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Research Article

Retractions and Homomorphisms on Some Operations of Graphs

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The aim of the present article is to introduce and study a new type of operations on graph, namely, edge graph. The relation between the homomorphisms and retractions on edge graphs is deduced. The limit retractions on the edge graphs are presented. Retractions on a finite number of edge graphs are obtained.

1. Introduction and Preliminaries

Graph theory is rapidly moving into the mainstream of mathematics. The prospects of further development in algebraic graph theory and important link with computational theory indicate the possibility of the subject quickly emerging at the forefront of mathematics. Its scientific and engineering applications, especially to communication science, computer technology, and system theory, have already been accorded a place of pride in applied mathematics. Graphs serve as mathematical models to analyze successfully many concrete real-world problems. A certain problem in physics, chemistry, genetics, psychology, sociology, and linguistics can be formulated as problems in graph theory. Also, many branches of mathematics such as game theory, group theory, matrix theory, probability, and topology have interactions with graph theory. Some puzzles and various problems of a practical nature have been instrumental in the development of various topics in graph theory. The theory of acyclic graphs was developed for solving problems of electrical networks and the study of trees was developed for enumerating isomers of organic compounds. This paper describes the operation of a graph from the viewpoint of an identification [1–10].

A graph is an ordered $G = (V(G), E(G))$, where $V(G) \neq \phi$, $E(G)$ is a set disjoint from $V(G)$, elements of $V(G)$ are called the vertices of G, and elements of $E(G)$ are called the edges. A graph is connected if, for every partition of its vertex set into two nonempty sets X and Y, there is an edge with one end in X and one end in Y; otherwise, the graph is disconnected. A graph H is said to be a subgraph of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph in which each pair of distinct vertices is adjacent is called a complete graph. A complete graph with n vertices is denoted by K_n [11]. The chromatic number $\chi(G)$ of a graph G is the minimum number of colors required for proper vertex coloring of G. A $m-$coloring of a graph G is a vertex coloring of G that uses at most $m-$colors. A graph G is said to be m-colorable if G admits a proper vertex coloring using at most m colors [11]. Let G and H be two graphs. A function $\phi : V(G) \longrightarrow V(H)$ is a homomorphism from G to H if it preserves edges, that is, if for any edge $[u, v]$ of G, $[\phi(u), \phi(v)]$ is an edge of H [12]. A retract of a graph G is a subgraph H of G such that there exists a homomorphism $r : G \longrightarrow H$, called retraction with $r(x) = x$ for any vertex x of H [7]. A core is a graph which does not retract to a proper subgraph [12].

2. The Main Results

Aiming at our study, we will introduce the following.

Definition 1. Let G_1 and G_2 be two connected graphs, where e_1 is an edge of G_1, e_2 is an edge of G_2, and $G_1 \cap G_2 = \phi$; then we define the edge graph $G_1 \underline{\vee} G_2$ by gluing together the two edges e_1 and e_2.

Theorem 2. *Let G_1 and G_2 be two connected graphs. Then $\chi(G_1 \underline{\vee} G_2) = \max\{\chi(G_1), \chi(G_2)\}$.*

Proof. Let $\chi(G_1 \underline{\vee} G_2) = m$. At that point, there exists an m-coloring ϖ of $G_1 \underline{\vee} G_2$. Since ϖ assigns different colors to every two adjacent vertices of G_1 and G_2, G_1 and G_2 are m-colorable and so $\chi(G_1) \leq \chi(G_1 \underline{\vee} G_2)$ and $\chi(G_2) \leq \chi(G_1 \underline{\vee} G_2)$. Also, using symmetry $\chi(G_1) \geq \chi(G_2)$. Beginning with an ideal coloring of $\chi(G_1)$, we can incorporate an ideal coloring of $\chi(G_2)$ by exchanging a pair of color names to make the coloring agree at two vertices of common edge graphs. This produces a proper coloring of $G_1 \underline{\vee} G_2$. □

Theorem 3. *The graphs G_1 and G_2 are subgraphs of $G_1 \underline{\vee} G_2$. Also, for any tree G_1 and G_2, the graph $G_1 \underline{\vee} G_2$ is also a tree.*

Proof. The proof of this theorem is clear. □

Theorem 4. *Suppose that $G_1, G_2, \ldots, G_n$ are connected graphs; then there is a sequence of nontrivial retractions $\{r_k : \underline{\vee}_{i=1}^{n} G_i \longrightarrow \underline{\vee}_{i=1}^{n} G_i, k = 1, 2 \ldots . n\}$, where $\underline{\vee}_{i=1}^{n} G_i$ are glued along the same edge such that $r_k(\underline{\vee}_{i=1}^{n} G_i)$ is a proper subgraph of $\underline{\vee}_{i=1}^{n} G_i$.*

Proof. Let $r_1 : \underline{\vee}_{i=1}^{n} G_i \longrightarrow \underline{\vee}_{i=1}^{n} G_i$ be a retraction from $\underline{\vee}_{i=1}^{n} G_i$ into itself and $r_1(\underline{\vee}_{i=1}^{n} G_i) = G_1 \underline{\vee} G_2 \underline{\vee} \ldots \underline{\vee} r_1(G_s) \ldots \underline{\vee} G_n$ for $s = 1, 2, \ldots, n$. Since $r_1(G_s)$ is a proper subgraph of G_s, it follows that $r_1(\underline{\vee}_{i=1}^{n} G_i)$ is a proper subgraph of $\underline{\vee}_{i=1}^{n} G_i$. Also, if $r_2(\underline{\vee}_{i=1}^{n} G_i) = G_1 \underline{\vee} G_2 \underline{\vee} \cdots \underline{\vee} r_2(G_s) \underline{\vee} \cdots \underline{\vee} r_2(G_k) \underline{\vee} \cdots \underline{\vee} G_n$ for $k = 1, 2, \ldots, n, s < k$ and $r_2(G_s), r_2(G_k)$ are subgraphs of G_s, G_k, respectively, then $r_2(\underline{\vee}_{i=1}^{n} G_i)$ is a proper subgraph of $\underline{\vee}_{i=1}^{n} G_i$. Moreover, by continuing this process if $r_n(\underline{\vee}_{i=1}^{n} G_i) = \underline{\vee}_{i=1}^{n} r_n(G_i)$, then $r_k(\underline{\vee}_{i=1}^{n} G_i)$ is a proper subgraph of $\underline{\vee}_{i=1}^{n} G_i$. □

Theorem 5. *Let G_1 and G_2 be two graphs; then there is a homomorphism $f : G_1 \longrightarrow G_2$ iff G_2 is a retract of $G_1 \underline{\vee} G_2$.*

Proof. Let $f : G_1 \longrightarrow G_2$ be a homomorphism. Since G_2 is subgraph of $G_1 \underline{\vee} G_2$, then there exists a homomorphism $r : G_1 \underline{\vee} G_2 \longrightarrow G_2$with $r(x) = x$, for any vertex x of G_2 and so G_2 is a retract of $G_1 \underline{\vee} G_2$. Conversely, assume that G_2 is a retract of $G_1 \underline{\vee} G_2$; thus $r : G_1 \underline{\vee} G_2 \longrightarrow G_2$ is a homomorphism with $r(x) = x$ for any vertex x of G_2, and so there is a homomorphism $f : G_1 \longrightarrow G_2$. □

Theorem 6. *Let G_1 and G_2 be connected graphs; then K_n is a retract of graph G_1 or G_2, iff K_n retract of $G_1 \underline{\vee} G_2$.*

Proof. Suppose G_1 and G_2 are connected graphs and K_n is a retract of graph G_1 or G_2. Then there is a homomorphism $r_1 : G_1 \longrightarrow K_n$ such that $r_1(x) = x$, or a homomorphism $r_2 : G_2 \longrightarrow K_n$ such that $r_2(x) = x$, for any vertex x of K_n. Since K_n is a core and it is subgraph of G_1 or G_2, it follows that K_n is subgraph of $G_1 \underline{\vee} G_2$ and so there is a homomorphism $r : G_1 \underline{\vee} G_2 \longrightarrow K_n$ such that $r(x) = x$, for any vertex x of K_n. Conversely, suppose K_n is a retract $G_1 \underline{\vee} G_2$; then K_n is subgraph of $G_1 \underline{\vee} G_2$ and $K_n \underline{\vee} K_n$ and so K_n is a retract of graphs G_1 or G_2. □

Theorem 7. *Let T be any tree of size n; then there is a sequence of nontrivial retractions $\{r_i, i = 1, 2, \ldots n\}$ such that $\lim_{n \longrightarrow \infty} r_n(r_{n-1}) \ldots (r_1(T) = K_2$.*

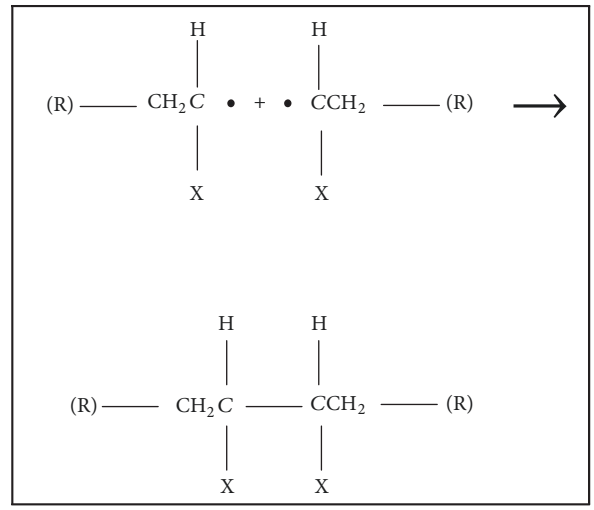

Figure 1

Proof. Consider the following sequence of retractions:

$r_1 : T \longrightarrow T_1$ is nontrivial retraction, where T_1 is subgraph of T and $1 \leq Size(T_1) \leq n - 1$,

$r_2 : r_1(T) \longrightarrow r_1(T_1)$, where $r_1(T_1)$ is subgraph of $r_1(T)$ and $1 \leq Size(r_1(T_1)) \leq n - 2$,

$\vdots$

$r_n : r_{n-1}(r_{n-2}) \ldots (r_1(T) \longrightarrow r_{n-1}(r_{n-2}) \ldots (r_1(T_1)$, where $r_{n-1}(r_{n-2}) \ldots (r_1(T_1)$ is subgraph of $r_{n-1}(r_{n-2}) \ldots (r_1(T)$, and $\lim_{n \longrightarrow \infty} r_n(r_{n-1}) \ldots (r_1(T)$ is a tree of size 1. Therefore, $\lim_{n \longrightarrow \infty} r_n(r_{n-1}) \ldots (r_1(T) = k_2$. □

Theorem 8. *Suppose that G_1 and G_2 are connected graphs; then $\lim_{n \longrightarrow \infty} r_n(G_1 \underline{\vee} G_2)) = \lim_{n \longrightarrow \infty} r_n(G_1) \underline{\vee} \lim_{n \longrightarrow \infty} r_n(G_1)$.*

Proof. If G_1 and G_2 are connected graphs, then we get the following induced subgraphs $\lim_{n \longrightarrow \infty} r_n(G_1 \underline{\vee} G_2)), \lim_{n \longrightarrow \infty} r_n(G_1), \lim_{n \longrightarrow \infty} r_n(G_1)$ and each of them is isomorphic to k_2. Since, $k_2 \approx k_2 \underline{\vee} k_2$, it follows that $\lim_{n \longrightarrow \infty} r_n(G_1 \underline{\vee} G_2)) = \lim_{n \longrightarrow \infty} r_n(G_1) \underline{\vee} \lim_{n \longrightarrow \infty} r_n(G_1)$. □

3. Some Applications in Chemistry and Biology

(i) A polymer is composed of many repeating units called monomers. Starch, cellulose, and proteins are natural polymers. Nylon and polyethylene are synthetic polymers. Polymerization is the process of joining monomers. Polymers may be formed by addition polymerization and one basic step in addition polymerization is combination as in Figure 1, which occurs when the polymer's growth is stopped by free electrons from two growing chains that join and form a single chain. The following diagram depicts combination, with the symbol (R) representing the rest of the chain. This is a representation type of connected two graphs into an edge graph.

(ii) Peptide bonds constitute the representation of an edge graph by linking two amino acids as in Figure 2, which is a

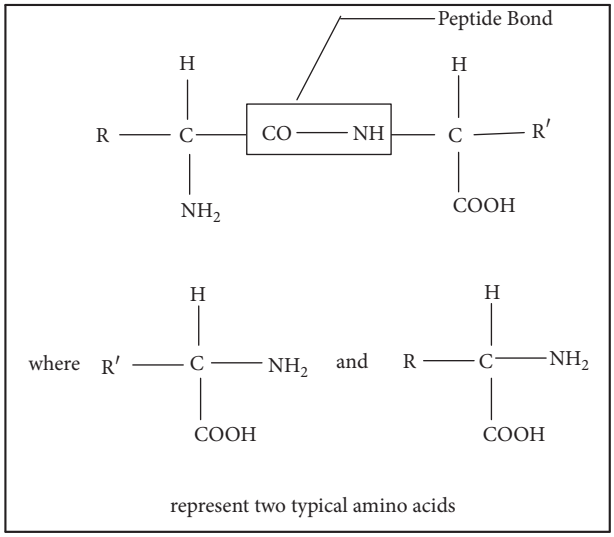

Figure 2

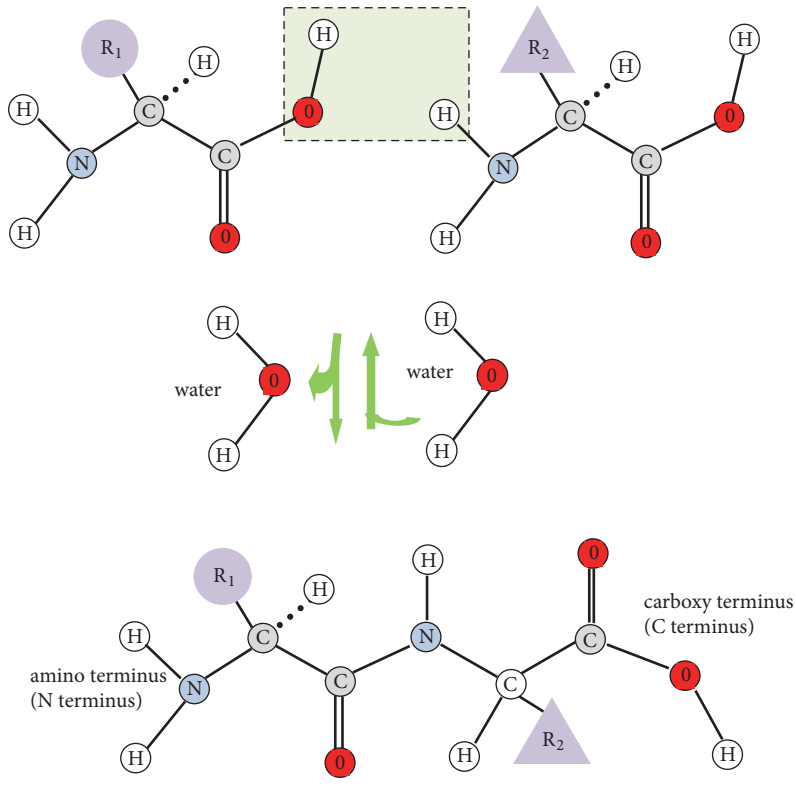

Figure 3

representation graph of connected two typical amino acids into an edge graph.

In Figure 3, peptide pond and formation hydrolysis: Formation (top to bottom) and hydrolysis from bottom to top of a peptide bonds require conceptually loss and addition, respectively, of a molecule of water. The actual chemical synthesis and hydrolysis of peptide bonds in the cell are enzymatically controlled processes that in the synthesis nearly always occur on the ribosome and are directed by an mRNA template. The end of a polypeptide with the free of amino group is known as the amino terminus (N terminus) and with the free carboxyl group is the carboxyl terminus (C terminus). This is a representation of connected two graphs into an edge graph.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that they have no conflicts of interest.

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