Research Article

Determinantal Representations of Solutions and Hermitian Solutions to Some System of Two-Sided Quaternion Matrix Equations

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Within the framework of the theory of quaternion row-column determinants previously introduced by the author, we derive determinantal representations (analogs of Cramer's rule) of solutions and Hermitian solutions to the system of two-sided quaternion matrix equations $A_1 X A_1^* = C_1$ and $A_2 X A_2^* = C_2$. Since the Moore-Penrose inverse is a necessary tool to solve matrix equations, we use determinantal representations of the Moore-Penrose inverse previously obtained by the theory of row-column determinants.

1. Introduction

The study of matrix equations and systems of matrix equations is an active research topic in matrix theory and its applications. Research on the classical system of two-sided matrix equations

$$\begin{aligned} A_1 X B_1 &= C_1, \\ A_2 X B_2 &= C_2 \end{aligned} \quad (1)$$

over the complex field, a principle domain, and the quaternion skew field has been actively ongoing for many years. For instance, Mitra [1, 2] gave necessary and sufficient conditions of system (3) over the complex field and the expression for its general solution. Navarra et al. [3] derived a new necessary and sufficient condition for the existence and a new representation of the general solution to (3) over the complex field and used the results to give a simple representation. Özgüler et al. [4] gave solutions to (3) over a principle domain. Wang [5] got solvability conditions of system (3) over the quaternion skew field and represented its general solution in terms of generalized inverses.

Since quaternion matrices play an important role in quantum mechanics, signal processing, and control theory, research on quaternion matrix equations and systems of

quaternion matrix equations, their general solutions, especially Hermitian solutions, has been actively developing for more recent years (see, e.g., [6–26]).

Throughout the paper, we denote the real number field by $\mathbb{R}$, the set of all $m \times n$ matrices over the quaternion algebra

$$\begin{aligned} \mathbb{H} &= \{a_0 + a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} \mid \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 \\ &= -1, a_0, a_1, a_2, a_3 \in \mathbb{R}\} \end{aligned} \quad (2)$$

by $\mathbb{H}^{m \times n}$, and by $\mathbb{H}_r^{m \times n}$ its subset of matrices of a rank r . Let $M(n, \mathbb{H})$ be the ring of $n \times n$ quaternion matrices. For $A \in \mathbb{H}^{n \times m}$, the symbol A^* stands for the conjugate transpose (Hermitian adjoint) matrix of A . The matrix $A = (a_{ij}) \in \mathbb{H}^{n \times n}$ is Hermitian if $A^* = A$.

Motivated by the wide application of quaternion matrix equations and in order to improve the theoretical development of solutions and Hermitian solutions to quaternion matrix equations, we consider a special case of (3), more specifically,

$$\begin{aligned} A_1 X A_1^* &= C_1, \\ A_2 X A_2^* &= C_2. \end{aligned} \quad (3)$$

Generalized inverses are useful tools used to solve matrix equations. The definition of the Moore-Penrose inverse matrix has been extended to quaternion matrices as follows.

Definition 1. The Moore-Penrose inverse of $\mathbf{A} \in \mathbb{H}^{m \times n}$, denoted by $\mathbf{A}^\dagger$, is the unique matrix $\mathbf{A}^\dagger \in \mathbb{H}^{n \times m}$ satisfying the following four equations:

$$\begin{aligned} (1) \quad & \mathbf{A} \mathbf{A}^\dagger \mathbf{A} = \mathbf{A}, \\ (2) \quad & \mathbf{A}^\dagger \mathbf{A} \mathbf{A}^\dagger = \mathbf{A}^\dagger, \\ (3) \quad & (\mathbf{A} \mathbf{A}^\dagger)^* = \mathbf{A} \mathbf{A}^\dagger, \\ (4) \quad & (\mathbf{A}^\dagger \mathbf{A})^* = \mathbf{A}^\dagger \mathbf{A}. \end{aligned} \quad (4)$$

The main goal of this paper is to derive determinantal representations of solutions and Hermitian solutions to system (1) over the quaternion skew field using previously obtained determinantal representations of the Moore-Penrose inverse. Evidently, determinantal representation of a solution gives a direct method of its finding analogous to the classical Cramer's rule that has important theoretical and practical significance [27].

Through the noncommutativity of the quaternion algebra when difficulties arise already in determining the quaternion determinant, the problem of the determinantal representation of generalized inverses only now can be solved due to the theory of column-row determinants introduced in [28, 29]. Within the framework of the theory of column-row determinants, determinantal representations of various kinds of generalized inverses (generalized inverses) solutions of quaternion matrix equations have been derived by the author (see, e.g., [30–39]) and by other researchers (see, e.g., [40–43]).

The paper is organized as follows. In Section 2, we start with preliminary introduction of row-column determinants, determinantal representations of the Moore-Penrose inverse previously obtained within the framework of the theory of row-column determinants, and Cramer's rules for the two-sided matrix equation and of its special cases, left- and right-sided equations. We derive some simplified expressions of general and partial solutions to (3), a solvability criterion and expressions of general and partial solutions to system (1), and determinantal representations (analogs of Cramer's rule) of its solution and Hermitian solution. A numerical example to illustrate the main results is considered in Section 4. Finally, the conclusion is drawn in Section 5.

2. Preliminaries

For $\mathbf{A} \in \mathbb{H}^{n \times n}$, we define n row determinants and n column determinants. Suppose S_n is the symmetric group on the set $I_n = \{1, \dots, n\}$.

Definition 2 (see [28]). The i th row determinant of $\mathbf{A} = (a_{ij}) \in \mathbb{H}^{n \times n}$ is defined for all $i = 1, \dots, n$ by putting

$$\begin{aligned} \text{rdet}_i \mathbf{A} &= \sum_{\sigma \in S_n} (-1)^{n-r} (a_{i i_{k_1}} a_{i i_{k_1+1}} \cdots a_{i i_{k_t+1}}) \cdots (a_{i i_{k_r}} \cdots a_{i i_{k_r+l_r}}), \\ \sigma &= (i i_{k_1} i_{k_1+1} \cdots i_{k_1+l_1}) (i_{k_2} i_{k_2+1} \cdots i_{k_2+l_2}) \cdots (i_{k_r} i_{k_r+1} \cdots i_{k_r+l_r}), \end{aligned} \quad (5)$$

where $i_{k_2} < i_{k_3} < \cdots < i_{k_r}$ and $i_{k_t} < i_{k_t+s}$ for all $t = 2, \dots, r$ and $s = 1, \dots, l_t$.

Definition 3 (see [28]). The j th column determinant of $\mathbf{A} = (a_{ij}) \in \mathbb{H}^{n \times n}$ is defined for all $j = 1, \dots, n$ by putting

$$\begin{aligned} \text{cdet}_j \mathbf{A} &= \sum_{\tau \in S_n} (-1)^{n-r} (a_{j k_r j_{k_r+l_r}} \cdots a_{j k_r+1 j_{k_r}}) \cdots (a_{j j_{k_1+1}} \cdots a_{j k_1+1 j_{k_1}} a_{j k_1 j}), \\ \tau &= (j_{k_r+l_r} \cdots j_{k_r+1} j_{k_r}) \cdots (j_{k_2+l_2} \cdots j_{k_2+1} j_{k_2}) (j_{k_1+l_1} \cdots j_{k_1+1} j_{k_1} j), \end{aligned} \quad (6)$$

where $j_{k_2} < j_{k_3} < \cdots < j_{k_r}$ and $j_{k_t} < j_{k_t+s}$ for $t = 2, \dots, r$ and $s = 1, \dots, l_t$.

Since [28] for Hermitian $\mathbf{A}$ we have

$$\begin{aligned} \text{rdet}_1 \mathbf{A} = \cdots = \text{rdet}_n \mathbf{A} = \text{cdet}_1 \mathbf{A} = \cdots = \text{cdet}_n \mathbf{A} \\ \in \mathbb{R}, \end{aligned} \quad (7)$$

the determinant of a Hermitian matrix is defined by putting $\det \mathbf{A} := \text{rdet}_i \mathbf{A} = \text{cdet}_i \mathbf{A}$ for all $i = 1, \dots, n$.

The properties of row and column determinants are completely explored in [29]. We note the following that will be required below.

Lemma 4. Let $\mathbf{A} \in \mathbb{H}^{m \times n}$. Then

$$\begin{aligned} \text{cdet}_i \mathbf{A}^* &= \overline{\text{rdet}_i \mathbf{A}}, \\ \text{rdet}_i \mathbf{A}^* &= \overline{\text{cdet}_i \mathbf{A}}. \end{aligned} \quad (8)$$

We shall use the following notations. Let $\alpha := \{\alpha_1, \dots, \alpha_k\} \subseteq \{1, \dots, m\}$ and $\beta := \{\beta_1, \dots, \beta_k\} \subseteq \{1, \dots, n\}$ be subsets of the order $1 \leq k \leq \min\{m, n\}$. Let $\mathbf{A}_\beta^\alpha$ be a submatrix of $\mathbf{A}$ whose rows are indexed by α and columns indexed by β . Similarly, let $\mathbf{A}_\alpha^\alpha$ be a principal submatrix of $\mathbf{A}$ whose rows and columns are indexed by α . If $\mathbf{A} \in M(n, \mathbb{H})$ is Hermitian, then $|\mathbf{A}|_\alpha^\alpha$ is the corresponding principal minor of $\det \mathbf{A}$. For $1 \leq k \leq n$, the collection of strictly increasing sequences of k integers chosen from $\{1, \dots, n\}$ is denoted by $L_{k,n} := \{\alpha : \alpha = (\alpha_1, \dots, \alpha_k), 1 \leq \alpha_1 < \cdots < \alpha_k \leq n\}$. For fixed $i \in \alpha$ and $j \in \beta$, let $L_{r,m}\{i\} := \{\alpha : \alpha \in L_{r,m}, i \in \alpha\}$, $J_{r,n}\{j\} := \{\beta : \beta \in L_{r,n}, j \in \beta\}$.

Let $\mathbf{a}_j$ be the j th column and $\mathbf{a}_i$ be the i th row of $\mathbf{A}$. Suppose $\mathbf{A}_{\cdot j}(\mathbf{b})$ denotes the matrix obtained from $\mathbf{A}$ by replacing its j th column with column $\mathbf{b}$, and $\mathbf{A}_i(\mathbf{b})$ denotes the matrix obtained from $\mathbf{A}$ by replacing its i th row with the row $\mathbf{b}$. Denote by $\mathbf{a}_{\cdot j}^*$ and $\mathbf{a}_i^*$ the j th column and the i th row of $\mathbf{A}^*$, respectively.

Theorem 5 (see [30]). If $\mathbf{A} \in \mathbb{H}^{m \times n}$, then the Moore-Penrose inverse $\mathbf{A}^\dagger = (a_{ij}^\dagger) \in \mathbb{H}^{n \times m}$ has the following determinantal representations,

$$a_{ij}^\dagger = \frac{\sum_{\beta \in J_{r,n}\{i\}} \text{cdet}_i((\mathbf{A}^* \mathbf{A})_{\cdot i}(\mathbf{a}_{\cdot j}^*))_\beta}{\sum_{\beta \in J_{r,n}} |\mathbf{A}^* \mathbf{A}|_\beta^\beta}, \quad (9)$$

and

$$a_{ij}^\dagger = \frac{\sum_{\alpha \in I_{r,m}\{j\}} \text{rdet}_j((\mathbf{A}\mathbf{A}^*)_j.(\mathbf{a}_i^\dagger))_\alpha^\alpha}{\sum_{\alpha \in I_{r,m}} |\mathbf{A}\mathbf{A}^*|_\alpha^\alpha}. \quad (10)$$

Remark 6. For an arbitrary full-rank matrix $\mathbf{A} \in \mathbb{H}_r^{m \times n}$, we put

$$\begin{aligned} \text{cdet}_i((\mathbf{A}^*\mathbf{A})_i.(\mathbf{d}_j)) &= \sum_{\beta \in J_{n,n}\{i\}} \text{cdet}_i((\mathbf{A}^*\mathbf{A})_i.(\mathbf{d}_j))_\beta^\beta, \\ \det(\mathbf{A}^*\mathbf{A}) &= \sum_{\beta \in J_{n,n}} |\mathbf{A}^*\mathbf{A}|_\beta^\beta \quad \text{when } r = n, \\ \text{rdet}_j((\mathbf{A}\mathbf{A}^*)_j.(\mathbf{d}_i)) &= \sum_{\alpha \in I_{m,m}\{j\}} \text{rdet}_j((\mathbf{A}\mathbf{A}^*)_j.(\mathbf{d}_i))_\alpha^\alpha, \\ \det(\mathbf{A}\mathbf{A}^*) &= \sum_{\alpha \in I_{m,m}} |\mathbf{A}\mathbf{A}^*|_\alpha^\alpha \quad \text{when } r = m, \end{aligned} \quad (11)$$

where a column vector $\mathbf{d}_j$ and a row vector $\mathbf{d}_i$ have appropriate sizes.

Remark 7. First note that $(\mathbf{A}^*)^\dagger = (\mathbf{A}^\dagger)^*$. Because of symbol equivalence, we shall use the denotation $\mathbf{A}^{\dagger,*} := (\mathbf{A}^*)^\dagger$ as well. So, by Lemma 4, for the Hermitian adjoint matrix $\mathbf{A}^* \in \mathbb{H}_r^{n \times m}$ determinantal representations of its Moore-Penrose inverse $(\mathbf{A}^*)^\dagger = ((a_{ij}^*)^\dagger) \in \mathbb{H}^{m \times n}$ are

$$(a_{ij}^*)^\dagger = \overline{(a_{ji})^\dagger} = \frac{\sum_{\alpha \in I_{r,n}\{j\}} \text{rdet}_j((\mathbf{A}^*\mathbf{A})_j.(\mathbf{a}_i))_\alpha^\alpha}{\sum_{\beta \in I_{r,n}} |\mathbf{A}^*\mathbf{A}|_\alpha^\alpha}, \quad (12)$$

and

$$(a_{ij}^*)^\dagger = \frac{\sum_{\beta \in J_{r,m}\{i\}} \text{cdet}_i((\mathbf{A}\mathbf{A}^*)_i.(\mathbf{a}_j))_\beta^\beta}{\sum_{\beta \in J_{r,m}} |\mathbf{A}\mathbf{A}^*|_\beta^\beta}. \quad (13)$$

Corollary 8. If $\mathbf{A} \in \mathbb{H}_r^{m \times n}$, then the projection matrix $\mathbf{A}^\dagger \mathbf{A} = \mathbf{P}_A = (p_{ij})_{n \times n}$ has the determinantal representation

$$p_{ij} = \frac{\sum_{\beta \in J_{r,n}\{i\}} \text{cdet}_i((\mathbf{A}^*\mathbf{A})_i.(\mathbf{a}_j))_\beta^\beta}{\sum_{\beta \in J_{r,n}} |\mathbf{A}^*\mathbf{A}|_\beta^\beta}, \quad (14)$$

where $\mathbf{a}_j$ is the j th column of $\mathbf{A}^* \mathbf{A} \in \mathbb{H}^{n \times n}$.

Corollary 9. If $\mathbf{A} \in \mathbb{H}_r^{m \times n}$, then the projection matrix $\mathbf{A}\mathbf{A}^\dagger = \mathbf{Q}_A = (q_{ij})_{m \times m}$ has the determinantal representation

$$q_{ij} = \frac{\sum_{\alpha \in I_{r,m}\{j\}} \text{rdet}_j((\mathbf{A}\mathbf{A}^*)_j.(\mathbf{a}_i))_\alpha^\alpha}{\sum_{\alpha \in I_{r,m}} |\mathbf{A}\mathbf{A}^*|_\alpha^\alpha}, \quad (15)$$

where $\mathbf{a}_i$ is the i th row of $\mathbf{A}\mathbf{A}^* \in \mathbb{H}^{m \times m}$.

The orthogonal projectors $\mathbf{L}_A := \mathbf{I} - \mathbf{A}^\dagger \mathbf{A}$ and $\mathbf{R}_A := \mathbf{I} - \mathbf{A}\mathbf{A}^\dagger$ induced by $\mathbf{A}$ will be used below.

Theorem 10 (see [44]). Let $\mathbf{A} \in \mathbb{H}^{m \times n}$, $\mathbf{B} \in \mathbb{H}^{r \times s}$, $\mathbf{C} \in \mathbb{H}^{m \times s}$ be known and $\mathbf{X} \in \mathbb{H}^{n \times r}$ be unknown. Then the matrix equation

$$\mathbf{A}\mathbf{X}\mathbf{B} = \mathbf{C} \quad (16)$$

is consistent if and only if $\mathbf{A}\mathbf{A}^\dagger \mathbf{C}\mathbf{B}^\dagger \mathbf{B} = \mathbf{C}$. In this case, its general solution can be expressed as

$$\mathbf{X} = \mathbf{A}^\dagger \mathbf{C}\mathbf{B}^\dagger + \mathbf{L}_A \mathbf{V} + \mathbf{W}\mathbf{R}_B, \quad (17)$$

where $\mathbf{V}$, $\mathbf{W}$ are arbitrary matrices over $\mathbb{H}$ with appropriate dimensions.

Theorem 11 (see [31]). Let $\mathbf{A} \in \mathbb{H}_{r_1}^{m \times n}$, $\mathbf{B} \in \mathbb{H}_{r_2}^{r \times s}$. Then the partial solution $\mathbf{X}^0 = \mathbf{A}^\dagger \mathbf{C}\mathbf{B}^\dagger = (x_{ij}^0) \in \mathbb{H}^{n \times r}$ to (16) has determinantal representations,

$$x_{ij}^0 = \frac{\sum_{\beta \in J_{r_1,n}\{i\}} \text{cdet}_i((\mathbf{A}^*\mathbf{A})_i.(\mathbf{d}_j^B))_\beta^\beta}{\sum_{\beta \in J_{r_1,n}} |\mathbf{A}^*\mathbf{A}|_\beta^\beta \sum_{\alpha \in I_{r_2,r}} |\mathbf{B}\mathbf{B}^*|_\alpha^\alpha}, \quad (18)$$

or

$$x_{ij}^0 = \frac{\sum_{\alpha \in I_{r_2,r}\{j\}} \text{rdet}_j((\mathbf{B}\mathbf{B}^*)_j.(\mathbf{d}_i^A))_\alpha^\alpha}{\sum_{\beta \in J_{r_1,n}} |\mathbf{A}^*\mathbf{A}|_\beta^\beta \sum_{\alpha \in I_{r_2,r}} |\mathbf{B}\mathbf{B}^*|_\alpha^\alpha}, \quad (19)$$

where

$$\begin{aligned} \mathbf{d}_j^B &= \left[\sum_{\alpha \in I_{r_2,r}\{j\}} \text{rdet}_j((\mathbf{B}\mathbf{B}^*)_j.(\tilde{\mathbf{c}}_k))_\alpha^\alpha \right] \in \mathbb{H}^{n \times 1}, \\ k &= 1, \dots, n, \end{aligned} \quad (20)$$

$$\begin{aligned} \mathbf{d}_i^A &= \left[\sum_{\beta \in J_{r_1,n}\{i\}} \text{cdet}_i((\mathbf{A}^*\mathbf{A})_i.(\tilde{\mathbf{c}}_l))_\beta^\beta \right] \in \mathbb{H}^{1 \times r}, \\ l &= 1, \dots, r, \end{aligned}$$

are the column vector and the row vector, respectively. $\tilde{\mathbf{c}}_i$ and $\tilde{\mathbf{c}}_j$ are the i th row and the j th column of $\tilde{\mathbf{C}} = \mathbf{A}^* \mathbf{C}\mathbf{B}^*$.

Corollary 12. Let $\mathbf{A} \in \mathbb{H}_k^{m \times n}$, $\mathbf{C} \in \mathbb{H}^{m \times s}$ be known and $\mathbf{X} \in \mathbb{H}^{n \times s}$ be unknown. Then the matrix equation $\mathbf{A}\mathbf{X} = \mathbf{C}$ is consistent if and only if $\mathbf{A}\mathbf{A}^\dagger \mathbf{C} = \mathbf{C}$. In this case, its general solution can be expressed as $\mathbf{X} = \mathbf{A}^\dagger \mathbf{C} + \mathbf{L}_A \mathbf{V}$, where $\mathbf{V}$ is an arbitrary matrix over $\mathbb{H}$ with appropriate dimensions. The partial solution $\mathbf{X}^0 = \mathbf{A}^\dagger \mathbf{C}$ has the following determinantal representation,

$$x_{ij}^0 = \frac{\sum_{\beta \in J_{k,n}\{i\}} \text{cdet}_i((\mathbf{A}^*\mathbf{A})_i.(\hat{\mathbf{c}}_j))_\beta^\beta}{\sum_{\beta \in J_{k,n}} |\mathbf{A}^*\mathbf{A}|_\beta^\beta}. \quad (21)$$

where $\hat{\mathbf{c}}_j$ is the j th column of $\hat{\mathbf{C}} = \mathbf{A}^* \mathbf{C}$.

Corollary 13. Let $\mathbf{B} \in \mathbb{H}_k^{r \times s}$, $\mathbf{C} \in \mathbb{H}^{n \times s}$ be given and $\mathbf{X} \in \mathbb{H}^{n \times r}$ be unknown. Then the equation $\mathbf{X}\mathbf{B} = \mathbf{C}$ is solvable if and only

if $C = CB^\dagger B$ and its general solution is $X = CB^\dagger + WR_B$, where W is any matrix with conformable dimension. Moreover, its partial solution $X = CB^\dagger$ has the determinantal representation,

$$x_{ij} = \frac{\sum_{\alpha \in I_{k,r}\{j\}} \text{rdet}_j((BB^*)_{j.}(\hat{c}_i.)^\alpha)}{\sum_{\alpha \in I_{k,r}} |BB^*|_\alpha^\alpha}. \quad (22)$$

where $\hat{c}_i$ is the i th row of $\hat{C} = CB^*$.

3. Cramer's Rules for the Solution and Hermitian Solution to System (3)

First, consider the general system (1).

Lemma 14 (see [5]). Let $A_1 \in \mathbb{H}^{m \times n}$, $B_1 \in \mathbb{H}^{r \times s}$, $C_1 \in \mathbb{H}^{m \times s}$, $A_2 \in \mathbb{H}^{k \times n}$, $B_2 \in \mathbb{H}^{r \times p}$, $C_2 \in \mathbb{H}^{k \times p}$ be given and $X \in \mathbb{H}^{n \times r}$ is to be determined. Put $H = A_2 L_{A_1}$, $N = R_{B_1} B_2$, $T = R_H A_2$, $F = B_2 L_N$. Then system (1) is consistent if and only if

$$A_i A_i^\dagger C_i B_i^\dagger B_i = C_i, \quad i = 1, 2; \quad (23)$$

$$T [A_2^\dagger C_2 B_2^\dagger - A_1^\dagger C_1 B_1^\dagger] F = 0. \quad (24)$$

In that case, the general solution to (1) can be expressed as the following:

$$\begin{aligned} X &= A_1^\dagger C_1 B_1^\dagger \\ &+ L_{A_1} H^\dagger A_2 L_T (A_2^\dagger C_2 B_2^\dagger - A_1^\dagger C_1 B_1^\dagger) B_2 B_2^\dagger \\ &+ T^\dagger T (A_2^\dagger C_2 B_2^\dagger - A_1^\dagger C_1 B_1^\dagger) B_2 N^\dagger R_{B_1} \\ &+ L_{A_1} (Z - H^\dagger H Z B_2 B_2^\dagger) - L_{A_1} H^\dagger A_2 L_T W N B_2^\dagger \\ &+ (W - T^\dagger T W N N^\dagger) R_{B_1}, \end{aligned} \quad (25)$$

where Z and W are arbitrary matrices over $\mathbb{H}$ with compatible dimensions.

Some simplification of (25) can be derived due to the quaternionic analogue of the following proposition.

Lemma 15 (see [45]). If $A \in \mathbb{H}^{n \times n}$ is Hermitian and idempotent, then the following equation holds for any matrix $B \in \mathbb{H}^{m \times n}$,

$$A(BA)^\dagger = (BA)^\dagger. \quad (26)$$

It is evident that if $A \in \mathbb{H}^{n \times n}$ is Hermitian and idempotent, then the following equation is true as well:

$$(AB)^\dagger A = (AB)^\dagger. \quad (27)$$

Since L_{A_1} , R_{B_1} , and R_H are projectors, then by (26) and (27), we have, respectively,

$$\begin{aligned} L_{A_1} H^\dagger &= L_{A_1} (A_2 L_{A_1})^\dagger = (A_2 L_{A_1})^\dagger = H^\dagger, \\ N^\dagger R_{B_1} &= (R_{B_1} B_2)^\dagger R_{B_1} = (R_{B_1} B_2)^\dagger = N^\dagger, \\ T^\dagger T &= (R_H A_2)^\dagger R_H A_2 = (R_H A_2)^\dagger A_2 = T^\dagger A_2, \\ L_T &= I - T^\dagger T = I - T^\dagger A_2. \end{aligned} \quad (28)$$

Using (28) and (23), we obtain the following expressions of (25):

$$\begin{aligned} X &= A_1^\dagger C_1 B_1^\dagger \\ &+ H^\dagger A_2 (I - T^\dagger A_2) (A_2^\dagger C_2 B_2^\dagger - A_1^\dagger C_1 B_1^\dagger) B_2 B_2^\dagger \\ &+ T^\dagger A_2 (A_2^\dagger C_2 B_2^\dagger - A_1^\dagger C_1 B_1^\dagger) B_2 N^\dagger \\ &+ L_{A_1} (Z - H^\dagger H Z B_2 B_2^\dagger) - H^\dagger A_2 L_T W N B_2^\dagger \\ &+ (W - T^\dagger T W N N^\dagger) R_{B_1} \\ &= A_1^\dagger C_1 B_1^\dagger + H^\dagger C_2 B_2^\dagger \\ &+ H^\dagger (A_2 T^\dagger - I) A_2 A_1^\dagger C_1 B_1^\dagger Q_{B_2} \\ &- H^\dagger A_2 T^\dagger C_2 B_2^\dagger + T^\dagger C_2 N^\dagger \\ &- T^\dagger A_2 A_1^\dagger C_1 B_1^\dagger B_2 N^\dagger + L_{A_1} (Z - H^\dagger H Z B_2 B_2^\dagger) \\ &- H^\dagger A_2 L_T W N B_2^\dagger + (W - T^\dagger T W N N^\dagger) R_{B_1}. \end{aligned} \quad (29)$$

By putting Z , W as zero-matrices in (29), we obtain the following partial solution of (25):

$$\begin{aligned} X &= A_1^\dagger C_1 B_1^\dagger + H^\dagger C_2 B_2^\dagger + T^\dagger C_2 N^\dagger \\ &+ H^\dagger A_2 T^\dagger A_2 A_1^\dagger C_1 B_1^\dagger Q_{B_2} - H^\dagger A_2 A_1^\dagger C_1 B_1^\dagger Q_{B_2} \\ &- H^\dagger A_2 T^\dagger C_2 B_2^\dagger - T^\dagger A_2 A_1^\dagger C_1 B_1^\dagger B_2 N^\dagger. \end{aligned} \quad (30)$$

Now consider system (1). Since

$$P_{A_i^*} = (A_i^*)^\dagger A_i^* = (A_i A_i^\dagger)^* = Q_{A_i}, \quad (31)$$

so $Q_{A_i^*} = P_{A_i}$, $L_{A_i^*} = I - P_{A_i^*} = I - Q_{A_i} = R_{A_i}$, and $R_{A_i^*} = L_{A_i}$ for $i = 1, 2$. Moreover, substituting $B_i = A_i^*$, we have $N = R_{A_1^*} A_2^* = (A_2 L_{A_1})^* = H^*$; similarly, $F = A_2^* L_{H^*} = (R_H A_2)^* = T^*$. Due to the above, we obtain the following analog of Lemma 14.

Lemma 16. Let $A_1 \in \mathbb{H}^{m \times n}$, $A_2 \in \mathbb{H}^{k \times n}$, $C_1 \in \mathbb{H}^{m \times m}$, $C_2 \in \mathbb{H}^{k \times k}$ be given and $X \in \mathbb{H}^{n \times n}$ is to be determined. Then system (3) is consistent if and only if

$$\begin{aligned} Q_{A_i} C_i Q_{A_i} &= C_i, \quad i = 1, 2; \\ T [A_2^\dagger C_2 A_2^{\dagger,*} - A_1^\dagger C_1 A_1^{\dagger,*}] T^* &= 0. \end{aligned} \quad (32)$$

In that case, the general solution to (3) can be expressed as follows:

$$\begin{aligned} \mathbf{X} = & \mathbf{A}_1^\dagger \mathbf{C}_1 (\mathbf{A}_1^*)^\dagger + \mathbf{H}^\dagger \mathbf{C}_2 (\mathbf{A}_2^*)^\dagger \\ & + \mathbf{H}^\dagger (\mathbf{A}_2 \mathbf{T}^\dagger - \mathbf{I}) \mathbf{A}_2 \mathbf{A}_1^\dagger \mathbf{C}_1 (\mathbf{A}_1^*)^\dagger \mathbf{P}_{A_2} \\ & - \mathbf{H}^\dagger \mathbf{A}_2 \mathbf{T}^\dagger \mathbf{C}_2 (\mathbf{A}_2^*)^\dagger + \mathbf{T}^\dagger \mathbf{C}_2 (\mathbf{H}^*)^\dagger \\ & - \mathbf{T}^\dagger \mathbf{A}_2 \mathbf{A}_1^\dagger \mathbf{C}_1 (\mathbf{A}_1^*)^\dagger \mathbf{A}_2^* (\mathbf{H}^*)^\dagger \\ & + \mathbf{L}_{A_1} (\mathbf{Z} - \mathbf{H}^\dagger \mathbf{H} \mathbf{Z} \mathbf{A}_2^* (\mathbf{A}_2^*)^\dagger) \\ & - \mathbf{H}^\dagger \mathbf{A}_2 \mathbf{L}_T \mathbf{W} \mathbf{H}^* (\mathbf{A}_2^*)^\dagger \\ & + (\mathbf{W} - \mathbf{T}^\dagger \mathbf{T} \mathbf{W} \mathbf{H}^* (\mathbf{H}^*)^\dagger) \mathbf{L}_{A_1}. \end{aligned} \quad (33)$$

$\mathbf{Z}$ and $\mathbf{W}$ are arbitrary matrices over $\mathbb{H}$ with compatible dimensions.

By putting $\mathbf{Z}$, $\mathbf{W}$ as zero-matrices, the partial solution of (3) is

$$\begin{aligned} \mathbf{X} = & \mathbf{A}_1^\dagger \mathbf{C}_1 (\mathbf{A}_1^*)^\dagger + \mathbf{H}^\dagger \mathbf{C}_2 (\mathbf{A}_2^*)^\dagger + \mathbf{T}^\dagger \mathbf{C}_2 (\mathbf{H}^*)^\dagger \\ & + \mathbf{H}^\dagger \mathbf{A}_2 \mathbf{T}^\dagger \mathbf{A}_2 \mathbf{A}_1^\dagger \mathbf{C}_1 (\mathbf{A}_1^*)^\dagger \mathbf{P}_{A_2} \end{aligned}$$

$$\begin{aligned} & - \mathbf{H}^\dagger \mathbf{A}_2 \mathbf{A}_1^\dagger \mathbf{C}_1 (\mathbf{A}_1^*)^\dagger \mathbf{P}_{A_2} - \mathbf{H}^\dagger \mathbf{A}_2 \mathbf{T}^\dagger \mathbf{C}_2 (\mathbf{A}_2^*)^\dagger \\ & - \mathbf{T}^\dagger \mathbf{A}_2 \mathbf{A}_1^\dagger \mathbf{C}_1 (\mathbf{A}_1^*)^\dagger \mathbf{A}_2^* (\mathbf{H}^*)^\dagger. \end{aligned} \quad (34)$$

Further we give determinantal representations of (34).

Let $\mathbf{A}_1 \in \mathbb{H}_{r_1}^{m \times n}$, $\mathbf{A}_2 \in \mathbb{H}_{r_2}^{k \times n}$, $\mathbf{C}_1 = (c_{ij}^{(1)}) \in \mathbb{H}^{m \times m}$, $\mathbf{C}_2 = (c_{ij}^{(2)}) \in \mathbb{H}^{k \times k}$, $\text{rank } \mathbf{H} = r_3$, and $\text{rank } \mathbf{T} = r_4$. There exist $\mathbf{A}_1^\dagger = (a_{ij}^{(1),\dagger}) \in \mathbb{H}^{n \times m}$, $(\mathbf{A}_1^*)^\dagger = (a_{ij}^{(1),*,\dagger}) \in \mathbb{H}^{m \times n}$, $\mathbf{A}_2^\dagger = (a_{ij}^{(2),\dagger}) \in \mathbb{H}^{n \times k}$, $(\mathbf{A}_2^*)^\dagger = (a_{ij}^{(2),*,\dagger}) \in \mathbb{H}^{k \times n}$, $\mathbf{H}^\dagger = (h_{ij}^\dagger) \in \mathbb{H}^{n \times k}$, and $\mathbf{T}^\dagger = (t_{ij}^\dagger) \in \mathbb{H}^{n \times k}$.

Consider each term of (34) separately.

(i) Denote $\mathbf{C}_{11} := \mathbf{A}_1^* \mathbf{C}_1 \mathbf{A}_1$. For the first term of (34) $\mathbf{X}_1 = \mathbf{A}_1^\dagger \mathbf{C}_1 (\mathbf{A}_1^*)^\dagger = (x_{ij}^{(1)})$, we have

$$x_{ij}^{(1)} = \sum_{l=1}^m \sum_{t=1}^m a_{il}^{(1),\dagger} c_{lt}^{(1)} a_{tj}^{(1),*,\dagger}. \quad (35)$$

By using determinantal representations (9) and (13) of the Moore-Penrose inverses $\mathbf{A}_1^\dagger$ and $(\mathbf{A}_1^*)^\dagger$, respectively, we obtain

$$x_{ij}^{(1)} = \frac{\sum_{l=1}^m \sum_{t=1}^m \sum_{\beta \in J_{r_1,n}\{i\}} \text{cdet}_i((\mathbf{A}_1^* \mathbf{A}_1)_{.i} (\mathbf{a}_l^{(1),*}))_\beta^\beta c_{lt}^{(1)} \sum_{\alpha \in I_{r_1,n}\{j\}} \text{rdet}_j((\mathbf{A}_1^* \mathbf{A}_1)_{.j} (\mathbf{a}_t^{(1)}))_\alpha^\alpha}{\sum_{\alpha \in I_{r_1,n}} |\mathbf{A}_1^* \mathbf{A}_1|_\alpha^\alpha \sum_{\beta \in J_{r_1,n}} |\mathbf{A}_1^* \mathbf{A}_1|_\beta^\beta} \quad (36)$$

Suppose $\mathbf{e}_l$ and $\mathbf{e}_l$ are the unit row vector and the unit column vector, respectively, such that all their components

are 0, except the l th components, which are 1. Since $\sum_{l=1}^m \sum_{t=1}^m a_{fl}^{(1),*} c_{lt}^{(1)} a_{ts}^{(1)} = c_{fs}^{(11)}$, then

$$x_{ij}^{(1)} = \frac{\sum_{f=1}^n \sum_{s=1}^n \sum_{\beta \in J_{r_1,n}\{i\}} \text{cdet}_i((\mathbf{A}_1^* \mathbf{A}_1)_{.i} (\mathbf{e}_f))_\beta^\beta c_{fs}^{(11)} \sum_{\alpha \in I_{r_1,n}\{j\}} \text{rdet}_j((\mathbf{A}_1^* \mathbf{A}_1)_{.j} (\mathbf{e}_s))_\alpha^\alpha}{\sum_{\alpha \in I_{r_1,n}} |\mathbf{A}_1^* \mathbf{A}_1|_\alpha^\alpha \sum_{\beta \in J_{r_1,n}} |\mathbf{A}_1^* \mathbf{A}_1|_\beta^\beta} \quad (37)$$

If we denote by

$$\begin{aligned} v_{is}^{(1)} &:= \sum_{f=1}^n \sum_{\beta \in J_{r_1,n}\{i\}} \text{cdet}_i((\mathbf{A}_1^* \mathbf{A}_1)_{.i} (\mathbf{e}_f))_\beta^\beta c_{fs}^{(11)} \\ &= \sum_{\beta \in J_{r_1,n}\{i\}} \text{cdet}_i((\mathbf{A}_1^* \mathbf{A}_1)_{.i} (\mathbf{c}_s^{(11)}))_\beta^\beta \end{aligned} \quad (38)$$

the s th component of a row vector $\mathbf{v}_i^{(1)} = [v_{i1}^{(1)}, \dots, v_{in}^{(1)}]$, then

$$\begin{aligned} & \sum_{s=1}^m v_{is}^{(1)} \sum_{\alpha \in I_{r_1,n}\{j\}} \text{rdet}_j((\mathbf{A}_1^* \mathbf{A}_1)_{.j} (\mathbf{e}_s))_\alpha^\alpha \\ &= \sum_{\alpha \in I_{r_1,n}\{j\}} \text{rdet}_j((\mathbf{A}_1^* \mathbf{A}_1)_{.j} (\mathbf{v}_i^{(1)}))_\alpha^\alpha. \end{aligned} \quad (39)$$

Further, it is evident that $\sum_{\beta \in J_{r_1,n}} |\mathbf{A}_1^* \mathbf{A}_1|_\beta^\beta = \sum_{\alpha \in I_{r_1,n}} |\mathbf{A}_1^* \mathbf{A}_1|_\alpha^\alpha$, so the first term of (34) has the determinantal representation

$$x_{ij}^{(1)} = \frac{\sum_{\alpha \in I_{r_1,n}\{j\}} \text{rdet}_j((\mathbf{A}_1^* \mathbf{A}_1)_{.j} (\mathbf{v}_i^{(1)}))_\alpha^\alpha}{\left(\sum_{\alpha \in I_{r_1,n}} |\mathbf{A}_1^* \mathbf{A}_1|_\alpha^\alpha \right)^2}, \quad (40)$$

where

$$\mathbf{v}_i^{(1)} = \left[\sum_{\beta \in J_{r_1,n}\{i\}} \text{cdet}_i((\mathbf{A}_1^* \mathbf{A}_1)_{.i} (\mathbf{c}_s^{(11)}))_\beta^\beta \right] \in \mathbb{H}^{1 \times n}, \quad (41)$$

$s = 1, \dots, n.$

If we denote by

$$\begin{aligned} v_{fj}^{(2)} &:= \sum_{s=1}^n c_{fs}^{(11)} \sum_{\alpha \in I_{r_1,n}\{j\}} \text{rdet}_j((\mathbf{A}_1^* \mathbf{A}_1)_j (\mathbf{e}_s))_\alpha^\alpha \\ &= \sum_{\alpha \in I_{r_1,n}\{j\}} \text{rdet}_j((\mathbf{A}_1^* \mathbf{A}_1)_j (\mathbf{c}_f^{(11)}))_\alpha^\alpha \end{aligned} \quad (42)$$

the f th component of a column vector $\mathbf{v}_j^{(2)} = [v_{1j}^{(2)}, \dots, v_{nj}^{(2)}]$, then

$$\begin{aligned} \sum_{f=1}^n \sum_{\beta \in I_{r_1,n}\{i\}} \text{cdet}_i((\mathbf{A}_1^* \mathbf{A}_1)_i (\mathbf{e}_f))_\beta^\beta v_{fj}^{(2)} \\ = \sum_{\beta \in I_{r_1,n}\{i\}} \text{cdet}_i((\mathbf{A}_1^* \mathbf{A}_1)_i (\mathbf{v}_j^{(2)}))_\beta^\beta. \end{aligned} \quad (43)$$

So, another determinantal representation of the first term of (34) is

$$x_{ij}^{(1)} = \frac{\sum_{\beta \in I_{r_1,n}\{i\}} \text{cdet}_i((\mathbf{A}_1^* \mathbf{A}_1)_i (\mathbf{v}_j^{(2)}))_\beta^\beta}{\left(\sum_{\beta \in I_{r_1,n}} |\mathbf{A}_1^* \mathbf{A}_1|_\beta^\beta\right)^2}, \quad (44)$$

where

$$\mathbf{v}_j^{(2)} = \left[\sum_{\alpha \in I_{r_1,n}\{j\}} \text{rdet}_j((\mathbf{A}_1^* \mathbf{A}_1)_j (\mathbf{c}_f^{(11)}))_\alpha^\alpha \right] \in \mathbb{H}^{n \times 1}, \quad (45)$$

$f = 1, \dots, n,$

are the column vector.

(ii) Similarly above, for the second term $\mathbf{X}_2 \mathbf{H}^\dagger \mathbf{C}_2 (\mathbf{A}_2^*)^\dagger = (\mathbf{x}_{ij}^{(2)})$ of (34), we have

$$x_{ij}^{(2)} = \frac{\sum_{\beta \in I_{r_3,n}\{i\}} \text{cdet}_i((\mathbf{H}^* \mathbf{H})_i (\mathbf{d}_j^{A_2}))_\beta^\beta}{\sum_{\beta \in I_{r_3,n}} |\mathbf{H}^* \mathbf{H}|_\beta^\beta \sum_{\alpha \in I_{r_2,n}} |\mathbf{A}_2^* \mathbf{A}_2|_\alpha^\alpha}, \quad (46)$$

or

$$x_{ij}^{(2)} = \frac{\sum_{\alpha \in I_{r_2,n}\{j\}} \text{rdet}_j((\mathbf{A}_2^* \mathbf{A}_2)_j (\mathbf{d}_i^H))_\alpha^\alpha}{\sum_{\beta \in I_{r_3,n}} |\mathbf{H}^* \mathbf{H}|_\beta^\beta \sum_{\alpha \in I_{r_2,n}} |\mathbf{A}_2^* \mathbf{A}_2|_\alpha^\alpha}, \quad (47)$$

where

$$\begin{aligned} \mathbf{d}_j^{A_2} &= \left[\sum_{\alpha \in I_{r_2,n}\{j\}} \text{rdet}_j((\mathbf{A}_2^* \mathbf{A}_2)_j (\mathbf{c}_q^{(21)}))_\alpha^\alpha \right] \in \mathbb{H}^{n \times 1}, \\ &q = 1, \dots, n, \\ \mathbf{d}_i^H &= \left[\sum_{\beta \in I_{r_3,n}\{i\}} \text{cdet}_i((\mathbf{H}^* \mathbf{H})_i (\mathbf{c}_l^{(21)}))_\beta^\beta \right] \in \mathbb{H}^{1 \times n}, \\ &l = 1, \dots, n, \end{aligned} \quad (48)$$

are the column vector and the row vector, respectively. $\mathbf{c}_q^{(21)}$ and $\mathbf{c}_l^{(21)}$ are the q th row and the l th column of $\mathbf{C}_{21} = \mathbf{H}^* \mathbf{C}_2 \mathbf{A}_2$. Note that $\mathbf{H}^* \mathbf{H} = (\mathbf{A}_2 \mathbf{L}_{A_1})^* \mathbf{A}_2 \mathbf{L}_{A_1} = \mathbf{L}_{A_1}^* \mathbf{A}_2^* \mathbf{A}_2 \mathbf{L}_{A_1}$.

(iii) The third term of (34) can be obtained similarly as well. So,

$$x_{ij}^{(3)} = \frac{\sum_{\beta \in I_{r_4,n}\{i\}} \text{cdet}_i((\mathbf{T}^* \mathbf{T})_i (\mathbf{d}_j^H))_\beta^\beta}{\sum_{\beta \in I_{r_4,n}} |\mathbf{T}^* \mathbf{T}|_\beta^\beta \sum_{\alpha \in I_{r_3,n}} |\mathbf{H}^* \mathbf{H}|_\alpha^\alpha}, \quad (49)$$

or

$$x_{ij}^{(3)} = \frac{\sum_{\alpha \in I_{r_3,n}\{j\}} \text{rdet}_j((\mathbf{H}^* \mathbf{H})_j (\mathbf{d}_i^T))_\alpha^\alpha}{\sum_{\beta \in I_{r_4,n}} |\mathbf{T}^* \mathbf{T}|_\beta^\beta \sum_{\alpha \in I_{r_3,n}} |\mathbf{H}^* \mathbf{H}|_\alpha^\alpha}, \quad (50)$$

where

$$\begin{aligned} \mathbf{d}_j^H &= \left[\sum_{\alpha \in I_{r_3,n}\{j\}} \text{rdet}_j((\mathbf{H}^* \mathbf{H})_j (\mathbf{c}_q^{(22)}))_\alpha^\alpha \right] \in \mathbb{H}^{n \times 1}, \\ &q = 1, \dots, n, \\ \mathbf{d}_i^T &= \left[\sum_{\beta \in I_{r_4,n}\{i\}} \text{cdet}_i((\mathbf{T}^* \mathbf{T})_i (\mathbf{c}_l^{(22)}))_\beta^\beta \right] \in \mathbb{H}^{1 \times n}, \\ &l = 1, \dots, n, \end{aligned} \quad (51)$$

are the column vector and the row vector, respectively. $\mathbf{c}_q^{(22)}$ are the q th row, and $\mathbf{c}_l^{(22)}$ are the l th column of $\mathbf{C}_{22} = \mathbf{T}^* \mathbf{C}_2 \mathbf{H}$. The following expressions give some simplifications in computing. Since $\mathbf{T}^* \mathbf{T} = (\mathbf{R}_H \mathbf{A}_2)^* (\mathbf{R}_H \mathbf{A}_2) = \mathbf{A}_2^* \mathbf{R}_H^* \mathbf{R}_H \mathbf{A}_2 = \mathbf{A}_2^* \mathbf{R}_H \mathbf{A}_2$ and $\mathbf{R}_H = \mathbf{I} - \mathbf{H} \mathbf{H}^\dagger = \mathbf{I} - \mathbf{A}_2 \mathbf{L}_{A_1} (\mathbf{A}_2 \mathbf{L}_{A_1})^\dagger = \mathbf{I} - \mathbf{A}_2 (\mathbf{A}_2 \mathbf{L}_{A_1})^\dagger$, then $\mathbf{T}^* \mathbf{T} = \mathbf{A}_2^* (\mathbf{I} - \mathbf{A}_2 (\mathbf{A}_2 \mathbf{L}_{A_1})^\dagger) \mathbf{A}_2$.

(iv) Using (9) for determinantal representations of $\mathbf{H}^\dagger$ and $\mathbf{T}^\dagger$ in the fourth term of (34), we obtain

$$x_{ij}^{(4)} = \frac{\sum_{q=1}^n \sum_{z=1}^n \sum_{f=1}^n \sum_{\beta \in I_{r_3,n}\{i\}} \text{cdet}_i((\mathbf{H}^* \mathbf{H})_i (\mathbf{a}_{.q}^{(2,H)}))_\beta^\beta \sum_{\beta \in I_{r_4,n}\{q\}} \text{cdet}_q((\mathbf{T}^* \mathbf{T})_q (\mathbf{a}_{.z}^{(2,T)}))_\beta^\beta x_{zf}^{(1)} p_{fj}}{\sum_{\beta \in I_{r_3,n}} |\mathbf{H}^* \mathbf{H}|_\beta^\beta \sum_{\beta \in I_{r_4,n}} |\mathbf{T}^* \mathbf{T}|_\beta^\beta}, \quad (52)$$

where $\mathbf{a}_{.i}^{(2,H)}$, $\mathbf{a}_{.i}^{(2,T)}$ are the i th columns of the matrices $\mathbf{H}^* \mathbf{A}_2$ and $\mathbf{T}^* \mathbf{A}_2$, respectively; $x_{zf}^{(1)}$ is the first term; p_{fj} is the

(fj) th element of $\mathbf{P}_{A_2}$ with determinantal representation by (14) as

$$p_{fj} = \frac{\sum_{\beta \in J_{r,n}\{f\}} \text{cdet}_f((\mathbf{A}_2^* \mathbf{A}_2)_{\cdot f} (\dot{\mathbf{a}}_{\cdot j}^{(2)}))_{\beta}^{\beta}}{\sum_{\beta \in J_{r,n}} |\mathbf{A}_2^* \mathbf{A}_2|_{\beta}^{\beta}}, \quad (53)$$

where $\dot{\mathbf{a}}_{\cdot j}^{(2)}$ is the j th column of $\mathbf{A}_2^* \mathbf{A}_2$. Note that $\mathbf{H}^* \mathbf{A}_2 = \mathbf{L}_{A_1} \mathbf{A}_2^* \mathbf{A}_2$ and $\mathbf{T}^* \mathbf{A}_2 = \mathbf{A}_2^* \mathbf{R}_H \mathbf{A}_2 = \mathbf{A}_2^* (\mathbf{I} - \mathbf{A}_2 (\mathbf{A}_2 \mathbf{L}_{A_1})^{\dagger}) \mathbf{A}_2$.

(v) Similarly to the previous case,

$$x_{ij}^{(5)} = \frac{\sum_{q=1}^n \sum_{f=1}^n \sum_{\beta \in J_{r_3,n}\{i\}} \text{cdet}_i((\mathbf{H}^* \mathbf{H})_{\cdot i} (\mathbf{a}_{\cdot q}^{(2,H)}))_{\beta}^{\beta} x_{qf}^{(1)} p_{fj}}{\sum_{\beta \in J_{r_3,n}} |\mathbf{H}^* \mathbf{H}|_{\beta}^{\beta}}, \quad (54)$$

(vi) Consider the sixth term by analogy to the fourth term. So,

$$x_{ij}^{(6)} = \frac{\sum_{q=1}^n \sum_{\beta \in J_{r_3,n}\{i\}} \text{cdet}_i((\mathbf{H}^* \mathbf{H})_{\cdot i} (\mathbf{a}_{\cdot q}^{(2,H)}))_{\beta}^{\beta} \varphi_{qj}}{\sum_{\beta \in J_{r_3,n}} |\mathbf{H}^* \mathbf{H}|_{\beta}^{\beta} \sum_{\beta \in J_{r_4,n}} |\mathbf{T}^* \mathbf{T}|_{\beta}^{\beta} \sum_{\beta \in J_{r_2,n}} |\mathbf{A}_2^* \mathbf{A}_2|_{\beta}^{\beta}}, \quad (55)$$

where

$$\begin{aligned} \varphi_{qj} &= \sum_{\beta \in J_{r_4,n}\{q\}} \text{cdet}_q((\mathbf{T}^* \mathbf{T})_{\cdot q} (\psi_{\cdot j}^{A_2}))_{\beta}^{\beta}, \\ \varphi_{qj} &= \sum_{\alpha \in J_{r_2,n}\{j\}} \text{rdet}_j((\mathbf{A}_2^* \mathbf{A}_2)_{\cdot j} (\psi_{\cdot q}^T))_{\alpha}^{\alpha}, \end{aligned} \quad (56)$$

and

$$\begin{aligned} \psi_{\cdot j}^{A_2} &= \left[\sum_{\alpha \in J_{r_2,n}\{f\}} \text{rdet}_j((\mathbf{A}_2^* \mathbf{A}_2)_{\cdot j} (\mathbf{c}_{\cdot \alpha}^{(23)}))_{\alpha}^{\alpha} \right] \in \mathbb{H}^{1 \times n}, \\ \psi_{\cdot q}^T &= \left[\sum_{\beta \in J_{r_4,n}\{q\}} \text{cdet}_q((\mathbf{T}^* \mathbf{T})_{\cdot q} (\mathbf{c}_{\cdot \beta}^{(23)}))_{\beta}^{\beta} \right] \in \mathbb{H}^{n \times 1}, \end{aligned} \quad \begin{aligned} q &= 1, \dots, n, \\ l &= 1, \dots, n, \end{aligned} \quad (57)$$

are the column vector and the row vector, respectively. $\mathbf{c}_{\cdot q}^{(23)}$ and $\mathbf{c}_{\cdot l}^{(23)}$ are the q th row and the l th column of $\mathbf{C}_{23} = \mathbf{T}^* \mathbf{C}_2 \mathbf{A}_2$.

(vii) Using (9) for determinantal representations of $\mathbf{T}^{\dagger}$ and (12) for $\mathbf{H}^{*,\dagger}$ in the seventh term of (34), we obtain

$$x_{ij}^{(7)} = \frac{\sum_{q=1}^n \sum_{f=1}^n \sum_{\beta \in J_{r_4,n}\{i\}} \text{cdet}_i((\mathbf{T}^* \mathbf{T})_{\cdot i} (\mathbf{a}_{\cdot q}^{(2,T)}))_{\beta}^{\beta} x_{qf}^{(1)} \sum_{\alpha \in J_{r_3,n}\{j\}} \text{rdet}_j((\mathbf{H}^* \mathbf{H})_{\cdot j} (\mathbf{a}_{\cdot \alpha}^{(2,H,*)}))_{\alpha}^{\alpha}}{\sum_{\beta \in J_{r_4,n}} |\mathbf{T}^* \mathbf{T}|_{\beta}^{\beta} \sum_{\alpha \in J_{r_3,n}} |\mathbf{H}^* \mathbf{H}|_{\alpha}^{\alpha}}, \quad (58)$$

where $\mathbf{a}_{\cdot q}^{(2,T)}$, $\mathbf{a}_{\cdot f}^{(2,H,*)}$ are the q th column of $\mathbf{T}^* \mathbf{A}_2$ and the f th row of $\mathbf{A}_2^* \mathbf{H} = \mathbf{A}_2^* \mathbf{A}_2 \mathbf{L}_{A_1}$, respectively.

Hence, we prove the following theorem.

Theorem 17. Let $\mathbf{A}_1 \in \mathbb{H}_{r_1}^{m \times n}$, $\mathbf{A}_2 \in \mathbb{H}_{r_2}^{k \times n}$, and $\text{rank } \mathbf{H} = \text{rank } (\mathbf{A}_2 \mathbf{L}_{A_1}) = r_3$, $\text{rank } \mathbf{T} = \text{rank } (\mathbf{R}_H \mathbf{A}_2) = r_4$. Then for the partial solution (34) to system (3), we have

$$x_{ij} = \sum_{\delta} x_{ij}^{(\delta)}, \quad (59)$$

where the term $x_{ij}^{(1)}$ has the determinantal representations (40) and (44); $x_{ij}^{(2)}$, (46) and (47); $x_{ij}^{(3)}$, (49) and (50); $x_{ij}^{(4)}$, (52); $x_{ij}^{(5)}$, (54); $x_{ij}^{(6)}$, (55); and $x_{ij}^{(7)}$, (58).

Due to Khatri and Mitra [46], the next lemma can be generalized to $\mathbb{H}$.

Lemma 18. Let $\mathbf{A} \in \mathbb{H}^{m \times n}$ and $\mathbf{B} \in \mathbb{H}^{m \times m}$ and $\mathbf{B} = \mathbf{B}^*$ be known and $\mathbf{X} \in \mathbb{H}^{n \times n}$ be unknown. Then the matrix equation

$$\mathbf{A} \mathbf{X} \mathbf{A}^* = \mathbf{B} \quad (60)$$

has a Hermitian solution if and only if $\mathbf{Q}_A \mathbf{B} = \mathbf{B}$. In that case, the general Hermitian solution of (60) is

$$\mathbf{X} = \mathbf{A}^{\dagger} \mathbf{B} \mathbf{A}^{\dagger,*} + \mathbf{L}_A \mathbf{V} + \mathbf{V}^* \mathbf{L}_A, \quad (61)$$

where $\mathbf{V} \in \mathbb{H}^{n \times n}$ is any matrix.

As it follows from the above, if $\text{rank } \mathbf{A} = r$, then the determinantal representation of the partial Hermitian solution $\mathbf{X} = \mathbf{A}^{\dagger} \mathbf{B} \mathbf{A}^{\dagger,*} = (x_{ij})$ is

$$\begin{aligned} x_{ij} &= \frac{\sum_{\alpha \in J_{r,n}\{j\}} \text{rdet}_j((\mathbf{A}^* \mathbf{A})_{\cdot j} (\mathbf{v}_{\cdot \alpha}^{(1)}))_{\alpha}^{\alpha}}{(\sum_{\alpha \in J_{r,n}} |\mathbf{A}^* \mathbf{A}|_{\alpha}^{\alpha})^2} \\ &= \frac{\sum_{\beta \in J_{r,n}\{i\}} \text{cdet}_i((\mathbf{A}^* \mathbf{A})_{\cdot i} (\mathbf{v}_{\cdot \beta}^{(2)}))_{\beta}^{\beta}}{(\sum_{\beta \in J_{r,n}} |\mathbf{A}^* \mathbf{A}|_{\beta}^{\beta})^2}, \end{aligned} \quad (62)$$

where

$$\begin{aligned} \mathbf{v}_{\cdot i}^{(1)} &= \left[\sum_{\beta \in J_{r,n}\{i\}} \text{cdet}_i((\mathbf{A}^* \mathbf{A})_{\cdot i} (\mathbf{b}_{\cdot \beta}^{(1)}))_{\beta}^{\beta} \right] \in \mathbb{H}^{1 \times n}, \\ \mathbf{v}_{\cdot j}^{(2)} &= \left[\sum_{\alpha \in J_{r,n}\{j\}} \text{rdet}_j((\mathbf{A}^* \mathbf{A})_{\cdot j} (\mathbf{b}_{\cdot \alpha}^{(1)}))_{\alpha}^{\alpha} \right] \in \mathbb{H}^{n \times 1}, \end{aligned} \quad \begin{aligned} s &= 1, \dots, n, \\ f &= 1, \dots, n, \end{aligned} \quad (63)$$

are the row vector and the column vector and $\mathbf{b}_{.s}^{(1)}$ and $\mathbf{b}_{f.}^{(1)}$ are the s th column and the f th row of $\mathbf{B}_1 := \mathbf{A}^* \mathbf{B} \mathbf{A}$, respectively.

The general Hermitian solution to system (3) can be expressed as $\mathbf{Y} = (1/2)(\mathbf{X} + \mathbf{X}^*)$, where $\mathbf{X}$ is an arbitrary solution of (3). Since by Lemma 18, the existence of Hermitian solutions (1) needs $\mathbf{C}_1$ and $\mathbf{C}_2$ to be Hermitian, then

$$\begin{aligned} \mathbf{X}^* &= (\mathbf{x}_{ij}^*) = (\overline{\mathbf{x}_{ji}}) \\ &= \mathbf{A}_1^\dagger \mathbf{C}_1 (\mathbf{A}_1^*)^\dagger + \mathbf{A}_2^\dagger \mathbf{C}_2 (\mathbf{H}^*)^\dagger + \mathbf{H}^\dagger \mathbf{C}_2 (\mathbf{T}^*)^\dagger \\ &\quad + \mathbf{P}_{A_2} (\mathbf{A}_1^*)^\dagger \mathbf{C}_1 \mathbf{A}_1^\dagger \mathbf{A}_2 (\mathbf{T}^*)^\dagger \mathbf{A}_2^* (\mathbf{H}^*)^\dagger \\ &\quad - \mathbf{P}_{A_2} (\mathbf{A}_1^*)^\dagger \mathbf{C}_1 \mathbf{A}_1^\dagger \mathbf{A}_2^* (\mathbf{H}^*)^\dagger \\ &\quad - \mathbf{A}_2^\dagger \mathbf{C}_2 (\mathbf{T}^*)^\dagger \mathbf{A}_2^* (\mathbf{H}^*)^\dagger \\ &\quad - \mathbf{H}^\dagger \mathbf{A}_2 \mathbf{A}_1^\dagger \mathbf{C}_1 (\mathbf{A}_1^*)^\dagger \mathbf{A}_2^* (\mathbf{T}^*)^\dagger. \end{aligned} \quad (64)$$

So, the determinantal representation of the partial Hermitian solution $\mathbf{Y} = (y_{ij})$ can be obtained as $y_{ij} = (1/2)(x_{ij} + \overline{x_{ji}})$ for all $i, j = 1, \dots, n$, where x_{ij} is determined by Theorem 17 and $\overline{x_{ji}} = \sum_\delta \overline{x_{ji}^{(\delta)}}$ so that

$$\begin{aligned} \text{(i)} \quad \overline{x_{ji}^{(1)}} &= x_{ij}^{(1)}; \\ \text{(ii)} \quad \overline{x_{ji}^{(2)}} &= \frac{\sum_{\alpha \in I_{r_3,n}\{j\}} \text{rdet}_j((\mathbf{H}^* \mathbf{H})_j. (\mathbf{d}_i^{A_2}))_\alpha^\alpha}{\sum_{\alpha \in I_{r_3,n}} |\mathbf{H}^* \mathbf{H}|_\alpha^\alpha \sum_{\beta \in I_{r_2,n}} |\mathbf{A}_2^* \mathbf{A}_2|_\beta^\beta} \\ &= \frac{\sum_{\beta \in I_{r_2,n}\{i\}} \text{cdet}_i((\mathbf{A}_2^* \mathbf{A}_2)_{.i} (\mathbf{d}_j^H))_\beta^\beta}{\sum_{\alpha \in I_{r_3,n}} |\mathbf{H}^* \mathbf{H}|_\alpha^\alpha \sum_{\beta \in I_{r_2,n}} |\mathbf{A}_2^* \mathbf{A}_2|_\beta^\beta}, \end{aligned} \quad (65)$$

where

where $\mathbf{a}_{z.}^{(2,T,*)}$ is the z row of $\mathbf{A}_2^* \mathbf{T}$ and $\mathbf{a}_{q.}^{(2,H,*)}$ is the q th row of $\mathbf{A}_2^* \mathbf{H}$, respectively; $x_{zf}^{(1)}$ can be obtained by (40) or (44); evidently, $\overline{p_{if}} = p_{if}$ and p_{if} is the (if) th element of $\mathbf{P}_{A_2}$ with determinantal representation by (14) as

$$p_{if} = \frac{\sum_{\beta \in I_{r,n}\{i\}} \text{cdet}_i((\mathbf{A}_2^* \mathbf{A}_2)_{.i} (\mathbf{a}_{.f}^{(2)}))_\beta^\beta}{\sum_{\beta \in I_{r,n}} |\mathbf{A}_2^* \mathbf{A}_2|_\beta^\beta}, \quad (70)$$

where $\mathbf{a}_{.f}^{(2)}$ is the f th column of $\mathbf{A}_2^* \mathbf{A}_2$;

$$\begin{aligned} \mathbf{d}_{i.}^{A_2} &= \left[\sum_{\beta \in I_{r_2,n}\{i\}} \text{cdet}_i((\mathbf{A}_2^* \mathbf{A}_2)_{.i} (\mathbf{c}_{.q}^{(21,*)}))_\beta^\delta \right] \in \mathbb{H}^{1 \times n}, \\ q &= 1, \dots, n, \\ \mathbf{d}_{.j}^H &= \left[\sum_{\alpha \in I_{r_3,n}\{j\}} \text{rdet}_j((\mathbf{H}^* \mathbf{H})_j. (\mathbf{c}_{.l}^{(21,*)}))_\alpha^\alpha \right] \in \mathbb{H}^{n \times 1}, \\ l &= 1, \dots, n, \end{aligned} \quad (66)$$

are the row vector and the column vector, respectively. $\mathbf{c}_{.q}^{(21,*)}$ and $\mathbf{c}_{l.}^{(21,*)}$ are the q th column and the l th row of $\mathbf{C}_{21}^* = \mathbf{A}_2^* \mathbf{C}_2 \mathbf{H}$;

$$\begin{aligned} \text{(iii)} \quad \overline{x_{ji}^{(3)}} &= \frac{\sum_{\alpha \in I_{r_4,n}\{j\}} \text{rdet}_j((\mathbf{T}^* \mathbf{T})_j. (\mathbf{d}_i^H))_\alpha^\alpha}{\sum_{\alpha \in I_{r_4,n}} |\mathbf{T}^* \mathbf{T}|_\alpha^\alpha \sum_{\beta \in I_{r_3,n}} |\mathbf{H}^* \mathbf{H}|_\beta^\beta} \\ &= \frac{\sum_{\beta \in I_{r_3,n}\{i\}} \text{cdet}_i((\mathbf{H}^* \mathbf{H})_{.i} (\mathbf{d}_j^T))_\beta^\beta}{\sum_{\alpha \in I_{r_4,n}} |\mathbf{T}^* \mathbf{T}|_\alpha^\alpha \sum_{\beta \in I_{r_3,n}} |\mathbf{H}^* \mathbf{H}|_\beta^\beta}, \end{aligned} \quad (67)$$

where

$$\begin{aligned} \mathbf{d}_{i.}^H &= \left[\sum_{\beta \in I_{r_3,n}\{i\}} \text{cdet}_i((\mathbf{H}^* \mathbf{H})_{.i} (\mathbf{c}_{.q}^{(22,*)}))_\beta^\beta \right] \in \mathbb{H}^{1 \times n}, \\ q &= 1, \dots, n, \\ \mathbf{d}_{.j}^T &= \left[\sum_{\alpha \in I_{r_4,n}\{j\}} \text{rdet}_j((\mathbf{T}^* \mathbf{T})_j. (\mathbf{c}_{.l}^{(22,*)}))_\alpha^\alpha \right] \in \mathbb{H}^{n \times 1}, \\ l &= 1, \dots, n, \end{aligned} \quad (68)$$

are the row vector and the column vector, respectively. $\mathbf{c}_{.q}^{(22,*)}$ are the q th column and $\mathbf{c}_{l.}^{(22,*)}$ are the l th row of $\mathbf{C}_{22}^* = \mathbf{H}^* \mathbf{C}_2 \mathbf{T}$;

$$\text{(iv)} \quad \overline{x_{ji}^{(4)}} = \frac{\sum_{q=1}^n \sum_{z=1}^n \sum_{f=1}^n \overline{p_{if}} x_{fz}^{(1)} \sum_{\alpha \in I_{r_4,n}\{q\}} \text{rdet}_q((\mathbf{T}^* \mathbf{T})_q. (\mathbf{a}_{z.}^{(2,T,*)}))_\alpha^\alpha \sum_{\alpha \in I_{r_3,n}\{i\}} \text{rdet}_i((\mathbf{H}^* \mathbf{H})_{.i} (\mathbf{a}_{q.}^{(2,H,*)}))_\alpha^\alpha}{\sum_{\alpha \in I_{r_3,n}} |\mathbf{H}^* \mathbf{H}|_\alpha^\alpha \sum_{\alpha \in I_{r_4,n}} |\mathbf{T}^* \mathbf{T}|_\alpha^\alpha}, \quad (69)$$

$$\begin{aligned} \text{(v)} \quad \overline{x_{ji}^{(5)}} &= \frac{\sum_{q=1}^n \sum_{f=1}^n p_{if} x_{fz}^{(1)} \sum_{\beta \in I_{r_3,n}\{j\}} \text{rdet}_j((\mathbf{H}^* \mathbf{H})_j. (\mathbf{a}_{q.}^{(2,H,*)}))_\beta^\beta}{\sum_{\alpha \in I_{r_3,n}} |\mathbf{H}^* \mathbf{H}|_\alpha^\alpha}, \end{aligned} \quad (71)$$

$$\begin{aligned} \text{(vi)} \quad \overline{x_{ji}^{(6)}} &= \frac{\sum_{q=1}^n \eta_{iq} \sum_{\alpha \in I_{r_3,n}\{i\}} \text{rdet}_i((\mathbf{H}^* \mathbf{H})_{.i} (\mathbf{a}_{q.}^{(2,H,*)}))_\alpha^\alpha}{\sum_{\alpha \in I_{r_3,n}} |\mathbf{H}^* \mathbf{H}|_\alpha^\alpha \sum_{\alpha \in I_{r_4,n}} |\mathbf{T}^* \mathbf{T}|_\alpha^\alpha \sum_{\beta \in I_{r_2,n}} |\mathbf{A}_2^* \mathbf{A}_2|_\beta^\beta}, \end{aligned}$$

where

$$\begin{aligned}\eta_{iq} &= \sum_{\alpha \in I_{4,n}\{q\}} \text{rdet}_q((\mathbf{T}^* \mathbf{T})_q (\zeta_i^{A_2}))_\alpha^\alpha \\ &= \sum_{\beta \in J_{r_2,n}\{i\}} \text{cdet}_i((\mathbf{A}_2^* \mathbf{A}_2)_{.i} (\zeta_q^T))_\beta^\beta,\end{aligned}\quad (72)$$

and

$$\zeta_i^{A_2} = \left[\sum_{\beta \in J_{r_2,n}\{i\}} \text{cdet}_i((\mathbf{A}_2^* \mathbf{A}_2)_{.i} (\mathbf{c}_{.q}^{(23),*}))_\beta^\beta \right] \in \mathbb{H}^{n \times 1},$$

$$q = 1, \dots, n,$$

$$(vii) \quad \overline{x_{ji}^{(7)}} = \frac{\sum_{q=1}^n \sum_{f=1}^n \sum_{\beta \in J_{r_3,n}\{i\}} \text{cdet}_i((\mathbf{H}^* \mathbf{H})_{.i} (\mathbf{a}_{.f}^{(2,H)}))_\beta^\beta x_{fg}^{(1)} \sum_{\alpha \in I_{4,n}\{j\}} \text{rdet}_j((\mathbf{T}^* \mathbf{T})_j (\mathbf{a}_q^{(2,T)}))_\alpha^\alpha}{\sum_{\alpha \in I_{4,n}} |\mathbf{T}^* \mathbf{T}|_\alpha^\alpha \sum_{\beta \in J_{r_3,n}} |\mathbf{H}^* \mathbf{H}|_\beta^\beta}, \quad (74)$$

where $\mathbf{a}_{.q}^{(2,T),*}$, $\mathbf{a}_f^{(2,H)}$ are the q th row of $\mathbf{A}_2^* \mathbf{T}$ and the f th column of $\mathbf{H}^* \mathbf{A}_2$, respectively.

4. An Example

In this section, we give an example to illustrate our results. Let us consider the system of matrix equations

$$\begin{aligned}\mathbf{A}_1 \mathbf{X} \mathbf{A}_1^* &= \mathbf{C}_1, \\ \mathbf{A}_2 \mathbf{X} \mathbf{A}_2^* &= \mathbf{C}_2\end{aligned}\quad (75)$$

where

$$\begin{aligned}\mathbf{A}_1 &= 0.5 \begin{bmatrix} 2 & -\mathbf{k} & \mathbf{i} \\ 2\mathbf{j} & -2\mathbf{i} & -2\mathbf{k} \end{bmatrix}, \\ \mathbf{C}_1 &= 0.5 \begin{bmatrix} \mathbf{k} & \mathbf{i} \\ \mathbf{i} & -\mathbf{k} \end{bmatrix}, \\ \mathbf{A}_2 &= \begin{bmatrix} \mathbf{j} & -\mathbf{k} & 1 \\ \mathbf{i} & 1 & \mathbf{k} \end{bmatrix}, \\ \mathbf{C}_2 &= \begin{bmatrix} -\mathbf{k} & 1 \\ \mathbf{i} & \mathbf{j} \end{bmatrix}.\end{aligned}\quad (76)$$

Since

$$\begin{aligned}\det(\mathbf{A}_1 \mathbf{A}_1^*) &= \det \begin{bmatrix} 1.5 & -2\mathbf{j} \\ 2\mathbf{j} & 3 \end{bmatrix} = 0.5, \\ \det(\mathbf{A}_2 \mathbf{A}_2^*) &= \det \begin{bmatrix} 3 & -\mathbf{k} \\ \mathbf{k} & 3 \end{bmatrix} = 8,\end{aligned}\quad (77)$$

then $\text{rank } \mathbf{A}_1 = 2$, $\text{rank } \mathbf{A}_2 = 2$. By Theorem 5, one can find

$$\begin{aligned}\zeta_{.q}^T &= \left[\sum_{\alpha \in I_{4,n}\{q\}} \text{rdet}_q((\mathbf{T}^* \mathbf{T})_q (\mathbf{c}_{.l}^{(23),*}))_\alpha^\alpha \right] \in \mathbb{H}^{1 \times n}, \\ l &= 1, \dots, n,\end{aligned}\quad (73)$$

are the row vector and the column vector, respectively. $\mathbf{c}_{.q}^{(23),*}$ and $\mathbf{c}_{l.}^{(23),*}$ are the q th column and the l th row of $\mathbf{C}_{23}^* = \mathbf{A}_2 \mathbf{C}_2 \mathbf{T}^*$;

$$\mathbf{A}_1^\dagger = \begin{bmatrix} 2 & \mathbf{j} \\ -\mathbf{k} & \mathbf{i} \\ \mathbf{i} & \mathbf{k} \end{bmatrix},$$

$$\mathbf{A}_1^\dagger \mathbf{A}_1 = 0.5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \mathbf{j} \\ 0 & -\mathbf{j} & 1 \end{bmatrix},$$

$$\mathbf{L}_{A_1} = 0.5 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -\mathbf{j} \\ 0 & \mathbf{j} & 1 \end{bmatrix},$$

$$\mathbf{A}_2^\dagger = 0.25 \begin{bmatrix} -2\mathbf{j} & -2\mathbf{i} \\ \mathbf{k} & 1 \\ 1 & -\mathbf{k} \end{bmatrix}, \quad (78)$$

$$\mathbf{H} = \mathbf{A}_2 \mathbf{L}_{A_1} = 0.5 \begin{bmatrix} 0 & \mathbf{j} - \mathbf{k} & 1 - \mathbf{i} \\ 0 & 1 - \mathbf{i} & -\mathbf{j} + \mathbf{k} \end{bmatrix},$$

$$\mathbf{H}^\dagger = 0.25 \begin{bmatrix} 0 & 0 \\ -\mathbf{j} + \mathbf{k} & 1 + \mathbf{i} \\ 1 + \mathbf{i} & \mathbf{j} - \mathbf{k} \end{bmatrix},$$

$$\mathbf{R}_H = 0.5 \begin{bmatrix} 1 & \mathbf{k} \\ -\mathbf{k} & 1 \end{bmatrix},$$

$$\mathbf{T} = \mathbf{R}_H \mathbf{A}_2 = \begin{bmatrix} \mathbf{j} & 0 & 0 \\ \mathbf{i} & 0 & 0 \end{bmatrix}.$$

Since $\mathbf{Q}_{A_1} = \mathbf{I}$, $\mathbf{Q}_{A_2} = \mathbf{I}$, $\mathbf{T} \mathbf{A}_2^\dagger \mathbf{C}_2 \mathbf{A}_2^{*,\dagger} \mathbf{T}^* = \mathbf{T} \mathbf{A}_1^\dagger \mathbf{C}_1 \mathbf{A}_1^{*,\dagger} \mathbf{T}^*$, then, by Lemma 16, system (75) is consistent.

First, we can find the solution of (75) by direct calculation. Since $\mathbf{X} = \sum_{\delta} \mathbf{X}_{\delta}$, where

$$\begin{aligned}
\mathbf{X}_1 &= \mathbf{A}_1^\dagger \mathbf{C}_1 (\mathbf{A}_1^*)^\dagger = 0.5 \begin{bmatrix} \mathbf{k} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
\mathbf{X}_2 &= \mathbf{H}^\dagger \mathbf{C}_2 (\mathbf{A}_2^*)^\dagger = 0.125 \begin{bmatrix} 0 & 0 & 0 \\ 2\mathbf{j} + 2\mathbf{k} & \mathbf{j} + \mathbf{k} & -1 + \mathbf{i} \\ -2 + 2\mathbf{i} & 1 + \mathbf{i} & -\mathbf{j} - \mathbf{k} \end{bmatrix}, \\
\mathbf{X}_3 &= \mathbf{T}^\dagger \mathbf{C}_2 (\mathbf{H}^*)^\dagger = 0.25 \begin{bmatrix} 0 & \mathbf{j} - \mathbf{k} & 1 - \mathbf{j} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
\mathbf{X}_4 &= \mathbf{H}^\dagger \mathbf{A}_2 \mathbf{T}^\dagger \mathbf{A}_2 \mathbf{A}_1^\dagger \mathbf{C}_1 (\mathbf{A}_1^*)^\dagger \mathbf{P}_{A_2} = \mathbf{0}, \\
\mathbf{X}_5 &= -\mathbf{H}^\dagger \mathbf{A}_2 \mathbf{A}_1^\dagger \mathbf{C}_1 (\mathbf{A}_1^*)^\dagger \mathbf{P}_{A_2} = \mathbf{0}, \\
\mathbf{X}_6 &= -\mathbf{H}^\dagger \mathbf{A}_2 \mathbf{T}^\dagger \mathbf{C}_2 (\mathbf{A}_2^*)^\dagger = \mathbf{0}, \\
\mathbf{X}_7 &= -\mathbf{T}^\dagger \mathbf{A}_2 \mathbf{A}_1^\dagger \mathbf{C}_1 (\mathbf{A}_1^*)^\dagger \mathbf{A}_2^* (\mathbf{H}^*)^\dagger = \mathbf{0},
\end{aligned} \tag{79}$$

then

$$\mathbf{X} = 0.125 \begin{bmatrix} 4\mathbf{k} & 2\mathbf{j} - 2\mathbf{k} & 2 - 2\mathbf{j} \\ 2\mathbf{j} + 2\mathbf{k} & \mathbf{j} + \mathbf{k} & -1 + \mathbf{i} \\ -2 + 2\mathbf{i} & 1 + \mathbf{i} & -\mathbf{j} - \mathbf{k} \end{bmatrix}. \tag{80}$$

Now, we find the solution of (75) by its determinantal representation by Theorem 17. Since

$$\begin{aligned}
\mathbf{C}_{11} &= \mathbf{A}_1^* \mathbf{C}_1 \mathbf{A}_1 = 0.125 \begin{bmatrix} 4\mathbf{k} & 12 & 12\mathbf{j} \\ -12 & 9\mathbf{k} & -9\mathbf{i} \\ 12\mathbf{i} & -9\mathbf{i} & -9\mathbf{k} \end{bmatrix}, \\
\mathbf{A}_1^* \mathbf{A}_1 &= 0.125 \begin{bmatrix} 16 & -12\mathbf{k} & 12\mathbf{i} \\ 12\mathbf{k} & 10 & 10\mathbf{j} \\ -12\mathbf{i} & -10\mathbf{j} & 10 \end{bmatrix},
\end{aligned} \tag{81}$$

then by (41)

$$\begin{aligned}
v_{11}^{(1)} &= \sum_{\beta \in I_{2,3}\{1\}} \text{cdet}_1 \left(((\mathbf{A}_1^* \mathbf{A}_1)_{\cdot 1} (\mathbf{c}_{\cdot 1}^{(11)}))^\beta \right) \\
&= \text{cdet}_1 \begin{bmatrix} 0.5\mathbf{k} & -1.5\mathbf{k} \\ -1.5 & 1.25 \end{bmatrix} + \text{cdet}_1 \begin{bmatrix} 0.5\mathbf{k} & 1.5\mathbf{i} \\ 1.5\mathbf{j} & 1.25 \end{bmatrix} \\
&= 0.5\mathbf{k}.
\end{aligned} \tag{82}$$

Similarly, $v_{12}^{(1)} = 0.375$, $v_{13}^{(1)} = 0.375\mathbf{j}$. So, $\mathbf{v}_1^{(1)} = [0.5\mathbf{k} \ 0.375 \ 0.375\mathbf{j}]$. Further, by (40),

$$\begin{aligned}
x_{11}^{(1)} &= \frac{\sum_{\alpha \in I_{2,3}\{1\}} \text{rdet}_1 \left(((\mathbf{A}_1^* \mathbf{A}_1)_{\cdot 1} (\mathbf{v}_1^{(1)}))^\alpha \right)}{\left(\sum_{\alpha \in I_{2,3}} |\mathbf{A}_1^* \mathbf{A}_1|_\alpha^\alpha \right)^2} \\
&= 4 \left(\text{rdet}_1 \begin{bmatrix} 0.5\mathbf{k} & 0.375 \\ 1.5\mathbf{k} & 1.25 \end{bmatrix} + \text{rdet}_1 \begin{bmatrix} 0.5\mathbf{k} & 0.375\mathbf{j} \\ -1.5\mathbf{j} & 1.25 \end{bmatrix} \right) \\
&= 0.5\mathbf{k}.
\end{aligned} \tag{83}$$

Since

$$\begin{aligned}
\mathbf{C}_{21} &= \mathbf{H}^* \mathbf{C}_2 \mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ \mathbf{j} + \mathbf{k} & \mathbf{j} + \mathbf{k} & -1 + \mathbf{i} \\ -1 + \mathbf{i} & -1 + \mathbf{i} & -\mathbf{j} - \mathbf{k} \end{bmatrix}, \\
\mathbf{C}_{22} &= \mathbf{T}^* \mathbf{C}_2 \mathbf{H} = \begin{bmatrix} 0 & \mathbf{j} - \mathbf{k} & 1 - \mathbf{i} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
\mathbf{H}^* \mathbf{H} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -\mathbf{j} \\ 0 & \mathbf{j} & 1 \end{bmatrix},
\end{aligned} \tag{84}$$

and $\text{rank } \mathbf{H} = \text{rank } \mathbf{T} = 1$, then

$$\begin{aligned}
\mathbf{d}_{i.}^H &= [0 \ 0 \ 0], \\
\mathbf{d}_{.j}^H &= [0 \ 0 \ 0]^T.
\end{aligned} \tag{85}$$

So, by (47) and by (49), $x_{11}^{(2)} = 0$ and $x_{11}^{(3)} = 0$, respectively. Moreover, $x_{11}^{(\delta)} = 0$ for all $\delta = 4, \dots, 7$. So, $x_{11} = x_{11}^{(1)} = 0.5\mathbf{k}$.

So, x_{11} obtained by Cramer's rule and the matrix method (80) are equal.

Similarly, we can obtain for all the remainder solutions.

Note that we used Maple with the package CLIFFORD in the calculations.

5. Conclusions

Within the framework of the theory of quaternion row-column determinants previously introduced by the author, we have derived determinantal representations (analogs of Cramer's rule) of the general and Hermitian solutions to the system of two-sided quaternion matrix equations $\mathbf{A}_1 \mathbf{X} \mathbf{A}_1^* = \mathbf{C}_1$ and $\mathbf{A}_2 \mathbf{X} \mathbf{A}_2^* = \mathbf{C}_2$. Since the Hermitian solution is $\mathbf{Y} = (1/2)(\mathbf{X} + \mathbf{X}^*)$, where $\mathbf{X}$ is an arbitrary solution, the determinantal representation of $\mathbf{X}^*$ is derived as well. To accomplish that goal, we have used the determinantal representations of the Moore-Penrose matrix inverse which were previously introduced by the author.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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