Research Article

On Katugampola Fourier Transform

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The aim of this article is to introduce a new definition for the Fourier transform. This new definition will be considered as one of the generalizations of the usual (classical) Fourier transform. We employ the new Katugampola derivative to obtain some properties of the Katugampola Fourier transform and find the relation between the Katugampola Fourier transform and the usual Fourier transform. The inversion formula and the convolution theorem for the Katugampola Fourier transform are considered.

1. Introduction

It is well known that fractional calculus is a generalization of the classical integer calculus, where several types of fractional derivatives are introduced and studied such as Riemann–Liouville, Caputo, Hadamard, Weyl, and Grünwald–Letnikov; for more details, one can see [1, 2, 3]. Unfortunately, all these fractional derivatives fail to satisfy some basic properties of the classical integer calculus such as product rule, quotient rule, chain rule, Roll's theorem, mean value theorem, and composition of two functions. Also, those fractional derivatives inherit nonlocality and most of them propose that the derivative of a constant is not zero. Those inconsistencies lead to some difficulties in the applications of fractional derivatives in physics, engineering, and real-world problems.

To overcome all the difficulties raised, Khalil et al. [4] introduced and investigated the so-called conformable fractional derivative, and also Katugampola [5, 6] introduced and studied a similar type of derivative, later called the Katugampola derivative and defined as follows.

Definition 1 [5]. Let $f: [0, \infty) \rightarrow \mathbb{R}$ and $t > 0$. Then, the Katugampola derivative of f of order α is defined by

$$D^\alpha(f)(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(te^{\varepsilon t^{-\alpha}}) - f(t)}{\varepsilon}, \quad (1)$$

where $t > 0$ and $\alpha \in (0, 1]$. If f is α -differentiable in some $(0, a)$, $a > 0$, and $\lim_{t \rightarrow 0^+} D^\alpha(f)(t)$ exists, then $D^\alpha(f)(0) = \lim_{t \rightarrow 0^+} D^\alpha(f)(t)$.

Definition 2 [5]. Let $\alpha \in (n, n+1]$, for some $n \in \mathbb{N}$, and f be an n -differentiable at $t > 0$. Then, the α -fractional derivative of f is defined by

$$D^\alpha f(t) = \lim_{\varepsilon \rightarrow 0} \frac{f^{(n)}(te^{\varepsilon t^{n-\alpha}}) - f^{(n)}(t)}{\varepsilon}, \quad (2)$$

if the limit exists.

Note that the Katugampola derivative satisfies product rule, quotient rule, and chain rule, and it is consistent in its properties with the classical calculus of integer order. In addition, we have the following theorem.

Theorem 1. Let $\alpha \in (n, n+1]$, for some $n \in \mathbb{N}$, and f be an $(n+1)$ -differentiable at $t > 0$. Then,

$$D^\alpha f(t) = t^{n+1-\alpha} f^{(n+1)}(t). \quad (3)$$

Proof.

$$\begin{aligned} D^\alpha(f)(t) &= \lim_{\varepsilon \rightarrow 0} \frac{f^{(n)}(te^{\varepsilon t^{n-\alpha}}) - f^{(n)}(t)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{f^{(n)}(t + \varepsilon t^{n-\alpha+1} + (\varepsilon^2 t^{2n-2\alpha+1}/2!) + (\varepsilon^3 t^{3n-3\alpha+1}/3!) + \dots) - f^{(n)}(t)}{\varepsilon}. \end{aligned} \quad (4)$$

Let $h = \varepsilon t^{n-\alpha+1} [1 + (\varepsilon t^{n-\alpha}/2!) + (\varepsilon^2 t^{2n-2\alpha}/3!) + \dots]$, then $h = \varepsilon t^{n-\alpha+1} [1 + O(\varepsilon)]$, where $h \rightarrow 0$ as $\varepsilon \rightarrow 0$. Hence,

$$\begin{aligned} D^\alpha(f)(t) &= t^{n-\alpha+1} \lim_{h \rightarrow 0} \frac{f^{(n)}(t+h) - f^{(n)}(t)}{h} \\ &= t^{n-\alpha+1} f^{(n+1)}(t). \end{aligned} \quad (5)$$

If $\alpha \in (n, n+1]$, for some $n \in \mathbb{N}$, and f be an $(n+1)$ -differentiable at $t > 0$. Then,

$$D^\alpha f(t) = t^{n+1-\alpha} f^{(n+1)}(t). \quad (6)$$

Note that, for $\alpha \in (0, 1]$, $t > 0$, we have $D^\alpha f(t) = t^{1-\alpha} (df/dt)(t)$.

The conformable and Katugampola derivatives have been investigated and applied to solve ordinary and partial differential equations of noninteger orders in physics, engineering, and other disciplines; some of these research works have been recently published by Anderson and Ulness [7], Cenesiz and Kurt [8], Silva et al. [9], Yavuz [10], Yavuz and Yaskiran [11], Abu Hammad and Khalil [12], Ilie et al. [13], and Kurt et al. [14], and many other valuable works can be found in the literature.

In this research work, we are intended to introduce and study the properties of the Katugampola Fourier transform based on the Katugampola derivative.

2. Katugampola Fourier Transform

From the literature, one can discover that several definitions of fractional Fourier transforms (not necessarily equivalent) have been introduced in recent years. They were motivated by their application to obtain solutions of the problems revealed from quantum mechanics, optics, signal processing, and others.

Negero [15] had studied applications of Fourier transform to partial differential equations. Also, Çenesiz and Kurt [8] introduced the definition of conformable Fourier transform. In this section, we define the Katugampola Fourier transform, obtain some properties of this transform, and find the relation between the Katugampola Fourier transform and the usual Fourier transform. We also obtain the formula of the inverse and the convolution theorem for Katugampola Fourier transform.

Definition 3. Let $\alpha \in (n, n+1]$ and $f(x)$ be a real valued function defined on $(-\infty, \infty)$. The Katugampola Fourier transform of $f(x)$ denoted by $F_\alpha\{f(x)\}(\kappa) = F_\alpha(\kappa)$, $\kappa \in \mathbb{R}$, is defined as

$$\begin{aligned} F_\alpha\{f(x)\}(\kappa) &= F_\alpha(\kappa) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\kappa(x^{\alpha-n}/(\alpha-n))} f(x) x^{\alpha-n-1} dx. \end{aligned} \quad (7)$$

Theorem 2. Let $\alpha \in (n, n+1]$ and $f(x)$ be an α -differentiable real valued function on $(-\infty, \infty)$, and $f(x)$ is $(n+1)$ -differentiable at $x > 0$, such that $f^{(n)}(b) = 0$ as $|b| \rightarrow \infty$, and then

$$F_\alpha\{D^\alpha(f)(x)\}(\kappa) = (i\kappa) F_\alpha\{f^{(n)}(x)\}. \quad (8)$$

Proof. By using Definition 3 and integration by parts, we have

$$\begin{aligned} F_\alpha\{D^\alpha(f)(x)\}(\kappa) &= F_\alpha\{x^{n+1-\alpha} f^{(n+1)}(x)\}(\kappa), \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{\alpha-n+1} f^{(n+1)}(x) e^{-i\kappa(x^{\alpha-n}/(\alpha-n))} x^{n-\alpha-1} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\kappa(x^{\alpha-n}/(\alpha-n))} f^{(n+1)}(x) dx, \\ &= \frac{1}{\sqrt{2\pi}} e^{-i\kappa(x^{\alpha-n}/(\alpha-n))} f^{(n)}(x) \Big|_{-\infty}^{\infty} + \frac{(i\kappa)}{\sqrt{2\pi}} \\ &\quad \cdot \int_{-\infty}^{\infty} e^{-i\kappa(x^{\alpha-n}/(\alpha-n))} f^{(n)}(x) x^{\alpha-n-1} dx. \end{aligned} \quad (9)$$

But $\lim_{|b| \rightarrow \infty} f^{(n)}(b) = 0$, and so

$$\begin{aligned} F_\alpha\{D^\alpha(f)(x)\}(\kappa) &= \frac{(i\kappa)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int e^{-i\kappa(x^{\alpha-n}/(\alpha-n))} f^{(n)}(x) x^{\alpha-n-1} dx \\ &= (i\kappa) F_\alpha\{f^{(n)}(x)\}. \end{aligned} \quad (10)$$

The following Lemma is the relation between the Katugampola Fourier transform and the usual Fourier transform.

Lemma 1. Let $\alpha \in (n, n+1]$ and $f: (-\infty, \infty) \rightarrow \mathbb{R}$ be a function which satisfies $F_\alpha\{f(x)\}(\kappa) = F_\alpha(\kappa)$, $\kappa \in \mathbb{R}$, property. Then,

$$F_\alpha\{f(x)\}(\kappa) = F\{f((\alpha-n)x)^{1/(\alpha-n)}\}(\kappa), \quad (11)$$

where $F\{f(x)\}(\kappa)$ denotes the usual Fourier transform defined by

$$F\{f(x)\}(\kappa) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\kappa x} dx. \quad (12)$$

Proof. We have

$$F_\alpha\{f(x)\}(\kappa) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\kappa(x^{\alpha-n}/(\alpha-n))} x^{\alpha-n-1} dx. \quad (13)$$

Then, by making the substitution $y = ((x^{\alpha-n})/(\alpha-n))$, $x = ((\alpha-n)y)^{1/(\alpha-n)}$, and $dy = x^{\alpha-n-1} dx$, we obtain

$$\begin{aligned} F_\alpha\{f(x)\}(\kappa) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\kappa y} f((\alpha-n)y)^{1/(\alpha-n)} dy \\ &= F\{f((\alpha-n)x)^{1/(\alpha-n)}\}(\kappa). \end{aligned} \quad (14)$$

Lemma 2. Let $\alpha \in (n, n+1]$ and $F_\alpha\{f(x)\}(\kappa)$ be the Katugampola Fourier transform of a function $f: (-\infty, \infty) \rightarrow \mathbb{R}$. Then, the inversion formula for Katugampola Fourier transform of $F_\alpha\{f(x)\}(\kappa)$ is as follows:

$$\begin{aligned} F_\alpha^{-1}\{F_\alpha\{f(x)\}(\kappa)\} &= f(x) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_\alpha\left\{f\left(\frac{x^{\alpha-n}}{\alpha-n}\right)\right\}(\kappa) \\ &\quad \cdot (\kappa) e^{i\kappa(x^{\alpha-n}/(\alpha-n))} f(x) d\kappa. \end{aligned} \quad (15)$$

Proof. The proof followed by applying the definition of the usual Fourier transform and Lemma 1.

Now, we list down some properties of the Katugampola Fourier transform in the theorem below.

Theorem 3. Let $\alpha \in (n, n+1]$, $f: (-\infty, \infty) \rightarrow \mathbb{R}$, and $F_\alpha\{f(x)\}(\kappa) = M_\alpha(\kappa)$, then we have the following:

- (i) For $\alpha = n+1$, we have $F_\alpha\{f(x-a)\}(\kappa) = e^{-ia\kappa} F\{f(x)\}$,
 - (ii) $F_\alpha\{f(ax)\}(\kappa) = \frac{1}{|a|^{\alpha-n}} M_\alpha\left\{\frac{\kappa}{(a)^{\alpha-n}}\right\}$,
 - (iii) $F_\alpha\{e^{-ia(x^{\alpha-n}/(\alpha-n))} f(x)\}(\kappa) = M_\alpha(\kappa - a)$,
 - (iv) $F_\alpha\{e^{-a(x^{\alpha-n}/(\alpha-n))^2}\}(\kappa) = \frac{1}{\sqrt{2a}} e^{-(\kappa/4a)^2}$,
 - (v) $F_\alpha\{e^{-a|x^{\alpha-n}/(\alpha-n)|}\}(\kappa) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + \kappa^2}$,
 - (vi) $F_\alpha\left\{f(x) \cos\left(a\left(\frac{x^{\alpha-n}}{\alpha-n}\right)\right)\right\}(\kappa) = \frac{1}{2} [M_\alpha(\kappa - a) + M_\alpha(\kappa + a)]$,
 - (vii) $F_\alpha\left\{\delta\left(\frac{x^{\alpha-n}}{\alpha-n}\right)\right\}(\kappa) = F\{\delta(x)\}(\kappa) = \frac{1}{\sqrt{2\pi}}$,
- (16)

where $\delta(x)$ is the Dirac delta function.

(viii) $F_\alpha\{f(x)\}(\kappa)$ is a liner operator. (17)

Proof. The proof is similar to the way as in the usual Fourier transform.

Lemma 3. Let $\alpha \in ((m-1)/m, 1]$, $m \in \mathbb{N}$, and $u(x, t)$ be an $m\alpha$ -differentiable real valued function defined on $(-\infty, \infty)$. Then,

$$F_\alpha\left\{\frac{\partial^{m\alpha}}{\partial x^{m\alpha}} u(x, t)\right\}(\kappa) = (i\kappa)^m F_\alpha\{u(x, t)\}(\kappa). \quad (18)$$

Proof. We can prove this theorem by mathematical induction on m .

For $m = 1$, we have

$$F_\alpha\left\{\frac{\partial^\alpha}{\partial x^\alpha} u(x, t)\right\}(\kappa) = (i\kappa) F_\alpha\{u(x, t)\}(\kappa), \quad (19)$$

which is true from Theorem 2 with $n = 0$.

Now, assume that the theorem is true for a particular value of m , say r . Then, we have

$$F_\alpha\left\{\frac{\partial^{r\alpha}}{\partial x^{r\alpha}} u(x, t)\right\}(\kappa) = (i\kappa)^r F_\alpha\{u(x, t)\}(\kappa). \quad (20)$$

Now, we need prove that the theorem is true for $r+1$; that is,

$$F_\alpha\left\{\frac{\partial^{(r+1)\alpha}}{\partial x^{(r+1)\alpha}} u(x, t)\right\}(\kappa) = (i\kappa)^{r+1} F_\alpha\{u(x, t)\}(\kappa), \quad (21)$$

and by using Theorem 2 and the assumption, we have

$$\begin{aligned} F_\beta\left\{\frac{\partial^{(r+1)\alpha}}{\partial x^{(r+1)\alpha}} u(x, t)\right\}(\kappa) &= F_\alpha\left\{\frac{\partial^\alpha}{\partial x^\alpha} \left(\frac{\partial^{r\alpha}}{\partial x^{r\alpha}} u(x, t)\right)\right\}(\kappa), \\ &= (i\kappa) F_\alpha\left\{\frac{\partial^{r\alpha}}{\partial x^{r\alpha}} u(x, t)\right\}(\kappa), \\ &= (i\kappa) (i\kappa)^r F_\alpha\{u(x, t)\}(\kappa), \\ &= (i\kappa)^{r+1} F_\alpha\{u(x, t)\}(\kappa). \end{aligned} \quad (22)$$

Therefore, the theorem is true for every positive integer value of m .

Theorem 4 (convolution theorem). Let $g(x)$ and $h(x)$ be arbitrary functions, where $f, g: (-\infty, \infty) \rightarrow \mathbb{R}$. Then,

$$\begin{aligned} F_\alpha\{(g * h)(x)\} &= F_\alpha\{g(x)\} F_\alpha\{h(x)\} \\ &= F_\alpha\{g\} F_\alpha\{h\}, \end{aligned} \quad (23)$$

where $g * h$ is the convolution of function $g(x)$ and $h(x)$ defined as

$$\begin{aligned}
(g * h)(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t)h(x-t)dt, \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-t)h(t)dt.
\end{aligned} \tag{24}$$

Proof. By using Lemma 1 and the definition of the Katugampola Fourier transform and changing the order of integration, we obtain the result.

Remark 1. Let $g(x)$ and $h(x)$ be arbitrary functions, and let

$$\begin{aligned}
F_{\alpha}^{-1}\{\widehat{g}(\kappa)\} &= g(x), \\
F_{\alpha}^{-1}\{\widehat{h}(\kappa)\} &= h(x),
\end{aligned} \tag{25}$$

where

$$\widehat{f}(\kappa) = F_{\alpha}\{f(x)\}(\kappa) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\kappa(x^{\alpha-n}/(\alpha-n))} f(x) x^{\alpha-n-1} dx. \tag{26}$$

Then,

$$\begin{aligned}
(g * h)(x) &= F_{\alpha}^{-1}\{F_{\alpha}\{(g * h)(x)\}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t)h(x-t)dt, \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-t)h(t)dt.
\end{aligned} \tag{27}$$

3. Katugampola Infinite Fourier Sine and Cosine Transforms

Here we consider the Katugampola infinite Fourier sine and cosine transforms with some of their properties. These transforms are convenient for problems over semi-infinite and some of finite intervals in a spatial variable in which the function or its derivative is prescribed on the boundary.

Definition 4. (Fourier sine transform). Let $\alpha \in (n, n+1]$ and $f(x)$ be a real valued function. The Katugampola infinite Fourier sine transform of a function $f: [0, \infty) \rightarrow \mathbb{R}$, denoted by $F_s^{\alpha}\{f(x)\}$, is defined as

$$F_s^{\alpha}\{f(x)\}(\kappa) = F_s^{\alpha}(\kappa) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin\left(\kappa \frac{x^{\alpha-n}}{\alpha-n}\right) x^{\alpha-n-1} dx. \tag{28}$$

Definition 5. (Fourier cosine transform). Let $\alpha \in (n, n+1]$ and $f(x)$ be a real valued function. The Katugampola infinite Fourier cosine transform of a function $f: [0, \infty) \rightarrow \mathbb{R}$, denoted by $F_c^{\alpha}\{f(x)\}$, is defined as

$$F_c^{\alpha}\{f(x)\}(\kappa) = F_c^{\alpha}(\kappa) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos\left(\kappa \frac{x^{\alpha-n}}{\alpha-n}\right) x^{\alpha-n-1} dx. \tag{29}$$

Remark 2. The transforms F_s^{α} and F_c^{α} are liner operators. They are

$$\begin{aligned}
\text{(i)} \quad F_s^{\alpha}\{af + bg\} &= aF_s^{\alpha}\{f\} + bF_s^{\alpha}\{g\}, \\
\text{(ii)} \quad F_c^{\alpha}\{af + bg\} &= aF_c^{\alpha}\{f\} + bF_c^{\alpha}\{g\}.
\end{aligned} \tag{30}$$

Theorem 5. Let $f: [0, \infty) \rightarrow \mathbb{R}$ be an α -differentiable real valued function and n -differentiable at $x > 0$, where $f^{(n)}(x) \rightarrow 0$ as $x \rightarrow \infty$ and $\alpha \in (n, n+1]$. Then,

$$\begin{aligned}
\text{(i)} \quad F_s^{\alpha}\{D^{\alpha}(f)(x)\}(\kappa) &= -\kappa F_c^{\alpha}(f^{(n)}(x)), \\
\text{(ii)} \quad F_c^{\alpha}\{D^{\alpha}(f)(x)\}(\kappa) &= \kappa F_s^{\alpha}(f^{(n)}(x)) - \sqrt{\frac{2}{\pi}} f^{(n)}(0).
\end{aligned} \tag{31}$$

Proof.

- (i) The proof follows by using Definition 4 and Theorem 1 and integration by parts.
- (ii) The proof follows by using Definition 5 and Theorem 1 and integration by parts.

4. Katugampola Finite Fourier Sine and Cosine Transforms

When the physical problem is defined on a finite domain, it is generally not suitable to use transformation with infinite range of integration. In such cases, the usage of finite Fourier transform is very advantageous.

In this section we shall discuss the Katugampola finite Fourier sine and cosine transforms and some of their properties.

Definition 6. (Katugampola finite Fourier sine transform). Let $\alpha \in (n, n+1]$ for some $n \in \mathbb{N}$ and $f(x)$ be a real valued function defined on $(0, L)$. The Katugampola finite Fourier sine transform of $f(x)$, $0 < x < L$ is defined as

$$F_s^{\alpha}(f(x))(\kappa) = F_s^{\alpha}(\kappa) = \int_0^L f(x) \sin\left(\kappa \pi \frac{x^{\alpha-n}}{L^{\alpha-n}}\right) x^{\alpha-n-1} dx. \tag{32}$$

The inverse Fourier Katugampola sine transform is defined as follows:

$$f(x) = \frac{2(\alpha-n)}{L^{\alpha-n}} \sum_{\kappa=1}^{\infty} F_s^{\alpha}(\kappa) \sin\left(\kappa \pi \frac{x^{\alpha-n}}{L^{\alpha-n}}\right). \tag{33}$$

Definition 7. (Katugampola Finite Fourier cosine transform). Let $\alpha \in (n, n+1]$ for some $n \in \mathbb{N}$ and $f(x)$ be a real valued function defined on $(0, L)$. The Katugampola finite Fourier cosine transform of $f(x)$, $0 < x < L$ is defined as

$$F_c^\alpha(f(x))(\kappa) = F_c^\alpha(\kappa) = \int_0^L f(x) \cos\left(\kappa \pi \frac{x^{\alpha-n}}{L^{\alpha-n}}\right) x^{\alpha-n-1} dx. \quad (34)$$

The inverse Fourier Katugampola cosine transform is defined as follows:

$$f(x) = \frac{(\alpha-n)}{L^{\alpha-n}} F_c^\alpha(0) + \frac{2(\alpha-n)}{L^{\alpha-n}} \sum_{\kappa=1}^{\infty} F_c^\alpha(\kappa) \cos\left(\kappa \pi \frac{x^{\alpha-n}}{L^{\alpha-n}}\right). \quad (35)$$

Theorem 6. Let $\alpha \in (n, n+1]$, for some $n \in \mathbb{N}$, and $u(x, t)$ be a real valued function of two variables $0 < x < L$ and $t > 0$ where $u(x, t)$ is α -differentiable and n -differentiable with respect to x . Then,

$$\begin{aligned} \text{(i)} \quad F_s^\alpha\left(\frac{\partial^\alpha u}{\partial x^\alpha}\right) &= -\frac{\kappa\pi(\alpha-n)}{L^{\alpha-n}} F_c^\alpha\{u^{(n)}(x, t)\}, \\ \text{(ii)} \quad F_c^\alpha\left(\frac{\partial^\alpha u}{\partial x^\alpha}\right) &= \frac{\kappa\pi(\alpha-n)}{L^{\alpha-n}} F_s^\alpha\{u^{(n)}(x, t)\} \\ &\quad - [u^{(n)}(0, t) - u^{(n)}(L, t) \cos(\kappa\pi)], \\ \text{(iii)} \quad F_s^\alpha\left(\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}}\right) &= -\frac{\kappa\pi(\alpha-n)}{L^{\alpha-n}} F_c^\alpha\left\{\frac{\partial^\alpha u^{(n)}}{\partial x^\alpha}\right\}, \\ &= -\left(\frac{\kappa\pi(\alpha-n)}{L^{\alpha-n}}\right)^2 F_s^\alpha\{u^{(2n)}(x, t)\} \\ &\quad - \frac{\kappa\pi(\alpha-n)}{L^{\alpha-n}} [u^{(2n)}(L, t) \cos(\kappa\pi) - u^{(2n)}(0, t)], \\ \text{(iv)} \quad F_c^\alpha\left(\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}}\right) &= -\left(\frac{\kappa\pi(\alpha-n)}{L^{\alpha-n}}\right)^2 F_s^\alpha\{u^{(2n)}(x, t)\} \\ &\quad - \left[\frac{\partial^\alpha u^{(n)}(0, t)}{\partial x^\alpha} - \frac{\partial^\alpha u^{(n)}(L, t)}{\partial x^\alpha} \cos(\kappa\pi)\right]. \end{aligned} \quad (36)$$

Proof.

- (i) The proof follows by using Definition 6 and Theorem 1 and integration by parts.
- (ii) The proof follows by using Definition 7 and Theorem 1 and integration by parts.
- (iii), (iv) By using parts (i) and (ii) above, we get the result.

Corollary 1. Let $\alpha \in (0, 1]$ and $u(x, t)$ be a real valued function of two variables $x > 0$ and $t > 0$. Then,

$$\begin{aligned} \text{(i)} \quad F_s^\alpha\left(\frac{\partial^\alpha u}{\partial x^\alpha}\right) &= -\frac{\kappa\pi\alpha}{L^\alpha} F_c^\alpha\{u(x, t)\}, \\ \text{(ii)} \quad F_c^\alpha\left(\frac{\partial^\alpha u}{\partial x^\alpha}\right) &= \frac{\kappa\pi\alpha}{L^\alpha} F_s^\alpha\{u(x, t)\} - [u(0, t) - u(L, t) \cos(\kappa\pi)], \\ \text{(iii)} \quad F_s^\alpha\left(\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}}\right) &= -\frac{\kappa\pi\alpha}{L^\alpha} F_c^\alpha\left\{\frac{\partial^\alpha u}{\partial x^\alpha}\right\}, \\ &= -\left(\frac{\kappa\pi\alpha}{L^\alpha}\right)^2 F_s^\alpha\{u(x, t)\} \\ &\quad - \frac{\kappa\pi\alpha}{L^\alpha} [u(L, t) \cos(\kappa\pi) - u(0, t)], \\ \text{(iv)} \quad F_c^\alpha\left(\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}}\right) &= -\left(\frac{\kappa\pi\alpha}{L^\alpha}\right)^2 F_c^\alpha\{u(x, t)\} \\ &\quad - \left[\frac{\partial^\alpha u(0, t)}{\partial x^\alpha} - \frac{\partial^\alpha u(L, t)}{\partial x^\alpha} \cos(\kappa\pi)\right]. \end{aligned} \quad (37)$$

Proof. The proof is direct from Theorem 6 by putting $n = 0$.

Remark 3. The results we obtained in Corollary 4.4 are similar to the results in ([3], pp. 137–138).

5. Conclusions

In this paper we obtained several results that have close resemblance to the results found in classical calculus. We defined both the Katugampola infinite and finite Fourier transforms and Fourier sine and cosine transforms. Also we established some properties of these transforms which are considered as generalizations to the usual transform.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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