Research Article

A Group Theory Approach towards Some Rational Difference Equations

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A full Lie point symmetry analysis of rational difference equations is performed. Nontrivial symmetries are derived, and exact solutions using these symmetries are obtained.

1. Introduction

Over a century ago, symmetries became a centre of interest of several authors after the work of Sophus Lie [1] on differential equations. He studied the continuous group of transformations that leaves the differential equations invariant. This concept of symmetries is strongly related to the existence of conservation laws, and the relationship between them has attracted great interest among researchers following the work of Noether [2]. The extension of this idea to difference equations is now well documented (see [3] and references herein). In [3], Hydon developed a symmetry-based algorithm enabling one to derive solutions of difference equations without making any special lucky guesses. Hydon emphasized on second-order difference equations, although his algorithm is valid for any order. When it comes to higher-order equations, computations can be cumbersome and extra ansatz may be needed to ease the calculations.

We aim to extend the work by Elsayed [4] where the author investigated the dynamics and solutions of

$$x_{n+1} = \frac{x_{n-5}}{\pm 1 \pm x_{n-1}x_{n-3}x_{n-5}}, \quad (1)$$

where the initial conditions $x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}$, and x_0 are arbitrary nonzero real numbers. For related work, see [5, 6]. One can notice that equation (1) is just a special case of a more general form

$$x_{n+1} = \frac{x_{n-5}}{a_n + b_n x_{n-1}x_{n-3}x_{n-5}}, \quad (2)$$

where $(a_n)_{n \in \mathbb{N}_0}$ and $(b_n)_{n \in \mathbb{N}_0}$ are arbitrary sequences. We will use a symmetry-based method to solve (2). Equivalently, we study

$$u_{n+6} = \frac{u_n}{A_n + B_n u_n u_{n+2} u_{n+4}}, \quad (3)$$

instead, where $(A_n)_{n \in \mathbb{N}_0}$ and $(B_n)_{n \in \mathbb{N}_0}$ are arbitrary sequences. This means we can only compare x_i with u_{i+5} . Furthermore, we use the same technique to obtain exact solutions of

$$x_{n+1} = \frac{x_{n-4}x_{n-3}x_{n-2}}{x_{n-1}x_n(\lambda + \mu x_{n-4}x_{n-3}x_{n-2})}, \quad (4)$$

where $\lambda, \mu \in \mathbb{R}$, and again we study

$$u_{n+5} = \frac{u_n u_{n+1} u_{n+2}}{u_{n+3} u_{n+4}(\lambda + \mu u_n u_{n+1} u_{n+2})}, \quad (5)$$

instead. Note that solutions of (4) are found in [7]; however, their method is completely different from ours.

2. Definitions and Notation

The definitions are taken from Hydon [3], and most of the notation follows from the same book.

Definition 1. A parameterized set of point transformations,

$$\Gamma_\varepsilon: x \mapsto \hat{x}(x; \varepsilon), \quad (6)$$

where $x = x_i$ and $i = 1, \dots, p$ are continuous variables, is a one-parameter local Lie group of transformations if the following conditions are satisfied:

- (1) Γ_0 is the identity map if $\hat{x} = x$ when $\varepsilon = 0$
- (2) $\Gamma_a \Gamma_b = \Gamma_{a+b}$ for every a and b sufficiently close to 0
- (3) Each $\hat{x}_i$ can be represented as a Taylor series (in a neighborhood of $\varepsilon = 0$ that is determined by x), and therefore

$$\hat{x}_i(x; \varepsilon) = x_i + \varepsilon \xi_i(x) + O(\varepsilon^2), \quad i = 1, \dots, p. \quad (7)$$

Consider the p th-order difference equation

$$u_{n+p} = \Omega(n, u_n, \dots, u_{n+p-1}), \quad (8)$$

for some function Ω . Assume the point transformations are of the form

$$\begin{aligned} \hat{n} &= n, \\ \hat{u}_n &= u_n + \varepsilon Q(n, u_n) + O(\varepsilon^2), \end{aligned} \quad (9)$$

with the corresponding infinitesimal symmetry generator

$$\begin{aligned} X &= Q(n, u_n) \frac{\partial}{\partial u_n} + SQ(n, u_n) \frac{\partial}{\partial u_{n+1}} + \dots \\ &+ S^{(p-1)}Q(n, u_n) \frac{\partial}{\partial u_{n+p-1}}, \end{aligned} \quad (10)$$

where S is the shift operator, i.e., $S: n \mapsto n+1$. The symmetry condition is defined as

$$\hat{u}_{n+p} = \Omega(n, \hat{u}_n, \hat{u}_{n+1}, \dots, \hat{u}_{n+p-1}), \quad (11)$$

whenever (8) is true. Substituting the Lie point symmetries (9) into the symmetry condition (11) leads to the linearized symmetry condition:

$$S^{(p)}Q - X\Omega = 0, \quad (12)$$

whenever (8) holds.

Definition 2. V_n is invariant under the Lie group of transformations (9) if and only if $XV_n = 0$.

We define the functions

$$\Theta^n(\theta_j, s, l) = \prod_{j=s}^l \theta_{2j+n}, \quad n = 0, 1, \quad (13)$$

and we adopt the standard conventions

$$\begin{aligned} \prod_{j=s}^l \theta_j &= 1, \quad \text{when } s > l, \\ \sum_{j=n}^{l_0} \theta_j &= 0, \quad \text{when } n > l_0. \end{aligned} \quad (14)$$

We refer the reader to [3] for a deeper understanding of the concept of symmetry analysis of difference equations.

3. Main Results

3.1. On the Difference Equation (3). Consider the sixth-order difference equations of the form (3), that is,

$$u_{n+6} = \frac{u_n}{A_n + B_n u_n u_{n+2} u_{n+4}}. \quad (15)$$

We impose the symmetry condition (12) and simplify the resulting equation to get

$$S^6 Q + \frac{B_n u_n^2 u_{n+2} S^4 Q + B_n u_n^2 u_{n+4} S^2 Q - A_n Q}{(B_n u_n u_{n+2} u_{n+4} + A_n)^2} = 0. \quad (16)$$

To solve for Q , we first differentiate (16) with respect to u_n (keeping Ω fixed and viewing u_{n+2} as a function of u_n, u_{n+4} , and Ω). This leads, after simplification, to

$$S^4 Q + u_{n+4} (S^2 Q)' - u_{n+4} Q' + \frac{2u_{n+4}}{u_n} Q = 0, \quad (17)$$

where $'$ denotes the derivative with respect to the independent variable. We then differentiate (17) with respect to u_n to get

$$Q''(n, u_n) - \frac{2}{u_n} Q'(n, u_n) + \frac{2}{u_n^2} Q(n, u_n) = 0. \quad (18)$$

The solutions of (18) are given by

$$Q(n, u_n) = \alpha_n u_n^2 + \beta_n u_n, \quad (19)$$

for some functions α_n and β_n of n . To obtain more information on α_n and β_n , we substitute (19) in (16) and we split the resulting equation to get the following:

$$u_n u_{n+2} u_{n+4}^2 : B_n \alpha_{n+4} = 0, \quad (20a)$$

$$u_n u_{n+2}^2 u_{n+4} : B_n \alpha_{n+2} = 0, \quad (20b)$$

$$u_n u_{n+2} u_{n+4} : B_n (\beta_{n+2} + \beta_{n+4} + \beta_{n+6}) = 0, \quad (20c)$$

$$u_n : -A_n \alpha_n + \alpha_{n+6} = 0, \quad (20d)$$

$$1 : A_n (\beta_n - \beta_{n+6}) = 0. \quad (20e)$$

These equations ((20a)–(20e)) reduce to

$$\begin{aligned} \alpha_n &= 0, \\ \beta_{n+4} + \beta_{n+2} + \beta_n &= 0. \end{aligned} \quad (21)$$

The expression of β_n in (21) is merely obtained by solving the corresponding characteristic equation $r^4 + r^2 + 1 = 0$ for r . Assuming that r_i , $i = 1, 2, 3, 4$, are the solutions of this characteristic equation, then β_n is a linear combination of the r_i^n 's. In other words, the solutions of (21) are

$$\begin{aligned} \alpha_n &= 0, \\ \beta_n &= (-1)^n \beta^n c_1 + (-1)^n \bar{\beta}^n c_2 + \bar{\beta}^n c_3 + \beta^n c_4, \end{aligned} \quad (22)$$

for some arbitrary constants c_i , $i = 1, \dots, 4$, where $\beta = \exp(\pi i/3)$. Thus, we obtain four characteristics given by

$$\begin{aligned} Q_1 &= (-1)^n \beta^n u_n, \\ Q_2 &= (-1)^n \bar{\beta}^n u_n, \\ Q_3 &= \bar{\beta}^n u_n, \\ Q_4 &= \beta^n u_n. \end{aligned} \quad (23)$$

The four corresponding symmetry generators X_1, X_2, X_3 , and X_4 are given by

$$\begin{aligned} X_1 &= (-1)^n \beta^n u_n \partial_{u_n} - (-1)^n \beta^{n+1} u_{n+1} \partial_{u_{n+1}} + (-1)^n \beta^{n+2} u_{n+2} \partial_{u_{n+2}} \\ &\quad - (-1)^n \beta^{n+3} u_{n+3} \partial_{u_{n+3}} + (-1)^n \beta^{n+4} u_{n+4} \partial_{u_{n+4}} - (-1)^n \beta^{n+5} u_{n+5} \partial_{u_{n+5}}, \end{aligned} \quad (24a)$$

$$\begin{aligned} X_2 &= (-1)^n \bar{\beta}^n u_n \partial_{u_n} - (-1)^n \bar{\beta}^{n+1} u_{n+1} \partial_{u_{n+1}} + (-1)^n \bar{\beta}^{n+2} u_{n+2} \partial_{u_{n+2}} \\ &\quad - (-1)^n \bar{\beta}^{n+3} u_{n+3} \partial_{u_{n+3}} + (-1)^n \bar{\beta}^{n+4} u_{n+4} \partial_{u_{n+4}} - (-1)^n \bar{\beta}^{n+5} u_{n+5} \partial_{u_{n+5}}, \end{aligned} \quad (24b)$$

$$\begin{aligned} X_3 &= \bar{\beta}^n u_n \partial_{u_n} + \bar{\beta}^{n+1} u_{n+1} \partial_{u_{n+1}} + \bar{\beta}^{n+2} u_{n+2} \partial_{u_{n+2}} + \bar{\beta}^{n+3} u_{n+3} \partial_{u_{n+3}} \\ &\quad + \bar{\beta}^{n+4} u_{n+4} \partial_{u_{n+4}} + \bar{\beta}^{n+5} u_{n+5} \partial_{u_{n+5}}, \end{aligned} \quad (24c)$$

$$\begin{aligned} X_4 &= \beta^n u_n \partial_{u_n} + \beta^{n+1} u_{n+1} \partial_{u_{n+1}} + \beta^{n+2} u_{n+2} \partial_{u_{n+2}} + \beta^{n+3} u_{n+3} \partial_{u_{n+3}} \\ &\quad + \beta^{n+4} u_{n+4} \partial_{u_{n+4}} + \beta^{n+5} u_{n+5} \partial_{u_{n+5}}. \end{aligned} \quad (24d)$$

Here, using Q_4 , we introduce the canonical coordinate [8]

$$s_n = \int \frac{du_n}{\beta^n u_n} = \frac{1}{\beta^n} \ln |u_n|. \quad (25)$$

In view of (21), we introduce the variable

$$r_n = \beta^{n+4} s_{n+4} + \beta^{n+2} s_{n+2} + \beta^n s_n. \quad (26)$$

It is easy to check that

$$X_i r_n = \beta^{n+4} + \beta^{n+2} + \beta^n = 0, \quad i = 1, \dots, 4. \quad (27)$$

Therefore, r_n is invariant under X_4 . It is advantageous to use

$$|\tilde{r}_n| = \exp(-r_n), \quad (28)$$

that is, $\tilde{r}_n = \pm 1/(u_n u_{n+2} u_{n+4})$. Here, we choose to use the plus sign and we show, using (3), that

$$\tilde{r}_{n+2} = A_n \tilde{r}_n + B_n. \quad (29)$$

Therefore,

$$\tilde{r}_{2n+k} = \tilde{r}_k \left(\prod_{k_1=0}^{n-1} A_{2k_1+k} \right) + \sum_{l=0}^{n-1} \left(B_{2l+k} \prod_{k_2=l+1}^{n-1} A_{2k_2+k} \right), \quad k = 0, 1. \quad (30)$$

It is worthwhile to mention that equation (30) gives the solution of (29) for all n . By reversing all the change of variables, we have

$$\begin{aligned} |u_n| &= \exp \left[(-\beta)^n c_5 + (\beta)^n c_6 + (\bar{\beta})^n c_7 + (-\bar{\beta})^n c_8 + (-\beta)^{n+1} \sum_{k_1=0}^{n-1} \frac{-i\sqrt{3}}{6} (-\bar{\beta})^{k_1} r_{k_1} + (\beta)^{n+1} \sum_{k_2=0}^{n-1} \frac{i\sqrt{3}}{6} (\bar{\beta})^{k_2} r_{k_2} \right. \\ &\quad \left. + (\bar{\beta})^{n+1} \sum_{k_3=0}^{n-1} \frac{-i\sqrt{3}}{6} \beta^{k_3} r_{k_3} + (-\bar{\beta})^{n+1} \sum_{k_4=0}^{n-1} \frac{i\sqrt{3}}{6} (-\beta)^{k_4} r_{k_4} \right] \\ &= \Gamma_n \exp \left(\sum_{k=0}^{n-1} \frac{\sqrt{3}}{3} [(-1)^{n+k} + 1] \operatorname{Im}[\gamma(n+1, k)] \ln |\tilde{r}_k| \right), \end{aligned} \quad (31)$$

where $\tilde{r}_k$ is given in (30), $\Gamma_n = \exp\{(-\beta)^n c_5 + \beta^n c_6 + \bar{\beta}^n c_7 + (-\bar{\beta})^n c_8\}$, and $\gamma(l, k) = \beta^l \bar{\beta}^k$.

Note. Equation (31) gives the solution of (3) in a unified manner.

We can simplify (31) further by splitting it into six categories. We have

$$|u_{6n}| = \Gamma_{6n} \exp\left(\sum_{k=0}^{6n-1} \frac{\sqrt{3}}{3} [(-1)^{6n+k} + 1] \operatorname{Im}[\gamma(6n+1, k)] \ln|\tilde{r}_k|\right),$$

$$u_{6n} = u_0 \prod_{s=0}^{n-1} \frac{\tilde{r}_{6s}}{\tilde{r}_{6s+2}}. \quad (32)$$

Similarly, after a straightforward but lengthy computation, we get

$$u_{6n+i} = u_i \prod_{s=0}^{n-1} \frac{\tilde{r}_{6s+i}}{\tilde{r}_{6s+2+i}}, \quad i = 0, \dots, 5. \quad (33)$$

Note. We can obtain (33) using (28) without considering the absolute value function.

Now, using (30) and (33), we obtain the solutions of (3) as follows:

$$u_{6n} = u_0 \prod_{s=0}^{n-1} \frac{(\prod_{k_1=0}^{3s-1} A_{2k_1}) + u_0 u_2 u_4 \sum_{l=0}^{3s-1} (B_{2l} \prod_{k_2=l+1}^{3s-1} A_{2k_2})}{(\prod_{k_1=0}^{3s} A_{2k_1}) + u_0 u_2 u_4 \sum_{l=0}^{3s} (B_{2l} \prod_{k_2=l+1}^{3s} A_{2k_2})}, \quad (34a)$$

$$u_{6n+1} = u_1 \frac{(\prod_{k_1=0}^{3s-1} A_{2k_1+1}) + u_1 u_3 u_5 \sum_{l=0}^{3s-1} (B_{2l+1} \prod_{k_2=l+1}^{3s-1} A_{2k_2+1})}{(\prod_{k_1=0}^{3s} A_{2k_1+1}) + u_1 u_3 u_5 \sum_{l=0}^{3s} (B_{2l+1} \prod_{k_2=l+1}^{3s} A_{2k_2+1})}, \quad (34b)$$

$$u_{6n+2} = u_2 \prod_{s=0}^{n-1} \frac{(\prod_{k_1=0}^{3s} A_{2k_1}) + u_0 u_2 u_4 \sum_{l=0}^{3s} (B_{2l} \prod_{k_2=l+1}^{3s} A_{2k_2})}{(\prod_{k_1=0}^{3s+1} A_{2k_1}) + u_0 u_2 u_4 \sum_{l=0}^{3s+1} (B_{2l} \prod_{k_2=l+1}^{3s+1} A_{2k_2})}, \quad (34c)$$

$$u_{6n+3} = u_3 \prod_{s=0}^{n-1} \frac{(\prod_{k_1=0}^{3s} A_{2k_1+1}) + u_1 u_3 u_5 \sum_{l=0}^{3s} (B_{2l+1} \prod_{k_2=l+1}^{3s} A_{2k_2+1})}{(\prod_{k_1=0}^{3s+1} A_{2k_1+1}) + u_1 u_3 u_5 \sum_{l=0}^{3s+1} (B_{2l+1} \prod_{k_2=l+1}^{3s+1} A_{2k_2+1})}, \quad (34d)$$

$$u_{6n+4} = u_4 \prod_{s=0}^{n-1} \frac{(\prod_{k_1=0}^{3s+1} A_{2k_1}) + u_0 u_2 u_4 \sum_{l=0}^{3s+1} (B_{2l} \prod_{k_2=l+1}^{3s+1} A_{2k_2})}{(\prod_{k_1=0}^{3s+2} A_{2k_1}) + u_0 u_2 u_4 \sum_{l=0}^{3s+2} (B_{2l} \prod_{k_2=l+1}^{3s+2} A_{2k_2})}, \quad (34e)$$

$$u_{6n+5} = u_5 \prod_{s=0}^{n-1} \frac{(\prod_{k_1=0}^{3s+1} A_{2k_1+1}) + u_1 u_3 u_5 \sum_{l=0}^{3s+1} (B_{2l+1} \prod_{k_2=l+1}^{3s+1} A_{2k_2+1})}{(\prod_{k_1=0}^{3s+2} A_{2k_1+1}) + u_1 u_3 u_5 \sum_{l=0}^{3s+2} (B_{2l+1} \prod_{k_2=l+1}^{3s+2} A_{2k_2+1})}, \quad (34f)$$

whenever the denominators do not vanish, i.e.,
 $u_0 u_2 u_4 \sum_{l=0}^s B_{2l} \Theta^0(A_{k_2}, l+1, s) \neq -\Theta^0(A_{k_1}, 0, s)$ and
 $u_1 u_3 u_5 \sum_{l=0}^s B_{2l+1} \Theta^1(A_{k_2}, l+1, s) \neq -\Theta^1(A_{k_1}, 0, s).$

The implication is that solutions of (2) are given by

$$x_{6n-5} = x_{-5} \prod_{s=0}^{n-1} \frac{(\prod_{k_1=0}^{3s-1} a_{2k_1}) + x_{-5} x_{-3} x_{-1} \sum_{l=0}^{3s-1} (b_{2l} \prod_{k_2=l+1}^{3s-1} a_{2k_2})}{(\prod_{k_1=0}^{3s} a_{2k_1}) + x_{-5} x_{-3} x_{-1} \sum_{l=0}^{3s} (b_{2l} \prod_{k_2=l+1}^{3s} a_{2k_2})}, \quad (35a)$$

$$x_{6n-4} = x_{-4} \prod_{s=0}^{n-1} \frac{(\prod_{k_1=0}^{3s-1} a_{2k_1+1}) + x_{-4} x_{-2} x_0 \sum_{l=0}^{3s-1} (b_{2l+1} \prod_{k_2=l+1}^{3s-1} a_{2k_2+1})}{(\prod_{k_1=0}^{3s} a_{2k_1+1}) + x_{-4} x_{-2} x_0 \sum_{l=0}^{3s} (b_{2l+1} \prod_{k_2=l+1}^{3s} a_{2k_2+1})}, \quad (35b)$$

$$x_{6n-3} = x_{-3} \prod_{s=0}^{n-1} \frac{(\prod_{k_1=0}^{3s} a_{2k_1}) + x_{-5} x_{-3} x_{-1} \sum_{l=0}^{3s} (b_{2l} \prod_{k_2=l+1}^{3s} a_{2k_2})}{(\prod_{k_1=0}^{3s+1} a_{2k_1}) + x_{-5} x_{-3} x_{-1} \sum_{l=0}^{3s+1} (b_{2l} \prod_{k_2=l+1}^{3s+1} a_{2k_2})}, \quad (35c)$$

$$x_{6n-2} = x_{-2} \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{3s} a_{2k_1+1} \right) + x_{-4} x_{-2} x_0 \sum_{l=0}^{3s} \left(b_{2l+1} \prod_{k_2=l+1}^{3s} a_{2k_2+1} \right)}{\left(\prod_{k_1=0}^{3s+1} a_{2k_1+1} \right) + x_{-4} x_{-2} x_0 \sum_{l=0}^{3s+1} \left(b_{2l+1} \prod_{k_2=l+1}^{3s+1} a_{2k_2+1} \right)}, \quad (35d)$$

$$x_{6n-1} = x_{-1} \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{3s+1} a_{2k_1} \right) + x_{-5} x_{-3} x_{-1} \sum_{l=0}^{3s+1} \left(b_{2l} \prod_{k_2=l+1}^{3s+1} a_{2k_2} \right)}{\left(\prod_{k_1=0}^{3s+2} a_{2k_1} \right) + x_{-5} x_{-3} x_{-1} \sum_{l=0}^{3s+2} \left(b_{2l} \prod_{k_2=l+1}^{3s+2} a_{2k_2} \right)}, \quad (35e)$$

$$x_{6n} = x_0 \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{3s+1} a_{2k_1+1} \right) + x_{-4} x_{-2} x_0 \sum_{l=0}^{3s+1} \left(b_{2l+1} \prod_{k_2=l+1}^{3s+1} a_{2k_2+1} \right)}{\left(\prod_{k_1=0}^{3s+2} a_{2k_1+1} \right) + x_{-4} x_{-2} x_0 \sum_{l=0}^{3s+2} \left(b_{2l+1} \prod_{k_2=l+1}^{3s+2} a_{2k_2+1} \right)}, \quad (35f)$$

whenever the denominators do not vanish, i.e., $x_{-5} x_{-3} x_{-1} \sum_{l=0}^s b_{2l} \Theta^0(a_{k_2}, l+1, s) \neq -\Theta^0(a_{k_1}, 0, s)$ and $x_{-4} x_{-2} x_0 \sum_{l=0}^s b_{2l+1} \Theta^1(a_{k_2}, l+1, s) \neq -\Theta^1(a_{k_1}, 0, s)$.

3.2. The Case Where (a_n) and (b_n) Are Two Periodic Sequences.

Let $a_n = (\lambda, \mu, \lambda, \mu, \lambda, \dots)$, $b_n = (\eta, \zeta, \eta, \zeta, \dots)$, $\Phi = x_{-5} x_{-3} x_{-1}$, and $\Psi = x_{-4} x_{-2} x_0$. The solution in this case is given by the following equations:

$$x_{6n-5} = x_{-5} \prod_{s=0}^{n-1} \frac{\lambda^{3s} + \eta \Phi \sum_{l=0}^{3s-1} \lambda^l}{\lambda^{3s+1} + \eta \Phi \sum_{l=0}^{3s} \lambda^l}, \quad (36a)$$

$$x_{6n-4} = x_{-4} \prod_{s=0}^{n-1} \frac{\mu^{3s} + \zeta \Psi \sum_{l=0}^{3s-1} \mu^l}{\mu^{3s+1} + \zeta \Psi \sum_{l=0}^{3s} \mu^l},$$

$$x_{6n-3} = x_{-3} \prod_{s=0}^{n-1} \frac{\lambda^{3s+1} + \eta \Phi \sum_{l=0}^{3s} \lambda^l}{\lambda^{3s+2} + \eta \Phi \sum_{l=0}^{3s+1} \lambda^l}, \quad (36b)$$

$$x_{6n-2} = x_{-2} \prod_{s=0}^{n-1} \frac{\mu^{3s+1} + \zeta \Psi \sum_{l=0}^{3s} \mu^l}{\mu^{3s+2} + \zeta \Psi \sum_{l=0}^{3s+1} \mu^l},$$

$$x_{6n-1} = x_{-1} \prod_{s=0}^{n-1} \frac{\lambda^{3s+2} + \eta \Phi \sum_{l=0}^{3s+1} \lambda^l}{\lambda^{3s+3} + \eta \Phi \sum_{l=0}^{3s+2} \lambda^l}, \quad (36c)$$

$$x_{6n} = x_0 \prod_{s=0}^{n-1} \frac{\mu^{3s+2} + \zeta \Psi \sum_{l=0}^{3s+1} \mu^l}{\mu^{3s+3} + \zeta \Psi \sum_{l=0}^{3s+2} \mu^l},$$

where $(\sum_{l=0}^{s-1} \lambda^l) \eta \Phi \neq -\lambda^s$ and $(\sum_{l=0}^{s-1} \lambda^l) \eta \Psi \neq -\lambda^s$, $s \leq 3n$.

3.3. The Case Where (a_n) and (b_n) Are Constants. Here, $a_n = \mu = \lambda$, $b_n = \eta = \zeta$, $\Phi = x_{-5} x_{-3} x_{-1}$, and $\Psi = x_{-4} x_{-2} x_0$. Equation (2) becomes $x_{n+1} = x_{n-5}/(\lambda + \eta x_{n-1} x_{n-3} x_{n-5})$.

3.3.1. $\lambda = 1$. The solution given in (35a)–(35f) simplifies to

$$x_{6n-5} = x_{-5} \prod_{s=0}^{n-1} \frac{1 + (3s) \eta \Phi}{1 + (3s+1) \eta \Phi}, \quad (37a)$$

$$x_{6n-4} = x_{-4} \prod_{s=0}^{n-1} \frac{1 + (3s) \eta \Psi}{1 + (3s+1) \eta \Psi},$$

$$x_{6n-3} = x_{-3} \prod_{s=0}^{n-1} \frac{1 + (3s+1) \eta \Phi}{1 + (3s+2) \eta \Phi}, \quad (37b)$$

$$x_{6n-2} = x_{-2} \prod_{s=0}^{n-1} \frac{1 + (3s+1) \eta \Psi}{1 + (3s+2) \eta \Psi},$$

$$x_{6n-1} = x_{-1} \prod_{s=0}^{n-1} \frac{1 + (3s+2) \eta \Phi}{1 + (3s+3) \eta \Phi}, \quad (37c)$$

$$x_{6n} = x_0 \prod_{s=0}^{n-1} \frac{1 + (3s+2) \eta \Psi}{1 + (3s+3) \eta \Psi},$$

where $s \eta \Phi \neq -1$ and $s \eta \Psi \neq -1$, $s \leq 3n$.

(i) If $\eta = 1$, then equations in (37a)–(37c) are exactly the ones obtained in Theorem 2.1 in [4] for

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-1} x_{n-3} x_{n-5}}, \quad (38)$$

and their restriction (the initial conditions are arbitrary nonzero positive real numbers) is a special case of our restriction (the initial conditions are arbitrary nonzero real numbers and $s\Phi \neq -1$, $s\Psi \neq -1$, and $0 \leq s \leq 3n$).

(ii) If $\eta = -1$, then equations in (37a)–(37c) are exactly the ones obtained in Theorem 3.1 in [4] for

$$x_{n+1} = \frac{x_{n-5}}{1 - x_{n-1} x_{n-3} x_{n-5}}, \quad (39)$$

and their restriction (the initial conditions are arbitrary nonzero real numbers and $j b d f \neq 1$ and $j a c e \neq 1$) coincides with our restriction (the initial conditions are arbitrary nonzero real numbers and $s\Phi \neq 1$, $s\Psi \neq 1$, and $0 \leq s \leq 3n$).

3.3.2. *The Case Where $\lambda \neq 1$.* In this case, the solution given in (35a)–(35f) simplifies to

$$\begin{aligned}
 x_{6n-5} &= x_{-5} \prod_{s=0}^{n-1} \frac{\lambda^{3s} + \eta\Phi((1-\lambda^{3s})/(1-\lambda))}{\lambda^{3s+1} + \eta\Phi((1-\lambda^{3s+1})/(1-\lambda))}, \\
 x_{6n-4} &= x_{-4} \prod_{s=0}^{n-1} \frac{\lambda^{3s} + \zeta\Psi((1-\lambda^{3s})/(1-\lambda))}{\lambda^{3s+1} + \zeta\Psi((1-\lambda^{3s+1})/(1-\lambda))}, \\
 x_{6n-3} &= x_{-3} \prod_{s=0}^{n-1} \frac{\lambda^{3s+1} + \eta\Phi((1-\lambda^{3s+1})/(1-\lambda))}{\lambda^{3s+2} + \eta\Phi((1-\lambda^{3s+2})/(1-\lambda))}, \\
 x_{6n-2} &= x_{-2} \prod_{s=0}^{n-1} \frac{\lambda^{3s+1} + \zeta\Psi((1-\lambda^{3s+1})/(1-\lambda))}{\lambda^{3s+2} + \zeta\Psi((1-\lambda^{3s+2})/(1-\lambda))}, \\
 x_{6n-1} &= x_{-1} \prod_{s=0}^{n-1} \frac{\lambda^{3s+2} + \eta\Phi((1-\lambda^{3s+2})/(1-\lambda))}{\lambda^{3s+3} + \eta\Phi((1-\lambda^{3s+3})/(1-\lambda))}, \\
 x_{6n} &= x_0 \prod_{s=0}^{n-1} \frac{\lambda^{3s+2} + \zeta\Psi((1-\lambda^{3s+2})/(1-\lambda))}{\lambda^{3s+3} + \zeta\Psi((1-\lambda^{3s+3})/(1-\lambda))},
 \end{aligned} \tag{40}$$

where $(1-\lambda^s)\eta\Phi \neq -\lambda^s(1-\lambda)$ and $(1-\lambda^s)\zeta\Psi \neq -\lambda^s(1-\lambda)$, $s \leq 3n$.

Note. If $\lambda = -1$, then the solution given in Section 3.3.2 simplifies to

$$\begin{aligned}
 x_{12n-5} &= x_{6(2n)-5} = x_{-5}, \\
 x_{12n-4} &= x_{-4}, \\
 x_{12n-3} &= x_{-3}, \\
 x_{12n-2} &= x_{-2}, \\
 x_{12n-1} &= x_{-1}, \\
 x_{12n} &= x_0, \\
 x_{12n+1} &= \frac{x_{-5}}{-1 + \eta\Phi}, \\
 x_{12n+2} &= \frac{x_{-4}}{-1 + \zeta\Psi}, \\
 x_{12n+3} &= x_{-3}(-1 + \eta\Phi), \\
 x_{12n+4} &= x_{-2}(-1 + \zeta\Psi), \\
 x_{12n+5} &= \frac{x_{-1}}{(-1 + \eta\Phi)}, \\
 x_{12n+6} &= \frac{x_0}{-1 + \zeta\Psi}, \\
 x_{12n+7} &= x_{6(2n+2)-5} = x_{-5}.
 \end{aligned} \tag{41}$$

(i) When setting $\eta = 1$ in Section 3.3.2, we get the result obtained in Theorem 4.1 in [4] for

$$x_{n+1} = \frac{x_{n-5}}{-1 + x_{n-1}x_{n-3}x_{n-5}}, \tag{42}$$

and their restriction coincides with our restriction (the initial conditions are arbitrary nonzero real numbers, $\Phi \neq 1$ and $\Psi \neq 1$).

(ii) When setting $\eta = -1$ in Section 3.3.2, we get the result obtained in Theorem 5.1 in [4] for

$$x_{n+1} = \frac{x_{n-5}}{-1 - x_{n-1}x_{n-3}x_{n-5}}, \tag{43}$$

and their restriction coincides with our restriction (the initial conditions are arbitrary nonzero real numbers, $\Phi \neq -1$ and $\Psi \neq -1$).

4. On the Difference Equation (5)

Consider the fifth-order difference equations of the form (5), i.e.,

$$u_{n+5} = \Omega = \frac{u_n u_{n+1} u_{n+2}}{u_{n+3} u_{n+4} (\lambda + \mu u_n u_{n+1} u_{n+2})}. \tag{44}$$

Here, the procedure for finding the characteristics of (5) is similar as above and is as follows:

- (i) Impose the symmetry condition (12) to (5)
- (ii) Differentiate with respect to u_n (keeping Ω fixed) and viewing u_{n+3} as a function of u_n, u_{n+1}, u_{n+2} , and Ω
- (iii) Differentiate with respect to u_n twice (keeping u_{n+1} fixed)
- (iv) Use the method of separation

After performing this series of operations, we obtain the following characteristics:

$$\begin{aligned}
 Q_5(n, u_n) &= \beta^n u_n, \\
 Q_6 &= \bar{\beta}^n u_n,
 \end{aligned} \tag{45}$$

where $\beta = \exp(-2\pi i/3)$. Thus, we obtain two characteristics with corresponding generators given by

$$\begin{aligned}
 X_5 &= \bar{\beta}^n u_n \partial u_n + \bar{\beta}^{n+1} u_{n+1} \partial u_{n+1} + \bar{\beta}^n u_{n+2} \partial u_{n+2} \\
 &\quad + \bar{\beta}^n u_{n+3} \partial u_{n+3} + \bar{\beta}^{n+1} u_{n+4} \partial u_{n+4},
 \end{aligned} \tag{46a}$$

$$\begin{aligned}
 X_6 &= \beta^n u_n \partial u_n + \beta^{n+1} u_{n+1} \partial u_{n+1} + \beta^n u_{n+2} \partial u_{n+2} \\
 &\quad + \beta^n u_{n+3} \partial u_{n+3} + \beta^{n+1} u_{n+4} \partial u_{n+4}.
 \end{aligned} \tag{46b}$$

From the characteristic equations

$$\frac{du_n}{\bar{\beta}^n u_n} = \frac{du_{n+1}}{\bar{\beta}^{n+1} u_{n+1}} = \frac{du_{n+2}}{\bar{\beta}^{n+2} u_{n+2}} = \frac{du_{n+3}}{\bar{\beta}^{n+3} u_{n+3}} = \frac{du_{n+4}}{\bar{\beta}^{n+4} u_{n+4}} \left(= \frac{V_n}{0} \right), \quad (47)$$

we obtain the invariants $c_1 = u_{n+1}^\beta / u_n$, $c_2 = u_{n+2}^{\bar{\beta}} / u_n$, $c_3 = u_n u_{n+3}$, $c_4 = u_{n+4}^\beta / u_n$, and $c_5 = V_n$. We readily notice that

$$S^3 \left(\frac{du_n}{\bar{\beta}^n u_n} = \frac{du_{n+1}}{\bar{\beta}^{n+1} u_{n+1}} \right) = \frac{du_{n+3}}{\bar{\beta}^{n+3} u_{n+3}} = \frac{du_{n+4}}{\bar{\beta}^{n+4} u_{n+4}}, \quad (48)$$

and we choose $V_n = c_1^{\bar{\beta}} c_2^\beta$, i.e.,

$$V_n = u_n u_{n+1} u_{n+2}. \quad (49)$$

By shifting (49) thrice, we get

$$V_{n+3} = \frac{V_n}{\lambda + \mu V_n}, \quad (50)$$

whose solution is given by

$$V_n = \begin{cases} \left(\frac{\mu [((-1)^{2/3}-1)n - (-1)^{1/3} + 1 + \beta]}{3((-1)^{2/3}-1)} + c_6 + \bar{\beta}^n c_7 + \beta^n c_8 \right)^{-1}, & \text{if } \lambda = 1, \\ (c_{10} \lambda^{n/3} + [(-1)^{2/3} \lambda^{1/3}]^n c_8 + [(-1)^{1/3} \lambda^{1/3}]^n c_9 + \frac{\mu}{1-\lambda})^{-1}, & \text{if } \lambda \neq 1. \end{cases} \quad (51)$$

The constants c_i , $i = 6, \dots, 10$, can be obtained from the following equations:

$$c_6 + c_7 + c_8 = \frac{1}{u_0 u_1 u_2} - \frac{\mu [-(-1)^{1/3} + 1 + \beta]}{3((-1)^{2/3}-1)}, \quad (52a)$$

$$c_6 + \bar{\beta} c_7 + \beta c_8 = \frac{1}{u_1 u_2 u_3} - \frac{\mu [\beta - (-1)^{1/3} + (-1)^{2/3}]}{3((-1)^{2/3}-1)}, \quad (52b)$$

$$c_6 + \beta c_7 + \bar{\beta} c_8 = \frac{1}{u_2 u_3 u_4} - \frac{\mu [-1 + 2(-1)^{2/3} - (-1)^{1/3} + \beta]}{3((-1)^{2/3}-1)}, \quad (52c)$$

$$c_{10} + c_8 + c_9 = \frac{1}{u_0 u_1 u_2} - \frac{\mu}{1-\lambda}, \quad (52d)$$

$$\lambda^{1/3} c_{10} + [(-1)^{2/3} \lambda^{1/3}] c_8 + [(-1)^{1/3} \lambda^{1/3}] c_9 = \frac{1}{u_1 u_2 u_3} - \frac{\mu}{1-\lambda}, \quad (52e)$$

$$\lambda^{2/3} c_{10} + [(-1)^{2/3} \lambda^{1/3}]^2 c_8 + [(-1)^{1/3} \lambda^{1/3}]^2 c_9 = \frac{1}{u_2 u_3 u_4} - \frac{\mu}{1-\lambda}. \quad (52f)$$

Thanks to (49), we can express u_n in terms of V_n as follows:

$$u_n = \exp \left(\beta^n c_{11} + \bar{\beta}^n c_{12} - \frac{2}{\sqrt{3}} \left[\sum_{k=0}^{n-1} \operatorname{Im}(\gamma(n, k)) \ln V_k \right] \right), \quad (53a)$$

where V_k is given in (51) with $\gamma(n, k) = \beta^n \bar{\beta}^{k+1}$. The constants c_{11} and c_{12} must satisfy

$$\begin{aligned} c_{11} + c_{12} &= \ln u_0, \\ \beta c_{11} + \bar{\beta} c_{12} &= \ln u_1. \end{aligned} \quad (53b)$$

Equations in (53a) and (53b) give the solutions of (5) in a unified manner.

For the sake of clarification, we split solutions (53a) to realize the solutions in the existing literature. Using (53a) and (53b), we have

$$u_{6n} = u_0 \prod_{s=1}^{2n} \frac{V_{3s-2}}{V_{3s-3}}. \quad (54)$$

Using the same approach, we show that

$$u_{6n+i} = u_i \prod_{s=1}^{2n} \frac{V_{3(s-1)+i+1}}{V_{3(s-1)+i}}, \quad i = 0, \dots, 5. \quad (55)$$

4.1. *The Case of $\lambda = 1$.* Equation (5) becomes

$$u_{n+5} = \frac{u_n u_{n+1} u_{n+2}}{u_{n+3} u_{n+4} (1 + \mu u_n u_{n+1} u_{n+2})}, \quad (56)$$

and we said earlier that the solution of (50), in this case, is (51), i.e.,

$$V_n = \left(\frac{\mu \left[(-1)^{2/3} - 1 \right] n - (-1)^{1/3} + 1 + \beta}{3 \left((-1)^{2/3} - 1 \right)} + c_6 + \beta^n c_7 + \beta^n c_8 \right)^{-1}. \quad (57)$$

We have

$$V_{3s} = \left(\frac{\mu \left[(-1)^{2/3} - 1 \right] (3s) - (-1)^{1/3} + 1 + \beta}{3 \left((-1)^{2/3} - 1 \right)} + c_6 + c_7 + c_8 \right)^{-1}, \quad (58)$$

and using (52a) in (58), we get

$$V_{3s} = \frac{u_0 u_1 u_2}{1 + \mu s u_0 u_1 u_2}. \quad (59)$$

Using the same approach, we show that

$$V_{3s} = \frac{u_0 u_1 u_2}{1 + \mu s u_0 u_1 u_2},$$

$$V_{3s+1} = \frac{u_1 u_2 u_3}{1 + \mu s u_1 u_2 u_3}, \quad (60)$$

$$V_{3s+2} = \frac{u_2 u_3 u_4}{1 + \mu s u_2 u_3 u_4}.$$

Let $a = x_{-4}$, $b = x_{-3}$, $c = x_{-2}$, and $d = x_{-1}$ and $A = u_0$, $B = u_1$, $C = u_2$, $D = u_3$, and $E = u_4$. Using (60) in (55), we obtain the solution of (56) as follows:

$$\begin{aligned} u_{6n} &= \frac{D^{2n}}{A^{2n-1}} \prod_{s=1}^{2n-1} \frac{1 + \mu s ABC}{1 + \mu s BCD}, \\ u_{6n+1} &= \frac{E^{2n}}{B^{2n-1}} \prod_{s=1}^{2n-1} \frac{1 + \mu s BCD}{1 + \mu s CDE}, \\ u_{6n+2} &= \frac{CA^{2n} B^{2n}}{D^{2n} E^{2n}} \prod_{s=0}^{2n-1} \frac{1 + \mu s CDE}{1 + \mu (s+1) ABC}, \\ u_{6n+3} &= \frac{D^{2n+1}}{A^{2n}} \prod_{s=1}^{2n} \frac{1 + \mu s ABC}{1 + \mu s BCD}, \\ u_{6n+4} &= \frac{E^{2n+1}}{B^{2n}} \prod_{s=0}^{2n} \frac{1 + \mu s BCD}{1 + \mu s CDE}, \\ u_{6n+5} &= \frac{C(AB)^{2n+1}}{(DE)^{2n+1}} \prod_{s=0}^{2n} \frac{1 + \mu s CDE}{1 + \mu (s+1) ABC}, \end{aligned} \quad (61)$$

whenever the denominators do not vanish.

4.2. *The Case of $\lambda \neq 1$.* In this case, as we found earlier, the solution of (50) is given by (51), i.e.,

$$V_n = \left(c_{10} \lambda^{n/3} + [(-1)^{2/3} \lambda^{1/3}]^n c_8 + [(-1)^{1/3} \lambda^{1/3}]^n c_9 + \frac{\mu}{1 - \lambda} \right)^{-1}. \quad (62)$$

Using this, we get

$$V_{3s} = \left(c_{10} \lambda^s + c_8 \lambda^s + c_9 \lambda^s + \frac{\mu}{1 - \lambda} \right)^{-1}, \quad (63)$$

and using (52d) in (63), we find

$$V_{3s} = \frac{ABC}{\lambda^s + \mu ABC ((1 - \lambda^s)/(1 - \lambda))}. \quad (64)$$

Using the same approach, we show that

$$V_{3s+i} = \frac{u_i u_{i+1} u_{i+2}}{\lambda^s + \mu u_i u_{i+1} u_{i+2} ((1 - \lambda^s)/(1 - \lambda))}, \quad i = 0, 1, 2. \quad (65)$$

Using (65) in (55), we obtain the solution of (5) as follows:

$$\begin{aligned} u_{6n} &= \frac{D^{2n}}{A^{2n-1}} \prod_{s=2}^{2n} \frac{\lambda^{s-1} + \mu \Delta_0 \sum_{j=0}^{s-2} \lambda^j}{\lambda^{s-1} + \mu \Delta_1 \sum_{j=0}^{s-2} \lambda^j}, \\ u_{6n+1} &= \frac{E^{2n}}{B^{2n-1}} \prod_{s=2}^{2n} \frac{\lambda^{s-1} + \mu \Delta_1 \sum_{j=0}^{s-2} \lambda^j}{\lambda^{s-1} + \mu \Delta_2 \sum_{j=0}^{s-2} \lambda^j}, \\ u_{6n+2} &= C \frac{\Delta_0^{2n}}{\Delta_1^{2n}} \frac{\prod_{s=2}^{2n} (\lambda^{s-1} + \mu \Delta_2 \sum_{j=0}^{s-2} \lambda^j)}{\prod_{s=1}^{2n} (\lambda^s + \mu \Delta_0 \sum_{j=0}^{s-1} \lambda^j)}, \\ u_{6n+3} &= \frac{D^{2n+1}}{A^{2n}} \prod_{s=2}^{2n+1} \frac{\lambda^{s-1} + \mu \Delta_0 \sum_{j=0}^{s-2} \lambda^j}{\lambda^{s-1} + \mu \Delta_1 \sum_{j=0}^{s-2} \lambda^j}, \\ u_{6n+4} &= E \frac{\Delta_2^{2n}}{\Delta_1^{2n}} \prod_{s=2}^{2n+1} \frac{\lambda^{s-1} + \mu \Delta_1 \sum_{j=0}^{s-2} \lambda^j}{\lambda^{s-1} + \mu \Delta_2 \sum_{j=0}^{s-2} \lambda^j}, \\ u_{6n+5} &= C \frac{\Delta_0^{2n+1}}{\Delta_1^{2n+1}} \frac{\prod_{s=2}^{2n+1} (\lambda^{s-1} + \mu \Delta_2 \sum_{j=0}^{s-2} \lambda^j)}{\prod_{s=1}^{2n+1} (\lambda^s + \mu \Delta_0 \sum_{j=0}^{s-1} \lambda^j)}, \end{aligned} \quad (66)$$

where $\Delta_i = u_i u_{i+1} u_{i+2}$. Equations in Section 4.2 give the exact solution of (5) for any real values of λ and μ provided that the denominators do not vanish.

Recall that we acted the shift operator on (4) to get (5). Hence, the solutions of (4) are obtained, using Section 4.2, as follows:

$$\begin{aligned}
x_{6n-4} &= \frac{d^{2n}}{a^{2n-1}} \prod_{s=2}^{2n} \frac{\lambda^{s-1} + \mu abc \sum_{j=0}^{s-2} \lambda^j}{\lambda^{s-1} + \mu bcd \sum_{j=0}^{s-2} \lambda^j}, \\
x_{6n-3} &= \frac{e^{2n}}{b^{2n-1}} \prod_{s=2}^{2n} \frac{\lambda^{s-1} + \mu bcd \sum_{j=0}^{s-2} \lambda^j}{\lambda^{s-1} + \mu cde \sum_{j=0}^{s-2} \lambda^j}, \\
x_{6n-2} &= \frac{c(ab)^{2n}}{(de)^{2n}} \frac{\prod_{s=2}^{2n} (\lambda^{s-1} + \mu cde \sum_{j=0}^{s-2} \lambda^j)}{\prod_{s=1}^{2n} (\lambda^s + \mu abc \sum_{j=0}^{s-1} \lambda^j)}, \\
x_{6n-1} &= \frac{d^{2n+1}}{a^{2n}} \prod_{s=2}^{2n+1} \frac{\lambda^{s-1} + \mu abc \sum_{j=0}^{s-2} \lambda^j}{\lambda^{s-1} + \mu bcd \sum_{j=0}^{s-2} \lambda^j}, \\
x_{6n} &= \frac{e^{2n+1}}{b^{2n}} \prod_{s=2}^{2n+1} \frac{\lambda^{s-1} + \mu bcd \sum_{j=0}^{s-2} \lambda^j}{\lambda^{s-1} + \mu cde \sum_{j=0}^{s-2} \lambda^j}, \\
x_{6n+1} &= \frac{c(ab)^{2n+1}}{(de)^{2n+1}} \frac{\prod_{s=2}^{2n+1} [\lambda^{s-1} + \mu cde \sum_{j=0}^{s-2} \lambda^j]}{\prod_{s=1}^{2n+1} (\lambda^s + \mu abc \sum_{j=0}^{s-1} \lambda^j)},
\end{aligned} \tag{67}$$

for any real values of λ and μ as long as the denominators do not vanish.

- (i) When $\lambda = 1$ and $\mu = 1$, equations in Section 4.2 yield the results obtained by Yasin Yazlik in Theorem 5 in [9] for

$$x_{n+1} = \frac{x_{n-2}x_{n-3}x_{n-4}}{x_n x_{n-1} (1 + x_{n-2}x_{n-3}x_{n-4})}, \quad n = 0, 1, 2, \dots, \tag{68}$$

where a, b, c, d , and e are positive real numbers.

- (ii) When $\lambda = 1$ and $\mu = -1$, equations in Section 4.2 yield the results obtained by Yasin Yazlik in Theorem 9 in [9] for

$$x_{n+1} = \frac{x_{n-2}x_{n-3}x_{n-4}}{x_n x_{n-1} (1 - x_{n-2}x_{n-3}x_{n-4})}, \quad n = 0, 1, 2, \dots, \tag{69}$$

where a, b, c, d , and e are positive real numbers with $abc \neq 1$ and $cde \neq 1$.

Note. There should not be a minus sign right after the expression of x_{3n-2} in Theorem 9 in [9].

- (i) When $\lambda = -1$ and $\mu = 1$, equations in Section 4.2 yield the results obtained by Yasin Yazlik in Theorem 7 in [9] for

$$x_{n+1} = \frac{x_{n-2}x_{n-3}x_{n-4}}{x_n x_{n-1} (-1 + x_{n-2}x_{n-3}x_{n-4})}, \quad n = 0, 1, 2, \dots, \tag{70}$$

where a, b, c, d , and e are nonzero real numbers with $abc \neq 1$, $bcd \neq 1$, and $cde \neq 1$.

- (ii) When $\lambda = -1$ and $\mu = -1$, equations in Section 4.2 yield the results obtained by Yasin Yazlik in Theorem 11 in [9] for

$$x_{n+1} = \frac{x_{n-2}x_{n-3}x_{n-4}}{x_n x_{n-1} (-1 - x_{n-2}x_{n-3}x_{n-4})}, \quad n = 0, 1, 2, \dots, \tag{71}$$

where a, b, c, d , and e are nonzero real numbers with $abc \neq -1$, $bcd \neq -1$, and $cde \neq -1$.

Note. There should be a minus sign right after the expression of x_{6n+1} in Theorem 11 in [9].

5. Conclusion

In this paper, we have obtained nontrivial symmetries of some rational ordinary difference equations and their exact solutions were obtained. Most importantly, we note that (21) gives a clear idea, without making any lucky guesses, for the most convenient choice of the invariant of (3).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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