\langle \mathbf{D}_{x_1}^\alpha u \rangle_{x_1, 0, \Omega}^\beta$ follow from the corresponding estimates of the functions $\mathcal{U}_j(x_1, x_2)$.

First, we describe the properties of the kernels $\mathcal{R}_2$ and $\mathcal{R}_3$.

Lemma 18. *Let $\varepsilon \in (\beta, 1)$, $x_1, x_2 \in [0, +\infty)$. Then there are the following estimates.*

(i)

$$\left| \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathcal{R}_2(x_1, x_2, -r_1, -r_2) dr_1 dr_2 \right| \leq C, \quad (185)$$

(ii)

$$\begin{aligned} & \int_0^{+\infty} dr_1 \int_0^{r_1} |\mathcal{R}_3(x_1, x_2, r_1, r_2)| |e^{\pm r_2} - 1|^\beta dr_2 \\ & + \int_0^{+\infty} |e^{\pm r_1} - 1|^\beta \left| \int_{-r_1}^{r_1} \mathcal{R}_3(x_1, x_2, r_1, r_2) dr_2 \right| dr_1 \quad (186) \\ & \leq C, \end{aligned}$$

(iii)

$$\begin{aligned} & \int_0^{+\infty} dr_2 \int_0^{r_2} |\mathcal{R}_3(x_1, x_2, r_1, r_2)| |e^{\pm r_1} - 1|^\beta dr_1 \\ & + \int_0^{+\infty} |e^{\pm r_2} - 1|^\beta \left| \int_{-r_2}^{r_2} \mathcal{R}_3(x_1, x_2, r_1, r_2) dr_1 \right| dr_2 \quad (187) \\ & \leq C, \end{aligned}$$

where the positive constant C is independent of ε , x_1 , and x_2 .

Proof. First, applying Proposition 11 with $b = 0$ and $\rho = +\infty$ and using definition (85), we have

$$\begin{aligned} & \int_{-\infty}^{+\infty} dr_1 \int_{-\infty}^{+\infty} dr_2 \mathcal{R}_2(x_1, x_2, -r_1, -r_2) \\ & = \int_{-\infty}^{+\infty} e^{\pi z/\alpha} K\left(-1 + \frac{iz}{\alpha}\right) e^{-iz \ln(x_1/x_2)} I_+^2(\varepsilon, z) I_-^2(\varepsilon, z) dz \\ & = \int_{|z|>2M} e^{\pi z/\alpha} K\left(-1 + \frac{iz}{\alpha}\right) e^{-iz \ln(x_1/x_2)} I_+^2(\varepsilon, z) I_-^2(\varepsilon, z) dz \\ & + B_0 \int_{|z|<2M} \frac{dz}{z} \int_0^1 \frac{\partial}{\partial q} e^{-izq \ln(x_1|r_2|/x_2|r_1|)} dq \\ & + \int_{-2M}^{2M} \frac{e^{-iz \ln(x_1|r_2|/x_2|r_1|)}}{z} \int_0^1 \frac{\partial}{\partial q} \left[e^{\pi qz/\alpha} qz K\left(-1 + \frac{izq}{\alpha}\right) \right. \\ & \quad \left. \times e^{iqz \ln(|r_2|/|r_1|)} I_+^2(\varepsilon, qz) I_-^2(\varepsilon, qz) \right] dq dz, \end{aligned} \quad (188)$$

where

$$\begin{aligned} B_0 &= \left[e^{\pi z/\alpha} z K\left(-1 + \frac{iz}{\alpha}\right) \right. \\ & \quad \left. \times e^{-iz \ln(x_1/x_2)} e^{iqz \ln(|r_2|/|r_1|)} I_+^2(\varepsilon, z) I_-^2(\varepsilon, z) \right] \Big|_{z=0}. \end{aligned} \quad (189)$$

Then, applying statement (iii) in Propositions 11 and 12 with aid (125) arrives at the first inequality of this proposition.

Further, we proceed with a detailed proof of statement (ii). The proof of statement (iii) is almost identical and is left to the interested reader.

At the beginning, we evaluate the first term in the left-hand side of the inequality in (ii)

$$\begin{aligned} \mathcal{J}_1 &:= \int_0^{+\infty} dr_1 \int_0^{r_1} |e^{\pm r_2} - 1|^\beta |\mathcal{R}_3(x_1, x_2, r_1, r_2)| dr_2. \end{aligned} \quad (190)$$

Simple calculations lead to

$$\mathcal{R}_3(x_1, x_2, r_1, r_2) e^{\varepsilon|r_1|+\varepsilon|r_2|} = \sum_{j=1}^3 \mathcal{R}_{3j}, \quad (191)$$

where we set

$$\begin{aligned} \mathcal{R}_{31} &= \int_{|z|>2M} B_1(z, r_1, r_2) e^{-iz \ln(x_1/x_2)} \frac{dz}{z}, \\ \mathcal{R}_{32} &= \int_0^1 dq \\ & \times \int_{|z|<2M} e^{-iz \ln(x_1|r_2|/x_2|r_1|)} \frac{\partial}{\partial qz} e^{iqz \ln(|r_2|/|r_1|)} \\ & \times B_1(qz, r_1, r_2) dz. \end{aligned} \quad (192)$$

At this point, we estimate each term $\mathcal{R}_{3j}$ separately. By Proposition 9 and estimates (125),

$$\begin{aligned} |\mathcal{R}_{31}| &\leq C(1 + |r_1|^{-1})(1 + |r_2|^{-1}) \\ & \times \int_{2M}^{+\infty} (1 + z^2) e^{-3\pi z/\alpha} \cosh \frac{\pi z}{2} dz \\ & < C(1 + |r_1|^{-1})(1 + |r_2|^{-1}) \end{aligned} \quad (193)$$

where the positive constant C is independent of r_i and x_i .

Concerning $\mathcal{R}_{32}$, keeping in mind Propositions 9 - 10 and estimates (125), we deduce

$$\begin{aligned} |\mathcal{R}_{32}| &\leq C(1 + |r_1|^{-1})(1 + |r_2|^{-1}) \int_0^1 dq \int_{|z|<2M} dz \\ & < C(1 + |r_1|^{-1})(1 + |r_2|^{-1}) \end{aligned} \quad (194)$$

where the positive constant C is independent of r_i and x_i .

In summary, we can conclude

$$\begin{aligned} \mathcal{J}_1 &\leq C \int_0^{+\infty} dr_1 (1 + |r_1|^{-1}) e^{-\varepsilon r_1} \\ & \times \int_0^{r_1} |e^{\pm r_2} - 1|^\beta e^{-\varepsilon r_2} (1 + |r_2|^{-1}) dr_2 < C. \end{aligned} \quad (195)$$

In order to obtain the last inequality, we use the condition $\varepsilon > \beta$.

Then, we are left to evaluate the second term in inequality (ii)

$$\begin{aligned} \mathcal{J}_2 &:= \int_0^{+\infty} dr_1 |e^{\pm r_1} - 1|^\beta \left| \int_{-r_1}^{r_1} \mathcal{R}_3(x_1, x_2, r_1, r_2) dr_2 \right|. \end{aligned} \quad (196)$$

To this end, coming to representation (191), we deduce

$$\mathcal{J}_2 \leq \mathcal{J}_{21} + \mathcal{J}_{22}, \quad (197)$$

with

$$\begin{aligned}\mathcal{J}_{21} &= \int_0^{+\infty} dr_1 |e^{\pm r_1} - 1|^\beta \left| \int_{-r_1}^{r_1} \mathcal{R}_{31} e^{-\varepsilon r_1 - \varepsilon |r_2|} dr_2 \right|, \\ \mathcal{J}_{22} &= \int_0^{+\infty} dr_1 |e^{\pm r_1} - 1|^\beta \left| \int_{-r_1}^{r_1} \mathcal{R}_{32} e^{-\varepsilon r_1 - \varepsilon |r_2|} dr_2 \right|.\end{aligned}\quad (198)$$

Proposition 9, statement (ii) in Proposition 11 (with $\varepsilon_1 = \varepsilon_2 = 1$, $\rho = r_1$), and estimate (125) lead to

$$\begin{aligned}\left| \int_{-r_1}^{r_1} \mathcal{R}_{31} e^{-\varepsilon r_1 - \varepsilon |r_2|} dr_2 \right| &\leq C [1 + r_1 + r_1^{-1}] \\ &\times \int_{|z| > 2M} \left| K \left(-1 + \frac{iz}{\alpha} \right) \right| e^{\pi z / \alpha} (1 + z^2)^2 \\ &\times \cosh^2 \frac{\pi z}{2} dz < C [1 + r_1 + r_1^{-1}].\end{aligned}\quad (199)$$

Thus, we have

$$|\mathcal{J}_{21}| \leq C \int_0^{+\infty} e^{-\varepsilon r_1} |e^{\pm r_1} - 1|^\beta [1 + r_1 + r_1^{-1}] < C. \quad (200)$$

Finally, Propositions 9 and 10 provide inequality

$$\begin{aligned}\left| \int_{-r_1}^{r_1} \mathcal{R}_{32} e^{-\varepsilon r_1 - \varepsilon |r_2|} dr_2 \right| \\ \leq C [1 + r_1 + r_1^{-1}] \int_0^1 dq \int_0^{2M} dz \\ < C [1 + r_1 + r_1^{-1}],\end{aligned}\quad (201)$$

which guarantees that

$$\begin{aligned}|\mathcal{J}_{22}| &\leq C \int_0^{+\infty} e^{-\varepsilon r_1} |e^{\pm r_1} - 1|^\beta [1 + r_1 + r_1^{-1}] dr_1 \\ &< C.\end{aligned}\quad (202)$$

Then, collecting this inequality with the estimate for $|\mathcal{J}_{21}|$ yields

$$\mathcal{J}_2 \leq C \quad (203)$$

with the positive constant C independent of x_1 and x_2 .

The last inequality together with the analogous estimate for $|\mathcal{J}_1|$ completes the proof of statement (ii). This finishes the proof of Lemma 18. $\square$

Now we estimate the function $|\mathbf{D}_{x_1}^\alpha u|$.

Lemma 19. *Let assumptions of Lemma 14 hold. Then*

$$\begin{aligned}\sup_{\overline{\Omega}} r^{-\beta}(x) (1 + r^2(x))^{(s_1 + \alpha)/2} |\mathbf{D}_{x_1}^\alpha u| \\ \leq C(R_0) \|f\|_{E_\beta^\beta(\overline{\Omega})}, \\ \sup_{\overline{\Omega}} d^{-\beta}(x) (1 + d(x))^{s_1 + \alpha} |\mathbf{D}_{x_1}^\alpha u| \\ \leq C(R_0) \|f\|_{C_\beta^\beta(\overline{\Omega})}.\end{aligned}\quad (204)$$

Proof. Here we prove in detail the estimate

$$\sup_{\overline{\Omega}} r^{-\beta}(x) |\mathbf{D}_{x_1}^\alpha u| \leq C \|f\|_{E_\beta^\beta(\overline{\Omega})}. \quad (205)$$

The rest inequalities are obtained with the same techniques and with the recasting the arguments in Lemma 14.

By representations (176) and (181), we have

$$|\mathbf{D}_{x_1}^\alpha u| \leq C \sum_{j=1}^3 |\mathcal{U}_j| \leq C [|f(x_1, x_2)| + |\mathcal{U}_2| + |\mathcal{U}_3|]. \quad (206)$$

Further, we estimate $|\mathcal{U}_j|$ separately.

By statement (i) in Lemma 18,

$$|\mathcal{U}_2| \leq C r^\beta(x) \sup_{\overline{\Omega}} r^{-\beta}(x) |f|. \quad (207)$$

Concerning $\mathcal{U}_3$, we rewrite this function in the form analogous to (164)

$$|\mathcal{U}_3| \leq \sum_{j=1}^8 |\mathcal{U}_{3j}|, \quad (208)$$

where we put

$$\begin{aligned}\mathcal{U}_{31} &= \int_0^{+\infty} dr_1 \\ &\times \int_{-r_1}^{r_1} [f(x_1 e^{-r_1}, x_2 e^{-r_2}) - f(x_1 e^{-r_1}, x_2)] \\ &\times \mathcal{R}_3(x_1, x_2, r_1, r_2) dr_2, \\ \mathcal{U}_{32} &= \int_0^{+\infty} dr_1 [f(x_1 e^{-r_1}, x_2) - f(x_1, x_2)] \\ &\times \int_{-r_1}^{r_1} \mathcal{R}_3(x_1, x_2, r_1, r_2) dr_2, \\ \mathcal{U}_{33} &= \int_{-\infty}^0 dr_1 \\ &\times \int_{r_1}^{-r_1} [f(x_1 e^{-r_1}, x_2 e^{-r_2}) - f(x_1 e^{-r_1}, x_2)] \\ &\times \mathcal{R}_3(x_1, x_2, r_1, r_2) dr_2, \\ \mathcal{U}_{34} &= \int_{-\infty}^0 dr_1 [f(x_1 e^{-r_1}, x_2) - f(x_1, x_2)] \\ &\times \int_{r_1}^{-r_1} \mathcal{R}_3(x_1, x_2, r_1, r_2) dr_2,\end{aligned}$$

$$\begin{aligned}
\mathcal{U}_{35} &= \int_0^{+\infty} dr_2 \\
&\times \int_{-r_2}^{r_2} [f(x_1 e^{-r_1}, x_2 e^{-r_2}) - f(x_1, x_2 e^{-r_2})] \\
&\times \mathcal{R}_3(x_1, x_2, r_1, r_2) dr_1, \\
\mathcal{U}_{36} &= \int_0^{+\infty} dr_2 [f(x_1, x_2 e^{-r_2}) - f(x_1, x_2)] \\
&\times \int_{-r_2}^{r_2} \mathcal{R}_3(x_1, x_2, r_1, r_2) dr_1, \\
\mathcal{U}_{37} &= \int_{-\infty}^0 dr_2 \\
&\times \int_{r_2}^{-r_2} [f(x_1 e^{-r_1}, x_2 e^{-r_2}) - f(x_1, x_2 e^{-r_2})] \\
&\times \mathcal{R}_3(x_1, x_2, r_1, r_2) dr_1, \\
\mathcal{U}_{38} &= \int_{-\infty}^0 dr_2 [f(x_1, x_2 e^{-r_2}) - f(x_1, x_2)] \\
&\times \int_{r_2}^{-r_2} \mathcal{R}_3(x_1, x_2, r_1, r_2) dr_1.
\end{aligned} \tag{209}$$

It is apparent that the estimates of each $\mathcal{U}_{3j}$ are simple consequence of Lemma 18. Thus, we deduce

$$\begin{aligned}
|\mathcal{U}_3| &\leq C \left(\langle f \rangle_{x_1, 0, \Omega}^{(\beta)} + \langle f \rangle_{x_2, 0, \Omega}^{(\beta)} \right) (x_1^\beta + x_2^\beta) \\
&\leq C r^\beta(x) \|f\|_{E_\beta^\beta(\overline{\Omega})}.
\end{aligned} \tag{210}$$

The last inequality with (206) and (207) provides

$$|\mathbf{D}_{x_1}^\alpha u| \leq C r^\beta(x) \|f\|_{E_\beta^\beta(\overline{\Omega})}. \tag{211}$$

This finishes the proof of Lemma 19. $\square$

6.2. Estimates on Hölder Seminorms of $\mathbf{D}_{x_1}^\alpha u(x_1, x_2)$. First of all, we return to representations (176) and (181) and evaluate each function $\mathcal{U}_i(x_1, x_2)$ separately.

Proposition 20. *Let assumptions of Lemma 14 hold. Then*

$$\begin{aligned}
&\sum_{i=1}^2 \left(\langle \mathcal{U}_1 \rangle_{x_i, 0, \Omega}^{(\beta)} + \langle \mathcal{U}_2 \rangle_{x_i, 0, \Omega}^{(\beta)} \right) (x_1^\beta + x_2^\beta) \\
&\leq C \|f\|_{E_\beta^\beta(\overline{\Omega})}.
\end{aligned} \tag{212}$$

Proof. It is apparent that (181) and properties of the function f ensure estimates of $\langle \mathcal{U}_1 \rangle_{x_i, 0, \Omega}^{(\beta)}$.

Next we evaluate the term $\langle \mathcal{U}_2 \rangle_{x_1, 0, \Omega}^{(\beta)}$. For simplicity consideration, we assume

$$\begin{aligned}
x_1, \bar{x}_1 &\in [0, +\infty) \\
\text{and } x_1 &< \bar{x}_1,
\end{aligned} \tag{213}$$

and put

$$\begin{aligned}
\Delta_1 x &= x_1 - \bar{x}_1, \\
\Delta_1 \mathcal{U}_2 &= \mathcal{U}_2(x_1, x_2) - \mathcal{U}_2(\bar{x}_1, x_2).
\end{aligned} \tag{214}$$

In virtue of representation (181) and Lemma 18, we have

$$\begin{aligned}
|\Delta_1 \mathcal{U}_2| &\leq |f(x_1, x_2) - f(\bar{x}_1, x_2)| \left| \int_{-\infty}^{+\infty} dr_1 \right. \\
&\times \int_{-\infty}^{+\infty} dr_2 \mathcal{R}_2(x_1, x_2, -r_1, -r_2) \Big| + |f(\bar{x}_1, x_2)| \\
&\times \left| \int_{-\infty}^{+\infty} dr_1 \int_{-\infty}^{+\infty} dr_2 [\mathcal{R}_2(x_1, x_2, -r_1, -r_2) \right. \\
&- \mathcal{R}_2(\bar{x}_1, x_2, -r_1, -r_2)] \Big| \leq C \langle f \rangle_{x_1, 0, \Omega}^{(\beta)} (\Delta x_1)^\beta \\
&+ |f(\bar{x}_1, x_2)| \left| \int_{-\infty}^{+\infty} dr_1 \right. \\
&\times \int_{-\infty}^{+\infty} dr_2 [\mathcal{R}_2(x_1, x_2, -r_1, -r_2) \\
&- \mathcal{R}_2(\bar{x}_1, x_2, -r_1, -r_2)] \Big|.
\end{aligned} \tag{215}$$

Then we are left to tackle the second term in the right-hand side of the last inequality. Simple calculations lead to

$$\begin{aligned}
&\mathcal{R}_2(x_1, x_2, -r_1, -r_2) - \mathcal{R}_2(\bar{x}_1, x_2, -r_1, -r_2) \\
&= \Delta_1 x \int_0^1 \frac{\partial}{\partial (q\Delta_1 x + \bar{x}_1)} \mathcal{R}_2(q\Delta_1 x + \bar{x}_1, x_2, \\
&-r_1, -r_2) dq = \Delta_1 x \int_0^1 dq e^{-\varepsilon|r_1| - \varepsilon|r_2|} \\
&\times \int_{-\infty}^{+\infty} \frac{e^{-iz \ln((q\Delta_1 x + \bar{x}_1)|r_2|/x_2|r_1|)} B_1(z, r_1, r_2)}{q\Delta_1 x + \bar{x}_1} dz.
\end{aligned} \tag{216}$$

At this point, recasting the proof of statement (i) in Lemma 18 arrives at

$$\begin{aligned}
&\left| \int_{-\infty}^{+\infty} dr_1 \int_{-\infty}^{+\infty} dr_2 [\mathcal{R}_2(x_1, x_2, -r_1, -r_2) \right. \\
&- \mathcal{R}_2(\bar{x}_1, x_2, -r_1, -r_2)] \Big| \leq C \Delta_1 x \int_0^1 \frac{dq}{q\Delta_1 x + \bar{x}_1} \\
&< C \frac{\Delta_1 x}{\bar{x}_1^\beta} \int_0^1 \frac{dq}{(q\Delta_1 x)^{1-\beta}} = C \frac{(\Delta_1 x)^\beta}{\bar{x}_1^\beta}.
\end{aligned} \tag{217}$$

In conclusion, we obtain the estimate

$$\begin{aligned}
&|f(\bar{x}_1, x_2)| \left| \int_{-\infty}^{+\infty} dr_1 \int_{-\infty}^{+\infty} dr_2 [\mathcal{R}_2(x_1, x_2, -r_1, -r_2) \right. \\
&- \mathcal{R}_2(\bar{x}_1, x_2, -r_1, -r_2)] \Big| \\
&\leq C \frac{|f(\bar{x}_1, x_2) - f(0, x_2)| (\Delta_1 x)^\beta}{\bar{x}_1^\beta}.
\end{aligned} \tag{218}$$

In order to reach the last inequality we use property (32) to the function f .

Summarizing, we have inequality

$$|\Delta_1 \mathcal{U}_2| \leq C \langle f \rangle_{x_1, 0, \Omega}^{(\beta)} \left((\Delta_1 x)^\beta \right)^\beta. \quad (219)$$

The same arguments in the case of the difference $\Delta_2 \mathcal{U}_2 = \mathcal{U}_2(x_1, x_2) - \mathcal{U}_2(x_1, \bar{x}_2)$ guarantee estimate

$$|\Delta_2 \mathcal{U}_2| \leq C \langle f \rangle_{x_2, 0, \Omega}^{(\beta)} |x_2 - \bar{x}_2|^\beta. \quad (220)$$

Thus, collecting estimates for $|\Delta_i \mathcal{U}_2|$ we reach the conclusion

$$\sum_{i=1}^2 \langle \mathcal{U}_2 \rangle_{x_i, 0, \Omega}^{(\beta)} \leq C \|f\|_{E_\beta^\beta(\Omega)}. \quad (221)$$

This completes the proof of Proposition 20. $\square$

Next we obtain the same results to the function $\mathcal{U}_3(x_1)$. For each point $x_1, \bar{x}_1, x_2, \bar{x}_2 \in [0, +\infty)$: $x_1 > \bar{x}_1$ and $x_2 > \bar{x}_2$, we put

$$\begin{aligned} \Delta_1 x &= x_1 - \bar{x}_1, \\ \Delta_2 x &= x_2 - \bar{x}_2. \end{aligned} \quad (222)$$

Further, we describe the properties of the kernel $\mathcal{R}_3(x_1, x_2, r_1, r_2)$.

Lemma 21. Let $\varepsilon \in (\beta, 1)$ and $\varepsilon_j, j = 1, 2, 3, 4$, be some positive numbers, $\varepsilon_j \in [0, 2]$, and let

$$\frac{\Delta_i x}{\bar{x}_j} < \frac{1}{3}, \quad i, j = 1, 2. \quad (223)$$

Then the kernel $\mathcal{R}_3$ possesses the following:

(i)

$$\begin{aligned} & \int_0^{\varepsilon_3} dr_1 \int_{-\varepsilon_1 r_1}^{\varepsilon_2 r_1} |e^{\pm r_2} - 1|^\beta |\mathcal{R}_3(x_1, x_2, r_1, r_2)| dr_2 \\ & + \int_0^{\varepsilon_3} dr_2 \int_{-\varepsilon_1 r_2}^{\varepsilon_2 r_2} |e^{\pm r_1} - 1|^\beta |\mathcal{R}_3(x_1, x_2, r_1, r_2)| dr_1 \\ & \leq C [\varepsilon_2^\beta + \varepsilon_1^\beta] \varepsilon_3^\beta, \end{aligned} \quad (224)$$

(ii)

$$\begin{aligned} & \int_0^{\varepsilon_3} dr_1 |e^{\pm r_1} - 1|^\beta \left| \int_{-\varepsilon_1 r_1}^{\varepsilon_2 r_1} \mathcal{R}_3(x_1, x_2, r_1, r_2) dr_2 \right| \\ & + \int_0^{\varepsilon_3} dr_2 |e^{\pm r_2} - 1|^\beta \left| \int_{-\varepsilon_1 r_2}^{\varepsilon_2 r_2} \mathcal{R}_3(x_1, x_2, r_1, r_2) dr_1 \right| \\ & \leq C \left[\frac{\varepsilon_2}{\varepsilon_1} + \frac{\varepsilon_1}{\varepsilon_2} + (\varepsilon_1 + \varepsilon_2) \varepsilon_3 \right] \varepsilon_3^\beta, \end{aligned} \quad (225)$$

(iii)

$$\begin{aligned} & \left| \iint_{\mathbb{R}^2} \mathcal{R}_3(x_1, x_2, r_1, r_2) dr_1 dr_2 \right| \\ & + \left| \int_{-\varepsilon_4}^{\varepsilon_3} \int_{-\varepsilon_1}^{\varepsilon_2} \mathcal{R}_3(x_1, x_2, r_1, r_2) dr_1 dr_2 \right| \\ & \leq C \left[1 + \left(1 + \frac{\varepsilon_2}{\varepsilon_1} + \frac{\varepsilon_1}{\varepsilon_2} \right) \left(1 + \frac{\varepsilon_3}{\varepsilon_4} + \frac{\varepsilon_4}{\varepsilon_3} \right) \right], \end{aligned} \quad (226)$$

(iv)

$$\begin{aligned} & \int_{\varepsilon_3}^{+\infty} dr_1 \int_0^{\varepsilon_2 r_1} |e^{\pm r_2} - 1|^\beta \left| \mathcal{R}_3 \left(x_1, x_2, r_1 \right. \right. \\ & \quad \left. \left. + \ln \left(1 + \frac{\Delta_1 x}{x_1} \right), r_2 \right) \right. \\ & \quad \left. - \mathcal{R}_3(\bar{x}_1, x_2, r_1, r_2) \right| dr_2 + \int_{\varepsilon_3}^{+\infty} dr_1 |e^{\pm r_1} \\ & \quad - 1|^\beta \left| \int_{-\varepsilon_1 r_1}^{\varepsilon_2 r_1} \left[\mathcal{R}_3 \left(x_1, x_2, r_1 + \ln \left(1 + \frac{\Delta_1 x}{x_1} \right), r_2 \right) \right. \right. \\ & \quad \left. \left. - \mathcal{R}_3(\bar{x}_1, x_2, r_1, r_2) \right] dr_2 \right| \leq C \left(1 + \varepsilon_2^\beta \right. \\ & \quad \left. + \frac{\varepsilon_2}{\varepsilon_1} + \frac{\varepsilon_1}{\varepsilon_2} \right) \varepsilon_3^{\beta-1} \ln \left(1 + \frac{\Delta_1 x}{x_1} \right), \\ & \int_{\varepsilon_3}^{+\infty} dr_2 \int_0^{\varepsilon_2 r_2} |e^{\pm r_1} - 1|^\beta \left| \mathcal{R}_3 \left(x_1, x_2, r_1 \right. \right. \\ & \quad \left. \left. + \ln \left(1 + \frac{\Delta_1 x}{x_1} \right), r_2 \right) \right. \\ & \quad \left. - \mathcal{R}_3(\bar{x}_1, x_2, r_1, r_2) \right| dr_1 + \int_{\varepsilon_3}^{+\infty} dr_2 |e^{\pm r_2} \\ & \quad - 1|^\beta \left| \int_{-\varepsilon_1 r_2}^{\varepsilon_2 r_2} \left[\mathcal{R}_3 \left(x_1, x_2, r_1 + \ln \left(1 + \frac{\Delta_1 x}{x_1} \right), r_2 \right) \right. \right. \\ & \quad \left. \left. - \mathcal{R}_3(\bar{x}_1, x_2, r_1, r_2) \right] dr_1 \right| \leq C \left(1 + \varepsilon_2^\beta \right. \\ & \quad \left. + \frac{\varepsilon_2}{\varepsilon_1} + \frac{\varepsilon_1}{\varepsilon_2} \right) \varepsilon_3^{\beta-1} \ln \left(1 + \frac{\Delta_1 x}{x_1} \right), \end{aligned} \quad (227)$$

where the positive constant C is independent of $x_i, \bar{x}_i, \varepsilon_j$.

Since the proof of this result is technically tedious and repeating certain steps in the proof of Lemma 18, we provide one in the Appendix.

Then, based on Lemma 21, we can conclude the following.

Lemma 22. Let assumptions of Lemma 14 hold. Then there is the estimate

$$\sum_{i=1}^2 \langle \mathcal{U}_3 \rangle_{x_i, 0, \Omega}^{(\beta)} \leq C \|f\|_{E_\beta^\beta(\Omega)}. \quad (228)$$

Proof. We provide here the estimate of $\langle \mathcal{U}_3 \rangle_{x_1, 0, \Omega}^{(\beta)}$. The case of $\langle \mathcal{U}_3 \rangle_{x_2, 0, \Omega}^{(\beta)}$ is treated in similar arguments.

It should be noted that it is enough to evaluate $\langle \mathcal{U}_3 \rangle_{x_1, 0, \Omega}^{(\beta)}$ in case of (223). Indeed, in the opposite case, its estimate follows from Lemma 19. Thus, we assume that (223) hold. Introducing the domain

$$\begin{aligned} \overline{D}_1(x_1, x_2) = \{ (y_1, y_2) \in \mathbb{R}^2 : |e^{-y_1} - x_1| \\ \leq 2\Delta_1 x, |e^{-y_2} - x_2| \leq 2\Delta_2 x, \} \end{aligned} \quad (229)$$

we rewrite the function $\mathcal{U}_3$ in a more suitable form:

$$\begin{aligned} \mathcal{U}_3 = \iint_{\overline{D}_1(x_1, x_2)} [f(e^{y_1}, e^{y_2}) - f(x_1, x_2)] \\ \times \mathcal{R}_3(x_1, x_2, \ln x_1 - y_1, \ln x_2 - y_2) dy_1 dy_2 \\ + \iint_{\mathbb{R}^2 \setminus \overline{D}_1(x_1, x_2)} [f(e^{y_1}, e^{y_2}) - f(x_1, x_2)] \\ \times \mathcal{R}_3(x_1, x_2, \ln x_1 - y_1, \ln x_2 - y_2) dy_1 dy_2. \end{aligned} \quad (230)$$

Then we consider the difference

$$\Delta_1 \mathcal{U}_3 := \mathcal{U}_3(x_1, x_2) - \mathcal{U}_3(\overline{x}_1, x_2) = \sum_{j=1}^4 \mathcal{V}_j, \quad (231)$$

where we set

$$\begin{aligned} \mathcal{V}_1 = \iint_{\overline{D}_1(x_1, x_2)} [f(e^{y_1}, e^{y_2}) - f(x_1, x_2)] \\ \times \mathcal{R}_3(x_1, x_2, \ln x_1 - y_1, \ln x_2 - y_2) dy_1 dy_2, \\ \mathcal{V}_2 = - \iint_{\overline{D}_1(x_1, x_2)} [f(e^{y_1}, e^{y_2}) - f(\overline{x}_1, x_2)] \\ \times \mathcal{R}_3(\overline{x}_1, x_2, \ln \overline{x}_1 - y_1, \ln x_2 - y_2) dy_1 dy_2, \\ \mathcal{V}_3 = \iint_{\mathbb{R}^2 \setminus \overline{D}_1(x_1, x_2)} [f(\overline{x}_1, x_2) - f(x_1, x_2)] \\ \times \mathcal{R}_3(x_1, x_2, \ln x_1 - y_1, \ln x_2 - y_2) \\ - \mathcal{R}_3(\overline{x}_1, x_2, \ln \overline{x}_1 - y_1, \ln x_2 - y_2) dy_1 dy_2. \end{aligned} \quad (232)$$

Then putting

$$\begin{aligned} a_1(x_1) &= \ln \left(1 + 2 \frac{\Delta_1 x}{x_1} \right), \\ a_1(\overline{x}_1) &= \ln \left(1 + \frac{\Delta_1 x}{\overline{x}_1} \right), \\ a_3(x_1) &= -\ln \left(1 - 2 \frac{\Delta_1 x}{x_1} \right), \\ a_3(\overline{x}_1) &= -\ln \left(1 - 3 \frac{\Delta_1 x}{\overline{x}_1} \right), \\ a_2 &= \ln \left(1 + \frac{\Delta_1 x}{x_2} \right), \\ a_4 &= -\ln \left(1 - 2 \frac{\Delta_1 x}{x_2} \right), \end{aligned} \quad (233)$$

and making change of variables (179), we reduce the domain $\overline{D}_1(x_1, x_2)$ to either $D_1(x_1, x_2)$ in the case of $y_1 = -r_1 + \ln x_1$ or $D_1(\overline{x}_1, x_2)$ if $y_1 = -r_1 + \ln \overline{x}_1$:

$$\begin{aligned} D_1(x_1, x_2) &= \{ (r_1, r_2) \in \mathbb{R}^2 : -a_3(x_1) < r_1 \\ &< a_1(x_1), -a_4 < r_2 < a_2 \}, \\ D_1(\overline{x}_1, x_2) &= \{ (r_1, r_2) \in \mathbb{R}^2 : -a_3(\overline{x}_1) < r_1 \\ &< a_1(\overline{x}_1), -a_4 < r_2 < a_2 \}. \end{aligned} \quad (234)$$

At this point, we estimate each term $\mathcal{V}_j$ separately. Change of variables (179) leads to

$$\begin{aligned} \mathcal{V}_1 &= \int_0^{a_1(x_1)} dr_1 \\ &\times \int_{(-a_4/a_1(x_1))r_1}^{(a_2/a_1(x_1))r_1} [f(x_1 e^{-r_1}, x_2 e^{-r_2}) - f(x_1, x_2)] \\ &\times \mathcal{R}_3(x_1, x_2, r_1, r_2) dr_2 + \int_{-a_3(x_1)}^0 dr_1 \\ &\times \int_{(a_4/a_3(x_1))r_1}^{(-a_2/a_3(x_1))r_1} [f(x_1 e^{-r_1}, x_2 e^{-r_2}) - f(x_1, x_2)] \\ &\times \mathcal{R}_3(x_1, x_2, r_1, r_2) dr_2 + \int_0^{a_2} dr_2 \\ &\times \int_{(-a_3(x_1)/a_2)r_2}^{(a_1(x_1)/a_2)r_2} [f(x_1 e^{-r_1}, x_2 e^{-r_2}) - f(x_1, x_2)] \\ &\times \mathcal{R}_3(x_1, x_2, r_1, r_2) dr_1 + \int_{-a_4}^0 dr_2 \\ &\times \int_{(a_3(x_1)/a_4)r_2}^{(-a_1(x_1)/a_4)r_2} [f(x_1 e^{-r_1}, x_2 e^{-r_2}) - f(x_1, x_2)] \\ &\times \mathcal{R}_3(x_1, x_2, r_1, r_2) dr_1. \end{aligned} \quad (235)$$

Applying decomposition like (164) to each term in the representation above, then statements (i) and (ii) in Lemma 21 provide estimate

$$|\mathcal{V}_1| \leq C (\Delta_1 x)^\beta \left(\langle f \rangle_{x_1,0,\Omega}^{(\beta)} + \langle f \rangle_{x_2,0,\Omega}^{(\beta)} \right). \quad (236)$$

Indeed, let us verify this with the example of the first term in the representation to $\mathcal{V}_1$. Standard calculations lead to

$$\begin{aligned} & \int_0^{a_1(x_1)} dr_1 \int_{(-a_4/a_1(x_1))r_1}^{(a_2/a_1(x_1))r_1} [f(x_1 e^{-r_1}, x_2 e^{-r_2}) \\ & - f(x_1, x_2)] \mathcal{R}_3(x_1, x_2, r_1, r_2) dr_2 \\ & = \int_0^{a_1(x_1)} dr_1 \int_{(-a_4/a_1(x_1))r_1}^{(a_2/a_1(x_1))r_1} [f(x_1 e^{-r_1}, x_2 e^{-r_2}) \\ & - f(x_1 e^{-r_1}, x_2)] \mathcal{R}_3(x_1, x_2, r_1, r_2) dr_2 \\ & + \int_0^{a_1(x_1)} dr_1 [f(x_1 e^{-r_1}, x_2) - f(x_1, x_2)] \\ & \times \int_{(-a_4/a_1(x_1))r_1}^{(a_2/a_1(x_1))r_1} \mathcal{R}_3(x_1, x_2, r_1, r_2) dr_2 \end{aligned} \quad (237)$$

Then, properties of the function f and Lemma 21 with $\varepsilon_3 = a_1(x_1)$, $\varepsilon_2 = a_2/a_1(x_1)$, $\varepsilon_1 = a_4/a_1(x_1)$ yield

$$\begin{aligned} & \left| \int_0^{a_1(x_1)} dr_1 \right. \\ & \times \int_{(-a_4/a_1(x_1))r_1}^{(a_2/a_1(x_1))r_1} [f(x_1 e^{-r_1}, x_2 e^{-r_2}) - f(x_1, x_2)] \\ & \times \mathcal{R}_3(x_1, x_2, r_1, r_2) dr_2 \left. \right| \leq C \left(\langle f \rangle_{x_1,0,\Omega}^{(\beta)} \right. \\ & + \langle f \rangle_{x_2,0,\Omega}^{(\beta)} \left(x_2^\beta a_2^\beta + x_2^\beta a_4^\beta + x_1^\beta a_1^\beta(x_1) \left[\frac{a_2}{a_4} \right. \right. \\ & \left. \left. + \frac{a_4}{a_2} \right] \right) \leq C (\Delta_1 x)^\beta \left(\langle f \rangle_{x_1,0,\Omega}^{(\beta)} + \langle f \rangle_{x_2,0,\Omega}^{(\beta)} \right). \end{aligned} \quad (238)$$

To reach the last inequality we apply (223) to (233). The remaining terms in the representation of $\mathcal{V}_1$ are evaluated in the same way.

Concerning $\mathcal{V}_2$, we repeat the arguments above with changing $D(x_1, x_2)$ by $D(\bar{x}_1, x_2)$ and obtain the estimate like (236) to the function $\mathcal{V}_2$.

Coming to $\mathcal{V}_3$, change of variables (179), Lemma 21, and condition (223) arrive at

$$\begin{aligned} |\mathcal{V}_3| &= |f(\bar{x}_1, x_2) - f(x_1, x_2)| \\ & \times \left| \iint_{\mathbb{R}^2} \mathcal{R}_3(x_1, x_2, r_1, r_2) dr_1 dr_2 \right. \\ & \left. + \iint_{D_1(x_1, x_2)} \mathcal{R}_3(x_1, x_2, r_1, r_2) dr_1 dr_2 \right| \\ & \leq C (\Delta_1 x)^\beta \langle f \rangle_{x_1,0,\Omega}^{(\beta)}. \end{aligned} \quad (239)$$

Finally, we are left to estimate $\mathcal{V}_4$. Change of variables (179) and statement (iv) in Lemma 21 lead to

$$|\mathcal{V}_4| \leq C (\Delta_1 x)^\beta \left[\langle f \rangle_{x_1,0,\Omega}^{(\beta)} + \langle f \rangle_{x_2,0,\Omega}^{(\beta)} \right]. \quad (240)$$

Collecting this inequality with (236) and (239), we get the bound

$$|\Delta_1 \mathcal{U}_3| \leq C (\Delta_1 x)^\beta \|f\|_{E_\beta^\beta(\bar{\Omega})}. \quad (241)$$

This finishes the proof of Lemma 22. $\square$

At this point, we come back to $\langle \mathbf{D}_{x_1}^\alpha u \rangle_{x_1,0,\Omega}^{(\beta)}$. Keeping in mind Proposition 20, Lemma 22, and representations (176) and (181), we deduce the inequality

$$\langle \mathbf{D}_{x_1}^\alpha u \rangle_{x_1,0,\Omega}^{(\beta)} + \langle \mathbf{D}_{x_1}^\alpha u \rangle_{x_2,0,\Omega}^{(\beta)} \leq C \|f\|_{E_\beta^\beta(\bar{\Omega})}. \quad (242)$$

Further, recasting arguments in Proposition 20 and Lemmas 22 and 14 and applying Proposition 8, we assert the result.

Lemma 23. *Let assumption of Lemma 14 hold. Then*

$$\begin{aligned} & \|\mathbf{D}_{x_1}^\alpha u\|_{E_\beta^\beta(\bar{\Omega})} + \|\mathbf{D}_{x_1}^\alpha u\|_{E_{-s_1-\alpha}^\beta(\bar{\Omega})} \leq C(R_0) \|f\|_{E_\beta^\beta(\bar{\Omega})}, \\ & \|\mathbf{D}_{x_1}^\alpha u\|_{C_\beta^\beta(\bar{\Omega})} + \|\mathbf{D}_{x_1}^\alpha u\|_{C_{-s_1-\alpha}^\beta(\bar{\Omega})} \leq C(R_0) \|f\|_{C_\beta^\beta(\bar{\Omega})}. \end{aligned} \quad (243)$$

In order to obtain the same results to the functions $\mathbf{D}_{x_1}^{\alpha_1} \mathbf{D}_{x_2}^{\alpha_2} u$ and $\mathbf{D}_{x_2}^{\alpha_2} \mathbf{D}_{x_1}^{\alpha_1} u$, $1 < \alpha_1 + \alpha_2 \leq \alpha$, it is enough to repeat the arguments from this section. Thus, we have.

Remark 24. Under conditions of Lemma 23, the functions $\mathbf{D}_{x_1}^{\alpha_1} \mathbf{D}_{x_2}^{\alpha_2} u$, $\mathbf{D}_{x_2}^{\alpha_2} \mathbf{D}_{x_1}^{\alpha_1} u$, $1 < \alpha_1 + \alpha_2 \leq \alpha$, satisfy inequalities (23) and (27) and

$$\mathbf{D}_{x_1}^{\alpha_1} \mathbf{D}_{x_2}^{\alpha_2} u = \mathbf{D}_{x_2}^{\alpha_2} \mathbf{D}_{x_1}^{\alpha_1} u. \quad (244)$$

7. Proofs of Theorems 3 and 4

Under restrictions (30) and (31), the proof of Theorems 3 and 4 follows from the arguments of Sections 3–6. Indeed, Proposition 17 and Lemmas 16, 19, and 23 with Remark 24 provide existence of a solution u which satisfies estimates (22), (23) and (26), (27). Next, direct calculations with aid of representation (176) allow one to verify that the constructed solution u with (57) satisfies (1) and the corresponding boundary conditions. Further, the uniqueness of the solution follows from the coercive estimates. This completes the proof of Theorems 3 and 4 if (30) and (31) hold.

At this point, we remove restriction (30). To this end, we introduce the new function

$$W_1 = \frac{x_1}{\sqrt{x_1^2 + x_2^2}} \psi_1(x_1) + \frac{x_2}{\sqrt{x_1^2 + x_2^2}} \psi_2(x_2). \quad (245)$$

The following properties of the function W_1 could be easily checked with aid (2.2) in [37] and properties of the functions ψ_i and f (see (h2), (19)): if $\beta > \alpha - 1$, then

$$\text{either } W_1 \in \mathcal{M}_s^{\alpha+\beta}(\overline{\Omega}) \cap \mathcal{M}_{-s_1}^{\alpha+\beta}(\overline{\Omega}),$$

$$\text{or } W_1 \in \mathcal{N}_s^{\alpha+\beta}(\overline{\Omega}) \cap \mathcal{N}_{-s_1}^{\alpha+\beta}(\overline{\Omega}),$$

$$\begin{aligned} f(x_1, 0) - \mathbf{D}_{x_1}^\alpha W_1(x_1, x_2)|_{x_2=0} - \mathbf{D}_{x_2}^\alpha W_1(x_1, x_2)|_{x_2=0} \\ - \mathbf{D}_{x_1}^\alpha \psi_1(x_1) = 0, \end{aligned} \quad (246)$$

$$\begin{aligned} f(0, x_2) - \mathbf{D}_{x_1}^\alpha W_1(x_1, x_2)|_{x_1=0} - \mathbf{D}_{x_2}^\alpha W_1(x_1, x_2)|_{x_1=0} \\ - \mathbf{D}_{x_2}^\alpha \psi_2(x_2) = 0. \end{aligned}$$

After that, we look for the solution to (1), (2) in the form

$$u(x, t) = \mathcal{W} + W_1, \quad (247)$$

where the unknown function $\mathcal{W}$ solves the boundary value problem with homogenous boundary condition

$$\begin{aligned} \mathbf{D}_{x_1}^\alpha \mathcal{W} - \mathbf{D}_{x_2}^\alpha \mathcal{W} = \bar{f}(x_1, x_2) \quad \text{in } \Omega, \\ \mathcal{W} = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2, \end{aligned} \quad (248)$$

with

$$\begin{aligned} \bar{f}(x_1, x_2) = f(x_1, x_2) - \mathbf{D}_{x_1}^\alpha W_1(x_1, x_2) \\ - \mathbf{D}_{x_2}^\alpha W_1(x_1, x_2). \end{aligned} \quad (249)$$

It is apparent that the function $\bar{f}$ satisfies conditions (h3), (19). Thus, arguments of Sections 3–6 guarantee the one-valued solvability to (248) and, besides, the function $\mathcal{W}$ satisfies properties (22), (23), and (27) and (26). Then, we return the function u and obtain the one-to-one solvability of the original problem, where the solution u possesses the same properties as $\mathcal{W}$. Thus, Theorems 3 and 4 are proven in the absence of (30).

Finally, we are left to remove restriction (31). To this end, it is worth to remark that a function $f \in E_{s-\alpha}^\beta(\overline{\Omega})$ which satisfies condition (31) belongs to $f \in E_{-\bar{s}-\alpha}^\beta(\overline{\Omega})$ with any positive $\bar{s}$ and hence also with $\bar{s} = s_1, s_1 \in (0, 2-\alpha)$. Further, it is enough to repeat the arguments from Sections 3–7 in order to prove Theorem 3, if f satisfies (17). If f meets requirement (18), the same reasons hold. This finishes the proof of Theorems 3 and 4.

Remark 25. Requirement (30) could be removed without introducing the function W_1 . To this end, it is enough to recast the arguments of Sections 3–6 to the problems with the homogenous equation and nonhomogeneous boundary conditions

$$\mathbf{D}_{x_1}^\alpha u - \mathbf{D}_{x_2}^\alpha u = 0 \quad \text{in } \Omega, \quad (250)$$

subject either to the Dirichlet boundary condition (DBC)

$$\begin{aligned} u = \psi_1(x_1) \quad \text{on } \Gamma_1 \\ \text{and } u = \psi_2(x_2) \quad \text{on } \Gamma_2, \end{aligned} \quad (251)$$

or to the Neumann boundary condition (NBC)

$$\begin{aligned} \frac{\partial u}{\partial x_2} = \varphi_1(x_1) \quad \text{on } \Gamma_1 \\ \text{and } \frac{\partial u}{\partial x_1} = \varphi_2(x_2) \quad \text{on } \Gamma_2. \end{aligned} \quad (252)$$

Remark 26. Arguments from Sections 3–6 guarantee that the requirement on the weight s in the assumption (h1) can be changed by

$$s \geq \alpha + \beta. \quad (253)$$

Remark 27. Actually, with inessential modifications in the proofs, the very same results hold for the more general equation

$$\mathbf{D}_{x_1}^{\alpha_1} u + \mathbf{D}_{x_2}^{\alpha_2} u = f(x_1, x_2) \quad \text{in } \Omega, \quad (254)$$

where $1 < \alpha_1, \alpha_2 < 2$. The details are left to the interested reader.

Appendix

Proof of Lemma 21

To prove statement (i), we estimate the first term in this inequality. The second one is evaluated with the same arguments. Let us rewrite the first term in the form

$$\begin{aligned} \int_0^{\varepsilon_3} dr_1 \int_{-\varepsilon_1 r_1}^{\varepsilon_2 r_1} |e^{\pm r_2} - 1|^\beta |\mathcal{R}_3| dr_2 \\ \leq C \sum_{j=1}^2 \left[\int_0^{\varepsilon_3} dr_1 \int_0^{\varepsilon_2 r_1} r_2^\beta e^{\beta r_2} |\mathcal{R}_{3j}| dr_2 \right. \\ \left. + \int_0^{\varepsilon_3} dr_1 \int_0^{\varepsilon_1 r_1} r_2^\beta e^{\beta r_2} |\mathcal{R}_{3j}| dr_2 \right], \end{aligned} \quad (A.1)$$

where $\mathcal{R}_{3j}$ is defined in (191). After that, (191), (193), and (199) arrive at

$$\begin{aligned} \int_0^{\varepsilon_3} dr_1 \left| \int_{-\varepsilon_1 r_1}^{\varepsilon_2 r_1} |e^{\pm r_2} - 1|^\beta |\mathcal{R}_3| dr_2 \right| \\ \leq C \int_0^{\varepsilon_3} dr_1 \left[1 + r_1^{-1} \right] \\ \times \left[\int_0^{\varepsilon_2 r_1} r_2^\beta e^{-(\varepsilon-\beta)r_2} \left[1 + r_2^{-1} \right] dr_2 \right. \\ \left. + \int_0^{\varepsilon_1 r_1} r_2^\beta e^{-(\varepsilon-\beta)r_2} \left[1 + r_2^{-1} \right] dr_2 \right] \leq C \varepsilon_3^\beta (\varepsilon_2^\beta + \varepsilon_1^\beta) \end{aligned} \quad (A.2)$$

with the positive constant C being independent of $\varepsilon_i, x_j, \varepsilon$.

Note that to reach the last inequality, we use the fact that $\varepsilon > \beta$. Thus, claim (i) is proven.

Concerning statement (ii), we apply inequalities analogous to (199) and (201) and obtain

$$\begin{aligned} \left| \int_{-\varepsilon_1 r_1}^{\varepsilon_2 r_1} \mathcal{R}_3 dr_2 \right| &\leq \sum_{j=1}^2 \left| \int_{-\varepsilon_1 r_1}^{\varepsilon_2 r_1} \mathcal{R}_{3j} dr_2 \right| \\ &\leq C e^{-\varepsilon |r_1|} \left[1 + r_1^{-1} \right] \left[1 + r_1 (\varepsilon_1 + \varepsilon_2) + \frac{\varepsilon_1}{\varepsilon_2} + \frac{\varepsilon_2}{\varepsilon_1} \right]. \end{aligned} \quad (\text{A.3})$$

Then, keeping in mind this inequality, we deduce

$$\begin{aligned} \int_0^{\varepsilon_3} dr_1 |e^{\pm r_1} - 1|^\beta \left| \int_{-\varepsilon_1 r_1}^{\varepsilon_2 r_1} \mathcal{R}_3 dr_2 \right| \\ \leq C \int_0^{\varepsilon_3} r_1^{\beta-1} e^{-r_1(\varepsilon-\beta)} \left[1 + r_1 (\varepsilon_1 + \varepsilon_2) + \frac{\varepsilon_1}{\varepsilon_2} \right. \\ \left. + \frac{\varepsilon_2}{\varepsilon_1} \right] dr_1 \leq C \varepsilon_3^\beta \left[1 + \varepsilon_3 (\varepsilon_1 + \varepsilon_2) + \frac{\varepsilon_1}{\varepsilon_2} \right. \\ \left. + \frac{\varepsilon_2}{\varepsilon_1} \right]. \end{aligned} \quad (\text{A.4})$$

Next, recasting this argument in the case of the term $\int_0^{\varepsilon_3} dr_2 |e^{\pm r_2} - 1|^\beta \left| \int_{-\varepsilon_1 r_2}^{\varepsilon_2 r_2} \mathcal{R}_3 dr_1 \right|$ provides the analogous estimate to the second term in the left-hand side of (ii). This completes the proof of (ii).

It is apparent that statement (iii) follows from direct calculations and representation (A.1) and Propositions 11 and 12.

As for (iv), we prove the first inequality; the second one is obtained with the same arguments. Let us consider the difference

$$\begin{aligned} \Delta \mathcal{R}_3 &= \mathcal{R}_3 \left(x_1, x_2, r_1 + \ln \left(1 + \frac{\Delta_1 x}{x_1} \right), r_2 \right) \\ &\quad - \mathcal{R}_3 (\bar{x}_1, x_2, r_1, r_2) = \mathcal{J}_1 + \mathcal{J}_2, \end{aligned} \quad (\text{A.5})$$

where we put

$$\begin{aligned} \mathcal{J}_i &= \mathcal{R}_{3i} \left(x_1, x_2, r_1 + \ln \left(1 + \frac{\Delta_1 x}{x_1} \right), r_2 \right) \\ &\quad - \mathcal{R}_{3i} (\bar{x}_1, x_2, r_1, r_2) \end{aligned} \quad (\text{A.6})$$

for $i = 1, 2$.

Introducing new functions

$$\begin{aligned} K_1(z) &= zK \left(-1 + \frac{iz}{\alpha} \right) e^{\pi z/\alpha} e^{iz(r_2-r_1)}, \\ \Psi(z, r_2) &= \int_{-\infty}^{+\infty} \frac{\Gamma(1 + \varepsilon \operatorname{sgn} r_2 + i(p_2 - z))}{\Gamma(1 + \varepsilon \operatorname{sgn} r_2 + ip_2)} e^{ip_2 r_2} dp_2, \end{aligned} \quad (\text{A.7})$$

we rewrite the functions $\mathcal{J}_i$ in the suitable forms

$$\begin{aligned} \mathcal{J}_1 &= e^{-\varepsilon |r_2|} \ln \left(1 + \frac{\Delta_1 x}{x_1} \right) \\ &\quad \times \int_{|z| > 2M} \frac{K_1(z) e^{iz \ln(x_2/\bar{x}_1)} \Psi(z, r_2)}{z} \\ &\quad \times \int_0^1 dp \frac{\partial}{\partial r_1} e^{-\varepsilon(r_1 + p \ln(1 + \Delta_1 x/x_1))} \\ &\quad \times \int_{-\infty}^{+\infty} \frac{\Gamma(1 - \varepsilon - i(p_1 - z))}{\Gamma(1 - \varepsilon - ip_1)} \\ &\quad \times e^{-ip_1(r_1 + p \ln(1 + \Delta_1 x/x_1))} dp_1 dz, \\ \mathcal{J}_2 &= e^{-\varepsilon |r_2|} \ln \left(1 + \frac{\Delta_1 x}{x_1} \right) \int_{|z| < 2M} dz e^{iz \ln(x_2/\bar{x}_1)} \\ &\quad \times \int_0^1 dq \frac{\partial}{\partial qz} \left[K_1(qz) \Psi(qz, r_2) \right. \\ &\quad \times \int_0^1 dp \frac{\partial}{\partial r_1} e^{-\varepsilon(r_1 + p \ln(1 + \Delta_1 x/x_1))} \\ &\quad \times \int_{-\infty}^{+\infty} \frac{\Gamma(1 - \varepsilon - i(p_1 - qz))}{\Gamma(1 - \varepsilon - ip_1)} \\ &\quad \left. \times e^{-ip_1(r_1 + p \ln(1 + \Delta_1 x/x_1))} dp_1 \right] \end{aligned} \quad (\text{A.8})$$

Recasting the arguments of Propositions 9 and 10 arrives at

$$\begin{aligned} &\left| \frac{\partial}{\partial r_1} e^{-\varepsilon(r_1 + p \ln(1 + \Delta_1 x/x_1))} \int_{-\infty}^{+\infty} \frac{\Gamma(1 - \varepsilon - i(p_1 - z))}{\Gamma(1 - \varepsilon - ip_1)} \right. \\ &\quad \times e^{-ip_1(r_1 + p \ln(1 + \Delta_1 x/x_1))} dp_1 \Big| \\ &\quad + \left| \frac{\partial}{\partial qz} e^{iz \ln r_1} e^{-\varepsilon(r_1 + p \ln(1 + \Delta_1 x/x_1))} \right. \\ &\quad \times \int_{-\infty}^{+\infty} \frac{\Gamma(1 - \varepsilon - i(p_1 - qz))}{\Gamma(1 - \varepsilon - ip_1)} \\ &\quad \times e^{-ip_1(r_1 + p \ln(1 + \Delta_1 x/x_1))} dp_1 \Big| \\ &\leq \frac{C e^{-\varepsilon r_1}}{[r_1 + p \ln(1 + \Delta_1 x/x_1)]^2}. \end{aligned} \quad (\text{A.9})$$

By this inequality together with Propositions 9 and 10, we have

$$|\mathcal{J}_1| + |\mathcal{J}_2| \leq C \frac{e^{-\varepsilon |r_2|}}{r_1^2} (1 + |r_2|^{-1}). \quad (\text{A.10})$$

In conclusion, we obtain

$$\begin{aligned} & \int_{\varepsilon_3}^{+\infty} dr_1 \int_0^{\varepsilon_2 r_1} |e^{\pm r_2} - 1|^\beta |\Delta \mathcal{R}_3| dr_2 \\ & \leq C \ln \left(1 + \frac{\Delta_1 x}{x_1} \right) \int_{\varepsilon_3}^{+\infty} \frac{dr_1}{r_1^2} \int_0^{\varepsilon_2 r_1} r_2^{\beta-1} e^{-(\varepsilon-\beta)r_2} dr_2 \quad (\text{A.11}) \\ & \leq \frac{C \varepsilon_2^\beta}{\varepsilon_3^{1-\beta}} \ln \left(1 + \frac{\Delta_1 x}{x_1} \right). \end{aligned}$$

Finally, we estimate the term $\int_{\varepsilon_3}^{+\infty} dr_1 |e^{\pm r_1} - 1|^\beta \left| \int_{-\varepsilon_1 r_1}^{\varepsilon_2 r_1} \Delta \mathcal{R}_3 dr_2 \right|$. To this end, we again represent $\Delta \mathcal{R}_3$ via $\mathcal{J}_j$ and then using Propositions 11 and 12 deduce

$$\begin{aligned} & \int_{\varepsilon_3}^{+\infty} dr_1 |e^{\pm r_1} - 1|^\beta \left| \int_{-\varepsilon_1 r_1}^{\varepsilon_2 r_1} \Delta \mathcal{R}_3 dr_2 \right| \leq C \\ & \times \ln \left(1 + \frac{\Delta_1 x}{x_1} \right) \left[1 + \frac{\varepsilon_1}{\varepsilon_2} + \frac{\varepsilon_2}{\varepsilon_1} \right] \int_{\varepsilon_3}^{+\infty} \frac{e^{-(\varepsilon-\beta)r_1} dr_1}{r_1^{2-\beta}} \quad (\text{A.12}) \\ & \leq C \ln \left(1 + \frac{\Delta_1 x}{x_1} \right) \left[1 + \frac{\varepsilon_1}{\varepsilon_2} + \frac{\varepsilon_2}{\varepsilon_1} \right] \varepsilon_3^{\beta-1}. \end{aligned}$$

Collecting this inequality with (A.11), we obtain the first inequality in (iv). This completes the proof of Lemma 21.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

- [1] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of *North-Holland Mathematics Studies*, Elsevier Science, Amsterdam, 2006.
- [2] B. A. Carreras, V. E. Lynch, and G. M. Zaslavsky, "Anomalous diffusion and exit time distribution of particle tracers in plasma turbulence model," *Physics of Plasmas*, vol. 8, no. 12, pp. 5096–5103, 2001.
- [3] M. F. Shlesinger, B. J. West, and J. Klafter, "Lévy dynamics of enclaved diffusion: application to turbulence," *Physical Review Letters*, vol. 58, no. 11, pp. 1100–1103, 1987.
- [4] G. M. Zaslavsky, D. Stevens, and H. Weitzner, "Self-similar transport in incomplete chaos," *Physical Review E: Statistical, Nonlinear, and Soft Matter Physics*, vol. 48, no. 3, pp. 1683–1694, 1993.
- [5] D. A. Benson, S. W. Wheatcraft, and M. M. Meerschaert, "Application of a fractional advection-dispersion equation," *Water Resources Research*, vol. 36, no. 6, pp. 1403–1412, 2000.
- [6] D. A. Benson, S. W. Wheatcraft, and M. M. Meerschaert, "The fractional-order governing equation of Lévy motion," *Water Resources Research*, vol. 36, no. 6, pp. 1413–1423, 2000.
- [7] D. A. Benson, R. Schumer, M. M. Meerschaert, and S. W. Wheatcraft, "Fractional dispersion Lévy motion, and the MADE tracer tests," *Transport in Porous Media*, vol. 42, no. 1–2, pp. 211–240, 2001.
- [8] Z.-Q. Deng, V. P. Singh, and L. Bengtsson, "Numerical solution of fractional advection-dispersion equation," *Journal of Hydraulic Engineering*, vol. 130, no. 5, pp. 422–431, 2004.
- [9] B. Hunt, "Asymptotic solutions for one-dimensional dispersion in rivers," *Journal of Hydraulic Engineering*, vol. 132, no. 1, pp. 87–93, 2006.
- [10] S. Kim and M. L. Kavvas, "Generalized Fick's law and fractional ADE for pollutant transport in a river: detailed derivation," *Journal of Hydrologic Engineering*, vol. 11, no. 1, pp. 80–83, 2006.
- [11] W. Tan, C. Fu, C. Fu, W. Xie, and H. Cheng, "An anomalous subdiffusion model for calcium spark in cardiac myocytes," *Applied Physics Letters*, vol. 91, no. 18, p. 183901, 2007.
- [12] A. V. Chechkin, V. Y. Gonchar, J. Klafter, R. Metzler, and L. V. Tanatarov, "Lévy flights in a steep potential well," *Journal of Statistical Physics*, vol. 115, no. 5–6, pp. 1505–1535, 2004.
- [13] A. V. Chechkin, O. Y. Sliusarenko, R. Metzler, and J. Klafter, "Barrier crossing driven by Lévy noise: Universality and the role of noise intensity," *Physical Review E: Statistical, Nonlinear, and Soft Matter Physics*, vol. 75, no. 4, 2007.
- [14] V. A. Kondratiev, "Boundary value problems for elliptic equations in domains with conical or angular points," *Transactions of the Moscow Mathematical Society*, vol. 16, pp. 227–313, 1967.
- [15] V. A. Kondrat'ev and O. A. Oleinik, "Boundary-value problems for partial differential equations in non-smooth domains," *Russian Mathematical Surveys*, vol. 38, no. 2, pp. 1–86, 1983.
- [16] M. Borsuk and V. Kondratiev, *Elliptic Boundary Value Problems of Second Order in Piecewise Smooth Domains*, vol. 69, Elsevier Science B.V., Amsterdam, Netherlands, 2006.
- [17] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, vol. 24, Pitman, Boston, Mass, USA, 1985.
- [18] V. G. Maz'ya and B. A. Plamenevskii, "Estimates in L^p and an Hölder classes and the Miranda-Agmon maximum principle for solutions of elliptic boundary value problems in domains with singular points on the boundary," *American Mathematical Society Translations*, vol. 123, pp. 1–56, 1984.
- [19] G. Barles, C. Georgelin, and E. R. Jakobsen, "On Neumann and oblique derivatives boundary conditions for nonlocal elliptic equations," *Journal of Differential Equations*, vol. 256, no. 4, pp. 1368–1394, 2014.
- [20] G. Barles, E. Chasseigne, C. Georgelin, and E. R. Jakobsen, "On Neumann type problems for nonlocal equations set in a half space," *Transactions of the American Mathematical Society*, vol. 366, no. 9, pp. 4873–4917, 2014.
- [21] C. Bjorland, L. Caffarelli, and A. Figalli, "Nonlocal tug-of-war and the infinity fractional Laplacian," *Communications on Pure and Applied Mathematics*, vol. 65, no. 3, pp. 337–380, 2012.
- [22] L. Caffarelli and L. Silvestre, "Regularity theory for fully nonlinear integro-differential equations," *Communications on Pure and Applied Mathematics*, vol. 62, no. 5, pp. 597–638, 2009.
- [23] X. Ros-Oton and J. Serra, "The Dirichlet problem for the fractional Laplacian: regularity up to the boundary," *Journal de Mathématiques Pures et Appliquées*, vol. 101, no. 3, pp. 275–302, 2014.
- [24] R. Servadei and E. Valdinoci, "Weak and viscosity solutions of the fractional Laplace equation," *Publicacions Matemàtiques*, vol. 58, no. 1, pp. 133–154, 2014.

- [25] G. P. Lopushanska, “Basic boundary value problems for a certain equation with fractional derivatives,” *Ukrainian Mathematical Zhournal*, vol. 51, no. 1, pp. 48–59, 1999.
- [26] O. K. Masaeva, “Dirichlet problem for the generalized Laplace equation with the Caputo derivative,” *Journal of Differential Equations*, vol. 48, no. 3, pp. 449–454, 2012.
- [27] B. K. Turmetov, B. T. Torebek, and S. Ontuganova, “Some problems for fractional analogue of laplace equation,” *International Journal of Pure and Applied Mathematics*, vol. 94, no. 4, pp. 525–532, 2014.
- [28] B. K. Turmetov and B. T. Torebek, “On solvability of some boundary value problems for a fractional analogue of the Helmholtz equation,” *New York Journal of Mathematics*, vol. 20, pp. 1237–1251, 2014.
- [29] V. J. Ervin and J. P. Roop, “Variational formulation for the stationary fractional advection dispersion equation,” *Numerical Methods for Partial Differential Equations*, vol. 22, no. 3, pp. 558–576, 2006.
- [30] V. J. Ervin and J. P. Roop, “Variational solution of fractional advection dispersion equation on bounded domain in $\mathbb{R}^d$,” *Numerical Methods for Partial Differential Equations*, vol. 23, no. 2, pp. 256–281, 2007.
- [31] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer, Berlin, Germany, 2010.
- [32] N. V. Krylov, *Lecture on Elliptic and Parabolic Equations in Hölder Spaces*, vol. 12 of *Graduate Studies in Mathematics*, AMS, 1996.
- [33] N. Vasylyeva, “On the solvability of some nonclassical boundary-value problem for the Laplace equation in the plane corner,” *Advances in Differential Equations*, vol. 12, no. 10, pp. 1167–1200, 2007.
- [34] J. C. Lagarias, “Euler’s constant: Euler’s work and modern developments,” *Bulletin of the American Mathematical Society*, vol. 50, no. 4, pp. 527–628, 2013.
- [35] L. M. Milne-Thomson, *The Calculus of Finite Differences*, MacMillan, London, UK, 1951.
- [36] B. V. Bazaliy and A. Friedman, “The Hele-Shaw problem with surface tension in a half-plane: a model problem,” *Journal of Differential Equations*, vol. 216, no. 2, pp. 387–438, 2005.
- [37] H. Brunner, H. Han, and D. Yin, “The maximum principle for time-fractional diffusion equations and its application,” *Numerical Functional Analysis and Optimization*, vol. 36, no. 10, pp. 1307–1321, 2015.

Submit your manuscripts at
www.hindawi.com