Research Article

Hermite-Hadamard-Fejér Inequality Related to Generalized Convex Functions via Fractional Integrals

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This paper deals with Hermite-Hadamard-Fejér inequality for (η_1, η_2) -convex functions via fractional integrals. Some mid-point and trapezoid type inequalities related to Hermite-Hadamard inequality when the absolute value of derivative of considered function is (η_1, η_2) -convex functions are obtained. Furthermore, a refinement for classic Hermite-Hadamard inequality via fractional integrals is given when a positive (η_1, η_2) -convex function is increasing.

1. Introduction and Preliminaries

In 1906, L. Fejér [1] proved the following integral inequalities which are known in the literature as Fejér inequality:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) \int_a^b g(x) dx &\leq \int_a^b f(x) g(x) dx \\ &\leq \frac{f(a) + f(b)}{2} \int_a^b g(x) dx, \end{aligned} \quad (1)$$

where $f : [a, b] \rightarrow \mathbb{R}$ is convex and $g : [a, b] \rightarrow [0, \infty) = \mathbb{R}^+$ is integrable and symmetric to $x = ((a+b)/2)$ ($g(x) = g(a+b-x)$, $\forall x \in [a, b]$).

If we consider in (1) that $g \equiv 1$, then we recapture the known Hermite-Hadamard inequality.

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}. \quad (2)$$

On the other hand, the concept of (η_1, η_2) -convex has been introduced in [2, 3] as a generalization of preinvex functions [4–6] and η -convex functions [7–9]. We remind these introductory concepts in the following with some geometric interpretation.

(Preinvex Functions)

Definition 1. A set $I \subseteq \mathbb{R}$ is invex with respect to a real bifunction $\eta : I \times I \rightarrow \mathbb{R}$, if

$$x, y \in I, \lambda \in [0, 1] \implies y + \lambda\eta(x, y) \in I. \quad (3)$$

If I is an invex set with respect to η , then a function $f : I \rightarrow \mathbb{R}$ is said to be preinvex if $x, y \in I$ and $\lambda \in [0, 1]$ implies

$$f(y + \lambda\eta(x, y)) \leq \lambda f(x) + (1 - \lambda) f(y). \quad (4)$$

In fact, in an invex set for any $x, y \in I$, there is a path starting from y to $y + \eta(x, y)$ which is contained in I . The point x is not necessarily the end point of the path. If for every $x, y \in I$ we need that x should be an end point of the path, then I reduces to a convex set.

(η -Convex Functions)

Definition 2. Consider a convex set $I \subseteq \mathbb{R}$ and a bifunction $\eta : f(I) \times f(I) \rightarrow \mathbb{R}$. A function $f : I \rightarrow \mathbb{R}$ is called convex with respect to η (briefly η -convex), if

$$f(\lambda x + (1 - \lambda)y) \leq f(y) + \lambda\eta(f(x), f(y)), \quad (5)$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Geometrically, it says that if a function is η -convex on I , then, for any $x, y \in I$, its graph is on or under the path starting from $(y, f(y))$ and ending at $(x, f(y) + \eta(f(x), f(y)))$. If $f(x)$ should be the end point of the path for every $x, y \in I$, then we have $\eta(x, y) = x - y$ and the function reduces to a convex one.

$((\eta_1, \eta_2)$ -Convex Functions)

Definition 3. Let $I \subseteq \mathbb{R}$ be an invex set with respect to $\eta_1 : I \times I \rightarrow \mathbb{R}$. Consider $f : I \rightarrow \mathbb{R}$ and $\eta_2 : f(I) \times f(I) \rightarrow \mathbb{R}$. The function f is said to be (η_1, η_2) -convex if

$$f(x + \lambda \eta_1(y, x)) \leq f(x) + \lambda \eta_2(f(y), f(x)), \quad (6)$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Remark 4. An (η_1, η_2) -convex function reduces to

- (i) a η -convex function if we consider $\eta_1(x, y) = x - y$ for all $x, y \in I$,
- (ii) a preinvex function if we consider $\eta_2(x, y) = x - y$ for all $x, y \in f(I)$,
- (iii) a convex function if satisfying (i) and (ii).

Example 5 (see [2]). Consider the function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1; \\ 1, & x > 1. \end{cases} \quad (7)$$

Define two bifunctions $\eta_1 : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ and $\eta_2 : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by

$$\eta_1(x, y) = \begin{cases} -y, & 0 \leq y \leq 1; \\ x + y, & y > 1, \end{cases} \quad (8)$$

and

$$\eta_2(x, y) = \begin{cases} x + y, & x \leq y; \\ 2(x + y), & x > y. \end{cases} \quad (9)$$

Then f is an (η_1, η_2) -convex function. But f is not preinvex with respect to η_1 and it is not convex (consider $x = 0, y = 2$, and $\lambda > 0$).

Also we need the following short preliminaries about the fractional calculus theory which are used throughout the paper.

Definition 6 (see [10]). Consider $f \in L^1[a, b]$. The Riemann-Liouville integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a, \quad (10)$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b, \quad (11)$$

respectively, where

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt \quad (12)$$

is Gamma function and $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

Fejér inequality for convex functions related to fractional integrals has been obtained in [11] as the following theorem.

Theorem 7. Let $f : [a, b] \rightarrow \mathbb{R}$ be convex function with $a < b$ and $f \in L^1[a, b]$. If $g : [a, b] \rightarrow \mathbb{R}$ is nonnegative, integrable, and symmetric to $(a+b)/2$, then the following inequalities hold for fractional integrals:

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) [J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)] \\ & \leq [J_{a+}^\alpha (fg)(b) + J_{b-}^\alpha (fg)(a)] \\ & \leq \frac{f(a) + f(b)}{2} [J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)], \end{aligned} \quad (13)$$

with $\alpha > 0$.

Motivated by above works and results, in this paper, we obtain Fejér inequality for (η_1, η_2) -convex functions via fractional integrals. Also we give some mid-point and trapezoid-type inequalities related to Hermite-Hadamard inequality when the absolute value of the derivative of the considered function is (η_1, η_2) -convex functions. Furthermore, we prove that when a positive (η_1, η_2) -convex function is increasing, there exists a refinement for classic Hermite-Hadamard inequality via fractional integrals.

2. Fejér Inequality

In this section, we obtain (η_1, η_2) -convex version of the Fejér inequality related to fractional integrals. For convenience, we separate this inequality to the left and right.

Theorem 8 (Fejér's left inequality). Let $I \subseteq \mathbb{R}$ be an invex set with respect to η_1 such that

$$\begin{aligned} & \eta_1(x_2 + t_2 \eta_1(x_1, x_2), x_2 + t_1 \eta_1(x_1, x_2)) \\ & = (t_2 - t_1) \eta_1(x_1, x_2), \end{aligned} \quad (14)$$

for all $x_1, x_2 \in I$ and $t_1, t_2 \in [0, 1]$. Also let $f : I \rightarrow \mathbb{R}$ be an (η_1, η_2) -convex function, where η_2 is an integrable bifunction on $f(I) \times f(I)$. For any $a, b \in I$ with $\eta_1(b, a) > 0$, suppose that $f \in L^1[a, a + \eta_1(b, a)]$ and the function $g : [a, a + \eta_1(b, a)] \rightarrow \mathbb{R}^+$ is integrable and symmetric to $a + (1/2)\eta_1(b, a)$. Then, for $\alpha > 0$, the following inequality holds:

$$\begin{aligned} & f\left(\frac{2a + \eta_1(b, a)}{2}\right) [J_{a+}^\alpha g(a + \eta_1(b, a)) \\ & + J_{(a+\eta_1(b,a))^-}^\alpha g(a)] - \frac{1}{2\Gamma(\alpha)} \end{aligned}$$

$$\begin{aligned}
& \cdot \int_a^{a+\eta_1(b,a)} \left[(x-a)^{\alpha-1} + (a+\eta_1(b,a)-x)^{\alpha-1} \right] \\
& \cdot \eta_2(f(x), f(2a+\eta_1(b,a)-x)) g(x) dx \\
& \leq \left[J_{a+}^{\alpha}(fg)(a+\eta_1(b,a)) + J_{(a+\eta_1(b,a))^-}^{\alpha}(fg)(a) \right].
\end{aligned} \quad (15)$$

Proof. Using condition (14) and the (η_1, η_2) -convexity of f , we have

$$\begin{aligned}
f\left(\frac{2a+\eta_1(b,a)}{2}\right) &= f\left(\frac{2a+(1+t)\eta_1(b,a)}{2} - \frac{t}{2}\right. \\
&\quad \cdot \eta_1(b,a) \Bigg) = f\left(\frac{2a+(1+t)\eta_1(b,a)}{2} + \frac{1}{2}\eta_1\left(a\right. \right. \\
&\quad \left. \left. + \frac{(1-t)}{2}\eta_1(b,a), a + \frac{(1+t)}{2}\eta_1(b,a)\right)\right) \\
&= f\left(\frac{2a+(1+t)\eta_1(b,a)}{2} + \frac{1}{2}\right. \\
&\quad \cdot \eta_1\left(\frac{2a+\eta_1(b,a)-t\eta_1(b,a)}{2}, \right. \\
&\quad \left. \left. \frac{2a+\eta_1(b,a)+t\eta_1(b,a)}{2}\right)\right) \\
&\leq f\left(\frac{2a+(1+t)\eta_1(b,a)}{2}\right) + \frac{1}{2} \\
&\quad \cdot \eta_2\left(f\left(\frac{2a+(1-t)\eta_1(b,a)}{2}\right), \right. \\
&\quad \left. f\left(\frac{2a+(1+t)\eta_1(b,a)}{2}\right)\right),
\end{aligned} \quad (16)$$

and with the same argument as above we have

$$\begin{aligned}
f\left(\frac{2a+\eta_1(b,a)}{2}\right) &\leq f\left(\frac{2a+(1-t)\eta_1(b,a)}{2}\right) + \frac{1}{2} \\
&\quad \cdot \eta_2\left(f\left(\frac{2a+(1+t)\eta_1(b,a)}{2}\right), \right. \\
&\quad \left. f\left(\frac{2a+(1-t)\eta_1(b,a)}{2}\right)\right).
\end{aligned} \quad (17)$$

By the use of Definition 6 and two changes of variable

$$x = \frac{2a+(1-t)\eta_1(b,a)}{2}, \quad (18)$$

and

$$x = \frac{2a+(1+t)\eta_1(b,a)}{2}, \quad (19)$$

in (16) and (17), respectively, we obtain the following inequalities:

$$\begin{aligned}
K_1 &= \Gamma(\alpha) J_{(a+\eta_1(b,a))^-}^{\alpha} f(a) g(a) = \int_a^{a+\eta_1(b,a)} (x \\
&\quad - a)^{\alpha-1} f(x) g(x) dx = \int_a^{a+(1/2)\eta_1(b,a)} (x-a)^{\alpha-1} \\
&\quad \cdot f(x) g(x) dx + \int_{a+(1/2)\eta_1(b,a)}^{a+\eta_1(b,a)} (x-a)^{\alpha-1} f(x) \\
&\quad \cdot g(x) dx = \left(\frac{\eta_1(b,a)}{2}\right)^{\alpha} \left[\int_0^1 (1-t)^{\alpha-1} \right. \\
&\quad \cdot f\left(\frac{2a+(1-t)\eta_1(b,a)}{2}\right) \\
&\quad \cdot g\left(\frac{2a+(1-t)\eta_1(b,a)}{2}\right) dt + \int_0^1 (1+t)^{\alpha-1} \\
&\quad \cdot f\left(\frac{2a+(1+t)\eta_1(b,a)}{2}\right) \\
&\quad \cdot g\left(\frac{2a+(1+t)\eta_1(b,a)}{2}\right) dt \Bigg] \\
&\geq \left(\frac{\eta_1(b,a)}{2}\right)^{\alpha} \left[\int_0^1 (1-t)^{\alpha-1} f\left(\frac{2a+\eta_1(b,a)}{2}\right) \right. \\
&\quad \cdot g\left(\frac{2a+(1-t)\eta_1(b,a)}{2}\right) dt - \frac{1}{2} \int_0^1 (1 \\
&\quad - t)^{\alpha-1} \eta_2\left(f\left(\frac{2a+(1+t)\eta_1(b,a)}{2}\right), \right. \\
&\quad \left. f\left(\frac{2a+(1-t)\eta_1(b,a)}{2}\right)\right) \\
&\quad \times g\left(\frac{2a+(1-t)\eta_1(b,a)}{2}\right) dt + \int_0^1 (1+t)^{\alpha-1} \\
&\quad \cdot f\left(\frac{2a+\eta_1(b,a)}{2}\right) \\
&\quad \cdot g\left(\frac{2a+(1+t)\eta_1(b,a)}{2}\right) dt - \frac{1}{2} \int_0^1 (1 \\
&\quad + t)^{\alpha-1} \eta_2\left(f\left(\frac{2a+(1-t)\eta_1(b,a)}{2}\right), \right. \\
&\quad \left. f\left(\frac{2a+(1+t)\eta_1(b,a)}{2}\right)\right) \\
&\quad \times g\left(\frac{2a+(1+t)\eta_1(b,a)}{2}\right) dt \Bigg].
\end{aligned} \quad (20)$$

The simple form of (20) along with the fact that g is symmetric to $a+(1/2)\eta_1(b,a)$ leads to the following relations:

$$K_1 \geq \left(\frac{2a+\eta_1(b,a)}{2}\right)^{\alpha} \int_a^{a+(1/2)\eta_1(b,a)} (x-a)^{\alpha-1}$$

$$\begin{aligned}
& \cdot g(x) dx - \frac{1}{2} \\
& \cdot \int_a^{a+(1/2)\eta_1(b,a)} (a + \eta_1(b,a) - x)^{\alpha-1} \\
& \cdot \eta_2(f(x), f(2a + \eta_1(b,a) - x)) g(x) dx \\
& + f\left(\frac{2a + \eta_1(b,a)}{2}\right) \int_{a+(1/2)\eta_1(b,a)}^{a+\eta_1(b,a)} (x-a)^{\alpha-1} \\
& \cdot g(x) dx - \frac{1}{2} \int_{a+(1/2)\eta_1(b,a)}^{a+\eta_1(b,a)} (a + \eta_1(b,a) - x)^{\alpha-1} \\
& \cdot \eta_2(f(x), f(2a + \eta_1(b,a) - x)) g(x) dx \\
& = f\left(\frac{2a + \eta_1(b,a)}{2}\right) \int_a^{a+\eta_1(b,a)} (x-a)^{\alpha-1} g(x) dx \\
& - \frac{1}{2} \int_a^{a+\eta_1(b,a)} (a + \eta_1(b,a) - x)^{\alpha-1} \\
& \cdot \eta_2(f(x), f(2a + \eta_1(b,a) - x)) g(x) dx \\
& = \Gamma(\alpha) f\left(\frac{2a + \eta_1(b,a)}{2}\right) J_{(a+\eta_1(b,a))^-}^\alpha g(a) - \frac{1}{2} \\
& \cdot \int_a^{a+\eta_1(b,a)} (a + \eta_1(b,a) - x)^{\alpha-1} \\
& \cdot \eta_2(f(x), f(2a + \eta_1(b,a) - x)) g(x) dx.
\end{aligned} \tag{21}$$

Also with the same argument as above we have

$$\begin{aligned}
K_2 &= \Gamma(\alpha) J_{a^+}^\alpha f(a + \eta_1(b,a)) g(a + \eta_1(b,a)) \\
&= \int_a^{a+\eta_1(b,a)} (a + \eta_1(b,a) - x)^{\alpha-1} f(x) g(x) dx \\
&\geq \Gamma(\alpha) f\left(\frac{2a + \eta_1(b,a)}{2}\right) J_{a^+}^\alpha g(a + \eta_1(b,a)) - \frac{1}{2} \\
&\cdot \int_a^{a+\eta_1(b,a)} (x-a)^{\alpha-1} \\
&\cdot \eta_2(f(x), f(2a + \eta_1(b,a) - x)) g(x) dx.
\end{aligned} \tag{22}$$

Now adding K_1 to K_2 implies the result. $\square$

To obtain the right part of the Fejér inequality related to fractional integrals, we need a primary lemma.

Lemma 9. Let $I \subseteq \mathbb{R}$ be an invex set with respect to $\eta_1 : I \times I \rightarrow \mathbb{R}$ and $a, b \in I$ with $\eta_1(b,a) > 0$. If $g : [a, a + \eta_1(b,a)] \rightarrow \mathbb{R}$ is integrable and symmetric to $a + (1/2)\eta_1(b,a)$, then

$$\begin{aligned}
J_{a^+}^\alpha g(a + \eta_1(b,a)) &= J_{(a+\eta_1(b,a))^-}^\alpha g(a) \\
&= \frac{1}{2} [J_{a^+}^\alpha g(a + \eta_1(b,a)) + J_{(a+\eta_1(b,a))^-}^\alpha g(a)].
\end{aligned} \tag{23}$$

Proof. Since g is symmetric to $a + (1/2)\eta_1(b,a)$, we have $g(2a + \eta_1(b,a) - x) = g(x)$ for all $x \in [a, a + \eta_1(b,a)]$. Then

$$\begin{aligned}
& J_{a^+}^\alpha g(a + \eta_1(b,a)) \\
&= \frac{1}{\Gamma(\alpha)} \int_a^{a+\eta_1(b,a)} (a + \eta_1(b,a) - x)^{\alpha-1} g(x) dx \\
&= \frac{1}{\Gamma(\alpha)} \int_a^{a+\eta_1(b,a)} (t-a)^{\alpha-1} g(2a + \eta_1(b,a) - t) dt \\
&= \frac{1}{\Gamma(\alpha)} \int_a^{a+\eta_1(b,a)} (t-a)^{\alpha-1} g(t) dt \\
&= J_{(a+\eta_1(b,a))^-}^\alpha g(a).
\end{aligned} \tag{24}$$

$\square$

Theorem 10 (Fejér's right inequality). Let $I \subseteq \mathbb{R}$ be an invex set with respect to η_1 and let $f : I \rightarrow \mathbb{R}$ be an (η_1, η_2) -convex function, where η_2 is an integrable bifunction on $f(I) \times f(I)$. For any $a, b \in I$ with $\eta_1(b,a) > 0$, suppose that the function $g : [a, a + \eta_1(b,a)] \rightarrow \mathbb{R}^+$ is integrable and symmetric to $a + (1/2)\eta_1(b,a)$ and $f \in L^1[a, a + \eta_1(b,a)]$. Then, for $\alpha > 0$, the following inequality holds:

$$\begin{aligned}
& [J_{a^+}^\alpha (fg)(a + \eta_1(b,a)) + J_{(a+\eta_1(b,a))^-}^\alpha (fg)(a)] \\
& \leq \left(\frac{2f(a) + \eta_2(f(b), f(a))}{2} \right) \\
& \cdot [J_{a^+}^\alpha g(a + \eta_1(b,a)) + J_{(a+\eta_1(b,a))^-}^\alpha g(a)].
\end{aligned} \tag{25}$$

Proof. From (η_1, η_2) -convexity of f , using the changes of variables $x = a + t\eta_1(b,a)$ and $x = a + (1-t)\eta_1(b,a)$, respectively, we obtain the two following inequalities:

$$\begin{aligned}
& \frac{\Gamma(\alpha)}{\eta_1^\alpha(b,a)} J_{(a+\eta_1(b,a))^-}^\alpha f(a) g(a) = \frac{1}{\eta_1^\alpha(b,a)} \\
& \cdot \int_a^{a+\eta_1(b,a)} (x-a)^{\alpha-1} f(x) g(x) dx \\
& \leq \int_0^1 t^{\alpha-1} [f(a) + t\eta_2(f(b), f(a))] \\
& \cdot g(a + t\eta_1(b,a)) dt = f(a) \\
& \cdot \int_0^1 t^{\alpha-1} g(a + t\eta_1(b,a)) dt + \eta_2(f(b), f(a)) \\
& \cdot \int_0^1 t^\alpha g(a + t\eta_1(b,a)) dt,
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
& \frac{\Gamma(\alpha)}{\eta_1^\alpha(b,a)} J_{a^+}^\alpha f(a + \eta_1(b,a)) g(a + \eta_1(b,a)) \\
&= \frac{1}{\eta_1^\alpha(b,a)} \int_a^{a+\eta_1(b,a)} (a + \eta_1(b,a) - x)^{\alpha-1} f(x)
\end{aligned}$$

$$\begin{aligned}
& \cdot g(x) dx \\
& \leq \int_0^1 t^{\alpha-1} [f(a) + (1-t)\eta_2(f(b), f(a))] \\
& \cdot g(a + (1-t)\eta_1(b, a)) dt = f(a) \\
& \cdot \int_0^1 t^{\alpha-1} g(a + (1-t)\eta_1(b, a)) dt \\
& + \eta_2(f(b), f(a)) \int_0^1 t^{\alpha-1} (1-t) \\
& \cdot g(a + (1-t)\eta_1(b, a)) dt.
\end{aligned} \tag{27}$$

Now adding (26) to (27) with the fact that g is symmetric to $a + (1/2)\eta_1(b, a)$ implies that

$$\begin{aligned}
& \frac{\Gamma(\alpha)}{\eta_1^\alpha(b, a)} [J_{a^+}^\alpha f(a + \eta_1(b, a)) g(a + \eta_1(b, a)) \\
& + J_{(a+\eta_1(b, a))^-}^\alpha f(a) g(a)] \\
& \leq \left[2f(a) \int_0^1 t^{\alpha-1} g(a + t\eta_1(b, a)) dt \right. \\
& \left. + \eta_2(f(b), f(a)) \int_0^1 t^{\alpha-1} g(a + t\eta_1(b, a)) dt \right] \\
& = (2f(a) + \eta_2(f(b), f(a))) \\
& \cdot \int_0^1 t^{\alpha-1} g(a + t\eta_1(b, a)) dt = \frac{\Gamma(\alpha)}{\eta_1^\alpha(b, a)} (2f(a) \\
& + \eta_2(f(b), f(a))) J_{(a+\eta_1(b, a))^-}^\alpha g(a).
\end{aligned} \tag{28}$$

Now by the use of Lemma 9 we have

$$\begin{aligned}
& \frac{\Gamma(\alpha)}{\eta_1^\alpha(b, a)} [J_{a^+}^\alpha f(a + \eta_1(b, a)) g(a + \eta_1(b, a)) \\
& + J_{(a+\eta_1(b, a))^-}^\alpha f(a) g(a)] \\
& \leq \frac{\Gamma(\alpha)}{\eta_1^\alpha(b, a)} \left(\frac{2f(a) + \eta_2(f(b), f(a))}{2} \right) \\
& \cdot [J_{a^+}^\alpha g(a + \eta_1(b, a)) + J_{(a+\eta_1(b, a))^-}^\alpha g(a)],
\end{aligned} \tag{29}$$

which implies the respected inequality. $\square$

Corollary 11. If in Theorems 8 and 10 we consider

(i) $\alpha = 1$, then the following inequality holds, which is the classical form of Fejér inequality related to (η_1, η_2) -convex functions:

$$\begin{aligned}
& f\left(\frac{2a + \eta_1(b, a)}{2}\right) \int_a^{a+\eta_1(b, a)} g(x) dx - \frac{1}{2} \\
& \cdot \int_a^{a+\eta_1(b, a)} \eta_2(f(x), f(2a + \eta_1(b, a) - x))
\end{aligned}$$

$$\begin{aligned}
& \cdot g(x) dx \leq \int_a^{a+\eta_1(b, a)} f(x) g(x) dx \leq \left[f(a) \right. \\
& \left. + \frac{\eta_2(f(b), f(a))}{2} \right] \int_a^{a+\eta_1(b, a)} g(x) dx.
\end{aligned} \tag{30}$$

(ii) $g \equiv 1$, then we get Hermite-Hadamard inequality for (η_1, η_2) -convex functions as follows:

$$\begin{aligned}
& f\left(\frac{2a + \eta_1(b, a)}{2}\right) \leq \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \\
& \leq \left(\frac{2f(a) + \eta_2(f(b), f(a))}{2} \right),
\end{aligned} \tag{31}$$

which is a generalization of inequality (2.1) in [12].

Corollary 12. If in Theorems 8 and 10 we set $\eta_1(x, y) = x - y$ for all $x, y \in I$, then we obtain Fejér inequality for fractional integrals related to η -convex functions.

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right) [J_{a^+}^\alpha g(b) + J_{b^-}^\alpha g(a)] \\
& \leq [J_{a^+}^\alpha (fg)(b) + J_{b^-}^\alpha (fg)(a)] \\
& \leq \left(\frac{2f(a) + \eta_2(f(b), f(a))}{2} \right) \\
& \cdot [J_{a^+}^\alpha g(b) + J_{b^-}^\alpha g(a)].
\end{aligned} \tag{32}$$

Corollary 13. If in Theorems 8 and 10 we set $\eta_2(x, y) = x - y$ for all $x, y \in f(I)$, then we obtain Fejér inequality for fractional integrals related to preinvex functions.

$$\begin{aligned}
& f\left(\frac{2a + \eta_1(b, a)}{2}\right) \\
& \cdot [J_{a^+}^\alpha g(a + \eta_1(b, a)) + J_{(a+\eta_1(b, a))^-}^\alpha g(a)] \\
& \leq [J_{a^+}^\alpha (fg)(a + \eta_1(b, a)) + J_{(a+\eta_1(b, a))^-}^\alpha (fg)(a)] \\
& \leq \left(\frac{f(a) + f(b)}{2} \right) \\
& \cdot [J_{a^+}^\alpha g(a + \eta_1(b, a)) + J_{(a+\eta_1(b, a))^-}^\alpha g(a)].
\end{aligned} \tag{33}$$

Corollary 14 (see [11]). With all conditions of Corollaries 12 and 13, we have the classic Fejér inequality for fractional integrals.

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right) [J_{a^+}^\alpha g(b) + J_{b^-}^\alpha g(a)] \\
& \leq [J_{a^+}^\alpha (fg)(b) + J_{b^-}^\alpha (fg)(a)] \\
& \leq \left(\frac{f(a) + f(b)}{2} \right) [J_{a^+}^\alpha g(b) + J_{b^-}^\alpha g(a)].
\end{aligned} \tag{34}$$

Corollary 15 (see [12]). If in (34) we consider $g \equiv 1$, then we recapture Hermite-Hadamard inequality for fractional integrals in convex case.

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha(f)(b) + J_{b^-}^\alpha(f)(a)] \\ &\leq \frac{f(a) + f(b)}{2}. \end{aligned} \quad (35)$$

3. Mid-Point and Trapezoid-Type Inequalities

In this section, we obtain, respectively, the mid-point and trapezoid-type inequalities related to (31) when the absolute value of the derivative of the considered function is (η_1, η_2) -convex. In fact, by mid-point-type inequality we mean estimating the difference between left and middle parts of (31) and by trapezoid-type inequality we mean estimating the difference between right and middle parts of (31).

The following lemma is generalization of Lemma 1 obtained in [13] to the preinvex case.

Lemma 16. Let $I \subseteq \mathbb{R}$ be an open invex set with respect to $\eta_1 : I \times I \rightarrow \mathbb{R}$ and let $f : I \rightarrow \mathbb{R}$ be a differentiable function. For any $a, b \in I$ with $\eta_1(b, a) > 0$, if $f \in L^1[a, a + \eta_1(b, a)]$, then the following equality for fractional integrals holds:

$$\begin{aligned} &f\left(\frac{2a + \eta_1(b, a)}{2}\right) \\ &- \frac{\Gamma(\alpha+1)}{2\eta_1^\alpha(b, a)} [J_{a^+}^\alpha f(a + \eta_1(b, a)) + J_{(a+\eta_1(b, a))^-}^\alpha f(a)] \quad (36) \\ &= \frac{\eta_1(b, a)}{2} \sum_{k=1}^4 I_k, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_0^{1/2} t^\alpha f'(a + t\eta_1(b, a)) dt, \\ I_2 &= \int_0^{1/2} (-t^\alpha) f'(a + (1-t)\eta_1(b, a)) dt, \\ I_3 &= \int_{1/2}^1 (t^\alpha - 1) f'(a + t\eta_1(b, a)) dt, \\ I_4 &= \int_{1/2}^1 (1 - t^\alpha) f'(a + (1-t)\eta_1(b, a)) dt. \end{aligned} \quad (37)$$

Proof. Integrating by parts in I_1 implies that

$$\begin{aligned} I_1 &= \int_0^{1/2} t^\alpha f'(a + t\eta_1(b, a)) dt \\ &= \frac{t^\alpha f(a + t\eta_1(b, a))}{\eta_1(b, a)} \Big|_0^{1/2} \end{aligned}$$

$$\begin{aligned} &- \frac{\alpha}{\eta_1(b, a)} \int_0^{1/2} t^{\alpha-1} f(a + t\eta_1(b, a)) dt \\ &= \frac{2^{-\alpha}}{\eta_1(b, a)} f\left(\frac{2a + \eta_1(b, a)}{2}\right) \\ &- \frac{\alpha}{\eta_1(b, a)} \int_0^{1/2} t^{\alpha-1} f(a + t\eta_1(b, a)) dt. \end{aligned} \quad (38)$$

Similarly, we have

$$\begin{aligned} I_2 &= \frac{2^{-\alpha}}{\eta_1(b, a)} f\left(\frac{2a + \eta_1(b, a)}{2}\right) \\ &- \frac{\alpha}{\eta_1(b, a)} \int_0^{1/2} t^{\alpha-1} f(a + (1-t)\eta_1(b, a)) dt, \\ I_3 &= \int_{1/2}^1 (t^\alpha - 1) f'(a + t\eta_1(b, a)) dt \\ &= \frac{(t^\alpha - 1) f(a + t\eta_1(b, a))}{\eta_1(b, a)} \Big|_{1/2}^1 \\ &- \frac{\alpha}{\eta_1(b, a)} \int_{1/2}^1 t^{\alpha-1} f(a + t\eta_1(b, a)) dt \quad (39) \\ &= \frac{1 - 2^{-\alpha}}{\eta_1(b, a)} f\left(\frac{2a + \eta_1(b, a)}{2}\right) \\ &- \frac{\alpha}{\eta_1(b, a)} \int_{1/2}^1 t^{\alpha-1} f(a + t\eta_1(b, a)) dt, \\ I_4 &= \frac{1 - 2^{-\alpha}}{\eta_1(b, a)} f\left(\frac{2a + \eta_1(b, a)}{2}\right) \\ &- \frac{\alpha}{\eta_1(b, a)} \int_{1/2}^1 t^{\alpha-1} f(a + (1-t)\eta_1(b, a)) dt. \end{aligned}$$

Now, by adding all of above equalities, we get to the desired result.

$$\begin{aligned} I_1 + I_2 + I_3 + I_4 &= \frac{2}{\eta_1(b, a)} f\left(\frac{2a + \eta_1(b, a)}{2}\right) \\ &- \frac{\alpha}{\eta_1(b, a)} \left[\int_0^1 t^{\alpha-1} f(a + t\eta_1(b, a)) dt \right. \\ &+ \left. \int_0^1 t^{\alpha-1} f(a + (1-t)\eta_1(b, a)) dt \right] = \frac{2}{\eta_1(b, a)} \\ &\cdot f\left(\frac{2a + \eta_1(b, a)}{2}\right) \\ &- \frac{\Gamma(\alpha+1)}{\eta_1^{\alpha+1}(b, a)} [J_{(a+\eta_1(b, a))^-}^\alpha f(a) \\ &+ J_{a^+}^\alpha f(a + \eta_1(b, a))] . \end{aligned} \quad (40)$$

□

The mid-point-type inequality related to (31) is obtained in the following.

Theorem 17. Let $I \subseteq \mathbb{R}$ be an open invex set with respect to $\eta_1 : I \times I \rightarrow \mathbb{R}$ and let $f : I \rightarrow \mathbb{R}$ be a differentiable function. Suppose that $|f'|$ is an (η_1, η_2) -convex function on I and, for any $a, b \in I$ with $\eta_1(b, a) > 0$, $f' \in L^1[a, a + \eta_1(b, a)]$. Then

$$\begin{aligned} & \left| f\left(\frac{2a + \eta_1(b, a)}{2}\right) - \frac{\Gamma(\alpha + 1)}{2\eta_1^\alpha(b, a)} [J_{a^+}^\alpha f(a + \eta_1(b, a)) \right. \\ & \quad \left. + J_{(a+\eta_1(b, a))^-}^\alpha f(a)] \right| \leq \frac{\eta_1(b, a)}{2^{\alpha+1}(\alpha + 1)} \left\{ |f'(a)| \right. \\ & \quad \left. + |f'(b)| + \frac{1}{2} \eta_2(|f'(a)|, |f'(b)|) + \frac{1}{2} \right. \\ & \quad \left. \cdot \eta_2(|f'(b)|, |f'(a)|) \right\}, \end{aligned} \quad (41)$$

for $0 < \alpha \leq 1$.

Proof. Using Lemma 16, we get

$$\begin{aligned} & \left| f\left(\frac{2a + \eta_1(b, a)}{2}\right) - \frac{\Gamma(\alpha + 1)}{2\eta_1^\alpha(b, a)} [J_{a^+}^\alpha f(a + \eta_1(b, a)) \right. \\ & \quad \left. + J_{(a+\eta_1(b, a))^-}^\alpha f(a)] \right| \leq \frac{\eta_1(b, a)}{2} \sum_{k=1}^4 |I_k|. \end{aligned} \quad (42)$$

Now, using (η_1, η_2) -convexity of $|f'|$, we obtain

$$\begin{aligned} |I_1| & \leq \int_0^{1/2} t^\alpha |f'(a + t\eta_1(b, a))| dt \\ & \leq \int_0^{1/2} t^\alpha |f'(a)| dt \\ & \quad + \int_0^{1/2} t^{\alpha+1} \eta_2(|f'(b)|, |f'(a)|) dt \\ & = \frac{1}{2^{\alpha+1}(\alpha + 1)} |f'(a)| \\ & \quad + \frac{1}{2^{\alpha+2}(\alpha + 2)} \eta_2(|f'(b)|, |f'(a)|). \end{aligned} \quad (43)$$

Analogously,

$$\begin{aligned} |I_2| & \leq \frac{1}{2^{\alpha+1}(\alpha + 1)} |f'(b)| \\ & \quad + \frac{1}{2^{\alpha+2}(\alpha + 2)} \eta_2(|f'(a)|, |f'(b)|). \end{aligned} \quad (44)$$

Also, using the fact that

$$|t_1^\alpha - t_2^\alpha| \leq |t_1 - t_2|^\alpha \quad (45)$$

for all $\alpha \in (0, 1]$ and $t_1, t_2 \in [0, 1]$, we have

$$\begin{aligned} |I_3| & \leq \frac{1}{2^{\alpha+1}(\alpha + 1)} |f'(a)| \\ & \quad + \frac{\alpha + 3}{2^{\alpha+2}(\alpha + 1)(\alpha + 2)} \eta_2(|f'(b)|, |f'(a)|), \end{aligned} \quad (46)$$

and

$$\begin{aligned} |I_4| & \leq \frac{1}{2^{\alpha+1}(\alpha + 1)} |f'(b)| \\ & \quad + \frac{\alpha + 3}{2^{\alpha+2}(\alpha + 1)(\alpha + 2)} \eta_2(|f'(a)|, |f'(b)|). \end{aligned} \quad (47)$$

Now adding all of above inequalities implies the required result. $\square$

Corollary 18. If in Theorem 17 we consider $\eta_1(x, y) = x - y$ for all $x, y \in I$, then

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{(b)^-}^\alpha f(a)] \right| \\ & \leq \frac{(b-a)}{2^{\alpha+1}(\alpha + 1)} \left\{ |f'(a)| + |f'(b)| \right. \\ & \quad \left. + \frac{1}{2} \eta_2(|f'(a)|, |f'(b)|) \right. \\ & \quad \left. + \frac{1}{2} \eta_2(|f'(b)|, |f'(a)|) \right\}. \end{aligned} \quad (48)$$

Furthermore, if we set $\eta_2(x, y) = x - y$ for all $x, y \in f(I)$, then we recapture Theorem 2 in [13].

The following result has been obtained in [14].

Lemma 19. Let $I \subseteq \mathbb{R}$ be an open invex set with respect to $\eta_1 : I \times I \rightarrow \mathbb{R}$. Also, suppose that $f : I \rightarrow \mathbb{R}$ is a differentiable function. For any $a, b \in I$ with $\eta_1(b, a) > 0$, if $f \in L^1[a, a + \eta_1(b, a)]$, then the following equality for fractional integral holds:

$$\begin{aligned} & \frac{f(a) + f(a + \eta_1(b, a))}{2} \\ & \quad - \frac{\Gamma(\alpha + 1)}{2\eta_1^\alpha(b, a)} [J_{a^+}^\alpha f(a + \eta_1(b, a)) + J_{(a+\eta_1(b, a))^-}^\alpha f(a)] \\ & = \frac{\eta_1(b, a)}{2} \int_0^1 [t^\alpha - (1-t)^\alpha] f'(a + t\eta_1(b, a)) dt. \end{aligned} \quad (49)$$

Now we give the trapezoid-type inequality related to (31).

Theorem 20. Let $I \subseteq \mathbb{R}$ be an open invex set with respect to $\eta_1 : I \times I \rightarrow \mathbb{R}$ and let $f : I \rightarrow \mathbb{R}$ be a differentiable function. Suppose that $|f'|$ is an (η_1, η_2) -convex function on I and,

for any $a, b \in I$ with $\eta_1(b, a) > 0$, $f' \in L^1[a, a + \eta_1(b, a)]$. Then the following inequality for fractional integrals holds:

$$\begin{aligned} & \left| \frac{f(a) + f(a + \eta_1(b, a))}{2} \right. \\ & \quad - \frac{\Gamma(\alpha + 1)}{2\eta_1^\alpha(b, a)} [J_{a^+}^\alpha f(a + \eta_1(b, a)) \\ & \quad + J_{(a+\eta_1(b, a))^-}^\alpha f(a)] \Big| \leq \frac{\eta_1(b, a)}{2(\alpha + 1)} \left(1 - \frac{1}{2^\alpha} \right) \\ & \quad \cdot [2|f'(a)| + \eta_2(|f'(b)|, |f'(a)|)]. \end{aligned} \quad (50)$$

Proof. Using Lemma 19 and (η_1, η_2) -convexity of $|f'|$, we get

$$\begin{aligned} & \left| \frac{f(a) + f(a + \eta_1(b, a))}{2} \right. \\ & \quad - \frac{\Gamma(\alpha + 1)}{2\eta_1^\alpha(b, a)} [J_{a^+}^\alpha f(a + \eta_1(b, a)) \\ & \quad + J_{(a+\eta_1(b, a))^-}^\alpha f(a)] \Big| \leq \frac{\eta_1(b, a)}{2} \int_0^1 |t^\alpha - (1-t)^\alpha| \\ & \quad \cdot |f'(a + t\eta_1(b, a))| dt \leq \frac{\eta_1(b, a)}{2} \int_0^1 |t^\alpha \\ & \quad - (1-t)^\alpha| [|f'(a)| + t\eta_2(|f'(b)|, |f'(a)|)] dt \\ & = \frac{\eta_1(b, a)}{2} \left\{ \int_0^{1/2} [(1-t)^\alpha - t^\alpha] \right. \\ & \quad \cdot [|f'(a)| + t\eta_2(|f'(b)|, |f'(a)|)] dt \\ & \quad + \int_{1/2}^1 [t^\alpha - (1-t)^\alpha] \\ & \quad \cdot [|f'(a)| + t\eta_2(|f'(b)|, |f'(a)|)] dt \Big\} \\ & = \frac{\eta_1(b, a)}{2(\alpha + 1)} \left(1 - \frac{1}{2^\alpha} \right) [2|f'(a)| \\ & \quad + \eta_2(|f'(b)|, |f'(a)|)], \end{aligned} \quad (51)$$

where the last equality can be obtained after some calculations in corresponding integrals and utilizing them. $\square$

Corollary 21. If in Theorem 20 we consider

(i) $\eta_1(x, y) = x - y$ for all $x, y \in I$, then we obtain

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \\ & \leq \frac{b-a}{2(\alpha + 1)} \left(1 - \frac{1}{2^\alpha} \right) \\ & \quad \cdot [2|f'(a)| + \eta_2(|f'(b)|, |f'(a)|)]. \end{aligned} \quad (52)$$

(ii) $\eta_2(x, y) = x - y$ for all $x, y \in f(I)$, then we have Theorem 2.5 in [14].

(iii) conditions of (i) and (ii) together, then we recapture Theorem 3 in [12].

As the last result by using Theorem 2 in [15], we obtain a refinement of Hermite-Hadamard inequality in connection with fractional integrals related to the increasing (η_1, η_2) -convex functions.

Theorem 22 (see [15]). If f_1 and f_2 are positive increasing functions on $[0, 1]$, then

$$\int_0^1 f_1(x) dx \int_0^1 f_2(x) dx \leq \int_0^1 f_1(x) f_2(x) dx. \quad (53)$$

Also if f_1 and f_2 are positive decreasing functions on $[0, 1]$ and K is an upper bound for f_1 and f_2 , then $K - f_1$ and $K - f_2$ are positive increasing functions and we have

$$\begin{aligned} & \int_0^1 (K - f_1(x)) dx \int_0^1 (K - f_2(x)) dx \\ & \leq \int_0^1 (K - f_1(x)) (K - f_2(x)) dx, \end{aligned} \quad (54)$$

which gives again

$$\int_0^1 f_1(x) dx \int_0^1 f_2(x) dx \leq \int_0^1 f_1(x) f_2(x) dx. \quad (55)$$

Theorem 23. Let $I \subseteq \mathbb{R}$ be an invex set with respect to $\eta_1 : I \times I \rightarrow \mathbb{R}$. Also let $f : I \rightarrow \mathbb{R}$ be an increasing positive (η_1, η_2) -convex function. If, for each $a, b \in I$ with $\eta_1(b, a) > 0$, $f \in L^1[a, a + \eta_1(b, a)]$ and η_2 is integrable on $f(I) \times f(I)$, then, for $\alpha > 1$, the following inequalities hold:

$$\begin{aligned} & f\left(\frac{2a + \eta_1(b, a)}{2}\right) \leq \frac{1}{\eta_1(b, a)} \int_a^{a+\eta_1(b, a)} f(x) dx \\ & \leq \frac{\Gamma(\alpha + 1)}{2\eta_1^\alpha(b, a)} [J_{a^+}^\alpha f(a + \eta_1(b, a)) + J_{(a+\eta_1(b, a))^-}^\alpha f(a)] \\ & \leq f(a) + \frac{\eta_2(f(b), f(a))}{2}. \end{aligned} \quad (56)$$

Proof. From (η_1, η_2) -convexity of f , we have,

$$f(a + t\eta_1(b, a)) \leq f(a) + t\eta_2(f(b), f(a)), \quad (57)$$

and

$$\begin{aligned} & f(a + (1-t)\eta_1(b, a)) \\ & \leq f(a) + (1-t)\eta_2(f(b), f(a)). \end{aligned} \quad (58)$$

By adding these inequalities, we get

$$\begin{aligned} & f(a + t\eta_1(b, a)) + f(a + (1-t)\eta_1(b, a)) \\ & \leq 2f(a) + \eta_2(f(b), f(a)). \end{aligned} \quad (59)$$

Multiplying both sides of (59) by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0, 1]$ and using Theorem 22, we obtain

$$\begin{aligned} & \int_0^1 t^{\alpha-1} dt \cdot \int_0^1 f(a + t\eta_1(b, a)) dt + \int_0^1 t^{\alpha-1} dt \\ & \cdot \int_0^1 f(a + (1-t)\eta_1(b, a)) dt \\ & \leq \int_0^1 t^{\alpha-1} f(a + t\eta_1(b, a)) dt \\ & + \int_0^1 t^{\alpha-1} f(a + (1-t)\eta_1(b, a)) dt \\ & \leq [2f(a) + \eta_2(f(b), f(a))] \int_0^1 t^{\alpha-1} dt. \end{aligned} \quad (60)$$

Using the changes of variables $x = a + t\eta_1(b, a)$ and $x = a + (1-t)\eta_1(b, a)$, respectively, in above integrals, we have

$$\begin{aligned} & \frac{1}{\eta_1(b, a)} \int_0^1 t^{\alpha-1} dt \left[\int_a^{a+\eta_1(b, a)} f(x) dx \right. \\ & \left. + \int_a^{a+\eta_1(b, a)} f(2a + \eta_1(b, a) - x) dx \right] \\ & \leq \frac{\Gamma(\alpha)}{\eta_1^\alpha(b, a)} [J_{a^+}^\alpha f(a + \eta_1(b, a)) + J_{(a+\eta_1(b, a))^-}^\alpha f(a)] \\ & \leq [2f(a) + \eta_2(f(b), f(a))] \int_0^1 t^{\alpha-1} dt. \end{aligned} \quad (61)$$

Since

$$\int_0^1 t^{\alpha-1} dt = \frac{1}{\alpha}, \quad (62)$$

and

$$\begin{aligned} & \int_a^{a+\eta_1(b, a)} f(x) dx \\ & = \int_a^{a+\eta_1(b, a)} f(2a + \eta_1(b, a) - x) dx, \end{aligned} \quad (63)$$

we get

$$\begin{aligned} & \frac{1}{\eta_1(b, a)} \int_a^{a+\eta_1(b, a)} f(x) dx \\ & \leq \frac{\Gamma(\alpha+1)}{2\eta_1^\alpha(b, a)} [J_{a^+}^\alpha f(a + \eta_1(b, a)) + J_{(a+\eta_1(b, a))^-}^\alpha f(a)] \\ & \leq f(a) + \frac{\eta_2(f(b), f(a))}{2}. \end{aligned} \quad (64)$$

Furthermore, since f is (η_1, η_2) -convex, then

$$f\left(\frac{2a + \eta_1(b, a)}{2}\right) \leq \frac{1}{\eta_1(b, a)} \int_a^{a+\eta_1(b, a)} f(x) dx, \quad (65)$$

which completes the proof. $\square$

Corollary 24. Let $f : [a, b] \rightarrow \mathbb{R}$ be an increasing positive convex function. Then, for $\alpha > 1$, the following inequalities hold:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) & \leq \frac{1}{b-a} \int_a^b f(x) dx \\ & \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \\ & \leq \frac{f(a) + f(b)}{2}. \end{aligned} \quad (66)$$

4. Conclusions

The convexity of a function is the basis for many inequalities in mathematics. Note that, in new problems related to the convexity, generalized notions about convex functions are required to obtain applicable results. One of these generalizations is the notion of (η_1, η_2) -convex functions which can generalize many inequalities related to convex functions such as Hermite-Hadamard inequality, Fejér inequality, and trapezoid-type and mid-point-type inequalities.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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