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Research Article

Canonical Form of Reduced 3-by-3 Matrix with One Characteristic Root and with Some Zero Subdiagonal Elements

B. Z. Shavarovskii

Department of Algebra, Pidstryhach Institute for Applied Problems of Mechanics and Mathematics of National Academy of Sciences of Ukraine, Lviv 79060, Ukraine

Correspondence should be addressed to B. Z. Shavarovskii; bshavarovskii@gmail.com

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A canonical form for a reduced matrix of order 3 with one characteristic root and with some zero subdiagonal elements is constructed. Thus, the problem of classification with respect to semiscalar equivalence of a selected set of polynomial matrices is solved.

1. Introduction

Let a matrix $F(x) \in M(n, \mathbf{C}[x])$ have a unit first invariant factor and only one (without taking into account multiplicity) characteristic root. Without loss of generality, we assume that this uniquely characteristic root is zero. Consider the transformation $F(x) \longrightarrow PF(x)Q(x) = G(x)$, where $P \in GL(n, \mathbf{C})$, $Q(x) \in GL(n, \mathbf{C}[x])$. In accordance with [1] (see also [2]) matrices $F(x)$, $G(x)$ are called semiscalarly equivalent (abbreviation: ss.e.; notation: $F(x) \approx G(x)$). In [3], the author proved that in the class $\{PF(x)Q(x)\}$, where $F(x) \in M(3, \mathbf{C}[x])$, there exists a matrix

$$A(x) = \left\| \begin{matrix} 1 & 0 & 0 \\ a_1(x) & x^{k_1} & 0 \\ a_3(x) & a_2(x) & x^{k_2} \end{matrix} \right\|, \qquad (1)$$

which has the following properties:

(i) $\deg a_1(x) < k_1$, $\deg a_2(x), \deg a_3(x) < k_2$, $a_2(x) = x^{k_1} a_2'(x)$, $a_1(0) = a_2'(0) = a_3(0) = 0$ (see Proposition 1 [3]);

(ii) $co\deg a_3(x) \neq co\deg a_1(x), co\deg a_2'(x)$, if $co\deg a_3(x) < co\deg a_2(x)$ (see Propositions 4, 5 [3]);

(iii) $co\deg a_3(x) \neq 2co\deg a_1(x) + co\deg a_2'(x)$ and in $a_1(x)$ the monomial of the degree $2co\deg a_1(x)$ is absent, if $co\deg a_3(x) \geq co\deg a_2(x)$ (see Propositions 6, 7 [3]).

Here $co\deg$ denotes *the junior degree* of polynomial. *Junior degree* of polynomial $f(x) \in \mathbf{C}[x]$, $f(x) \neq 0$, is the least degree of the monomial (of nonzero coefficient) of this polynomial; notation $co\deg f$. The monomial of degree $co\deg f$ and its coefficient are called the *junior term* and *junior coefficient*, respectively. Denote by symbol $+\infty$ the junior degree of the polynomial $f(x) \equiv 0$. If both elements $a_1(x)$, $a_2(x)$ of the matrix $A(x)$ are nonzero, then we may take their junior coefficients to be identity elements. In the opposite case, we may take the junior coefficients of the nonzero subdiagonal elements of the matrix $A(x)$ to be one. Such matrix $A(x)$ in [3] is called the *reduced matrix*. The purpose of this paper is to construct the canonical form of the matrix $F(x) \in M(3, \mathbf{C})$ in the class $\{PF(x)Q(x)\}$ of ss.e. matrices. For this purpose we base our work on the reduced matrix $A(x)$ of the form (1). This article is a continuation of the work [3]. The case $a_1(x) = a_2(x) = a_3(x) \equiv 0$ is trivial. Then the Smith form is canonical for the matrix $F(x)$ in the class $\{PF(x)Q(x)\}$. If $k_1 = k_2$, then $a_2(x) \equiv 0$. This case is considered in the author's paper [4]. For this reason in the sequel we shall take $k_1 \neq k_2$. In this paper we consider the case, when some of the elements $a_1(x)$, $a_2(x)$, $a_3(x)$ of the matrix $A(x)$ are equal to zero and at least one of them is different from zero. Recall that the zero equality of some subdiagonal elements of the matrix $A(x)$ is an invariant (see

Proposition 2 [3]). A case in which all subdiagonal elements of the matrix $A(x)$ are nonzero will be the subject of another study. It should be noted that the problem of ss.e. matrices of the second order is solved in the article [5]. Some aspects of this problem for the matrices of the arbitrary order are considered in the papers [6–9]. We also add that the work close to [1, 2] is [10, 11].

2. The Canonical Form of a Reduced Matrix with Two Zero Subdiagonal Elements

To abbreviate in the sequel the expressions "monomial of degree q" and "coefficient of the monomial of degree q" of some polynomial we will use "q-monomial" and "q-coefficient," respectively. Further, by the symbol 0 we often denote, besides the zero element of the field $\mathbf{C}$, the rectangular zero matrix if it is clear from the context what we have in mind. By $\overline{0}$ we denote zero column of the necessary height.

Theorem 1. *Let in the reduced matrix $A(x)$ of the form (1) the conditions $a_i(x) \neq 0$, for some index i from set $\{1,2,3\}$ and $a_j(x) \equiv 0$ for the rest $j \in \{1,2,3\}$, $j \neq i$, be fulfilled. Then $A(x) \approx B(x)$, where in the reduced matrix*

$$B(x) = \begin{Vmatrix} 1 & 0 & 0 \\ b_1(x) & x^{k_1} & 0 \\ b_3(x) & b_2(x) & x^{k_2} \end{Vmatrix}, \tag{2}$$

the element $b_i(x) \neq 0$ does not contain n_i-monomial,

$$n_i = \begin{cases} 2co\deg a_i, & i = 1,3, \\ 2co\deg a_2' + k_1, & i = 2, \end{cases} \tag{3}$$

$a_2'(x) = a_2(x)/x^{k_1}$, $b_j(x) \equiv 0$. The matrix $B(x)$ is uniquely defined.

Proof.

Existence. Let the conditions of the Theorem hold true. If

$$n_i \geq \begin{cases} k_1, & i = 1, \\ k_2, & i = 2,3, \end{cases} \tag{4}$$

then in $a_i(x)$ the n_i-monomial is absent, since

$$\deg a_i(x) < \begin{cases} k_1, & i = 1, \\ k_2, & i = 2,3. \end{cases} \tag{5}$$

Suppose that

$$n_i < \begin{cases} k_1, & i = 1, \\ k_2, & i = 2,3. \end{cases} \tag{6}$$

Denote n_i-coefficient of the polynomial $a_i(x)$ by c. We will apply to the matrix $A(x)$ a semiscalarly equivalent transformation (ss.e.t.) with left transforming matrix of one of the forms:

$$S = \begin{Vmatrix} 1 & s_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \tag{7}$$

if $i = 1$;

$$S = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & s_{23} \\ 0 & 0 & 1 \end{Vmatrix}, \tag{8}$$

if $i = 2$, or

$$S = \begin{Vmatrix} 1 & 0 & s_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \tag{9}$$

if $i = 3$. In doing so every time instead of s_{12}, s_{23}, or s_{13} in the transforming matrix S we put c, that is,

$$c = \begin{cases} s_{12}, & i = 1, \\ s_{23}, & i = 2, \\ s_{13}, & i = 3. \end{cases} \tag{10}$$

In article [3], for the given matrices $A(x)$ of the form (1) and S one of the form (7), (8), or (9) is a method for finding the matrix $B(x)$ of the form (2) such that $A(x) \approx B(x)$. In doing so, we can find also the right transforming matrix. Depending on the form (7), (8), or (9) of the matrix S we shall say that the ss.e.t.-I, ss.e.t.-II, or ss.e.t.-III is applied to matrix $A(x)$, respectively.

If $i = 1$ and ss.e.t.-I is applied to matrix $A(x)$, then the elements $a_1(x) \neq 0, a_2(x), a_3(x) \equiv 0$ and $b_1(x), b_2(x), b_3(x)$ of the matrices $A(x)$ and $B(x)$ satisfy the following congruences:

$$b_1(x) - a_1(x)\left(1 - s_{12}b_1(x)\right) \equiv 0 \left(\operatorname{mod} x^{k_1}\right), \tag{11}$$

$$\begin{aligned} & b_2(x) - a_2(x) - s_{12}\left(a_1(x)a_2(x) - x^{k_1}a_3(x)\right) \\ & \quad \equiv 0 \left(\operatorname{mod} x^{k_2}\right), \end{aligned} \tag{12}$$

$$\begin{aligned} & b_3(x) - a_3(x) + s_{12}a_3(x)b_1(x) - a_2(x)r_{12}(x) \\ & \quad \equiv 0 \left(\operatorname{mod} x^{k_2}\right), \end{aligned} \tag{13}$$

where $r_{21}(x) = (b_1(x) - a_1(x)(1 - s_{12}b_1(x)))/x^{k_1} \in \mathbf{C}[x]$. From (12) and (13) we have that $b_2(x) = b_3(x) \equiv 0$. From (11) we obtain that $b_1(x) \neq 0$ and in $b_1(x)$ the n_1-monomial is absent.

If $i = 2$, then the matrix $B(x)$ is obtained by means of ss.e.t-II. Then the elements of $A(x)$ and $B(x)$ fulfill the following congruences:

$$a_1(x) + s_{23}a_3(x) - b_1(x) \equiv 0 \left(\operatorname{mod} x^{k_1}\right), \quad (14)$$

$$a_2(x) - b_2(x)\left(1 + s_{23}a_2'(x)\right) \equiv 0 \left(\operatorname{mod} x^{k_2}\right), \quad (15)$$

$$a_3(x) - b_3(x) - b_2(x) r_{21}(x) \equiv 0 \left(\operatorname{mod} x^{k_2}\right), \quad (16)$$

where $r_{21}(x) = (a_1(x) + s_{23}a_3(x) - b_1(x)))/x^{k_1} \in \mathbf{C}[x]$. From (14) and (16) it follows at once that $b_1(x) \equiv 0$, $r_{21}(x) \equiv 0$ and $b_3(x) \equiv 0$. From (15) we have that $b_2(x) \not\equiv 0$ and moreover $b_2(x)$ does not contain the n_2-monomial.

Let $i = 3$ and we pass from $A(x)$ to $B(x)$ with the help of ss.e.t.-III. From ss.e. of the matrices $A(x)$ and $B(x)$ it follows that their elements satisfy the congruences:

$$b_1(x) + s_{13}a_3(x) b_1(x) \equiv 0 \left(\operatorname{mod} x^{k_1}\right), \quad (17)$$

$$b_2(x) \equiv 0 \left(\operatorname{mod} x^{k_2}\right), \quad (18)$$

$$\begin{aligned} &a_3(x) - b_3(x) - s_{13}a_3(x) b_3(x) - b_2(x) r_{21}(x) \\ &\quad \equiv 0 \left(\operatorname{mod} x^{k_2}\right), \end{aligned} \quad (19)$$

where $r_{21}(x) = -b_1(x)(1 + s_{13}a_3(x))/x^{k_1} \in \mathbf{C}[x]$. From (17) and (18) we obtain $b_1(x) = b_2(x) \equiv 0$, $r_{21}(x) \equiv 0$. From (19) we have that $b_3(x) \not\equiv 0$ and in $b_3(x)$ the n_2-monomial is absent. Evidently, the obtained matrix $B(x)$ is reduced in each under consideration case to $i = 1, 2, 3$. The first part of the theorem is proved.

Observe that the equality $b_j(x) \equiv 0$, $j \neq i$, each time follows from Corollaries 2, 3 [3].

Uniqueness. It suffices to prove for some of the cases: $i = 1$, $i = 2$, or $i = 3$. The proof in two other cases is analogous. Assume in the reduced matrices $A(x)$ and $B(x)$ of the forms (1) and (2) we have $a_i(x), b_i(x) \not\equiv 0$, $a_j(x) = b_j(x) \equiv 0$, $i \neq j$ and in $a_i(x), b_i(x)$ the n_i-monomial is absent. If $i = 3$, then from $A(x) \approx B(x)$ we obtain the congruence (19), where $r_{21}(x) \equiv 0$. Comparing the n_3-coefficients in both sides of this congruence, we obtain $s_{13} = 0$. From this it follows that $a_3(x) = b_3(x)$, because $\deg a_3, \deg b_3 < k_2$. The theorem is proved completely. □

3. The Canonical Form of a Reduced Matrix with One Zero Subdiagonal Element

Let us now consider the ss.e.t. with the left transforming matrix of one of the next forms:

$$S = \begin{Vmatrix} 1 & s_{12} & 0 \\ 0 & 1 & s_{23} \\ 0 & 0 & 1 \end{Vmatrix}, \quad (20)$$

$$S = \begin{Vmatrix} 1 & s_{12} & s_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \quad (21)$$

$$S = \begin{Vmatrix} 1 & 0 & s_{13} \\ 0 & 1 & s_{23} \\ 0 & 0 & 1 \end{Vmatrix} \quad (22)$$

Such transformations are called ss.e.t-I-II, ss.e.t-I-III, and ss.e.t-II-III, respectively.

Let $a_1(x) = b_1(x) \equiv 0$, $a_2(x), a_3(x), b_2(x), b_3(x) \not\equiv 0$, $a_2(x) = x^{k_1}a_2'(x)$, $b_2(x) = x^{k_1}b_2'(x)$ in the reduced matrices $A(x)$ and $B(x)$ of the forms (1) and (2). We determine the polynomials $a_{32}(x)$ and $a_{33}(x)$ as follows:

$$\begin{aligned} a_{32}(x) &:\equiv a_3(x) a_2'(x) \left(\operatorname{mod} x^{k_2}\right), \\ a_{33}(x) &:\equiv \left(a_3(x)\right)^2 \left(\operatorname{mod} x^{k_2}\right). \end{aligned} \quad (23)$$

Next, we form the columns $\overline{a}_3$, $\overline{a}_{32}$, and $\overline{a}_{33}$ from the coefficients of the polynomials $a_3(x)$, $a_{32}(x)$, and $a_{33}(x)$, respectively. In doing so we place on the first positions in these columns the coefficients of the monomials of degree $co\deg a_3$. Below we arrange every succeeding coefficients in the order of growth of the degrees of the monomials, up to $(k_2 - 1)$-coefficient. At the same time we do not omit zero coefficients. Take into consideration the polynomials

$$\begin{aligned} a_{22}(x) &:\equiv \left(a_2'(x)\right)^2 \left(\operatorname{mod} x^{k_2-k_1}\right), \\ a_{23}(x) &:\equiv a_2'(x) a_3(x) \left(\operatorname{mod} x^{k_2-k_1}\right). \end{aligned} \quad (24)$$

With the coefficients of the polynomials $a_2'(x)$, $a_3(x)$, $a_{22}(x)$, and $a_{23}(x)$ we form the columns $\overline{a}_2$, $\overline{a}_{30}$, $\overline{a}_{22}$ and $\overline{a}_{23}$, respectively, of the height $k_2 - co\deg a_2$. Here we place on the first positions the coefficients of monomials of degree $co\deg a_2 - k_1$. Further we put all coefficients of the monomials of higher degrees up to degree $k_2 - k_1 - 1$. Let us build for $A(x)$ the matrices:

$$\begin{aligned} M_A &= \begin{Vmatrix} \overline{a}_3 & \multirow{2}{*}{M_{0A}} \\ \overline{a}_2 & \end{Vmatrix}, \\ M_{0A} &= \begin{Vmatrix} M_{3A} \\ M_{2A} \end{Vmatrix}, \\ M_{3A} &= \begin{Vmatrix} \overline{a}_{32} & \overline{a}_{33} & \overline{0} \end{Vmatrix}, \\ M_{2A} &= \begin{Vmatrix} \overline{a}_{22} & \overline{a}_{23} & \overline{a}_{30} \end{Vmatrix} \end{aligned} \quad (25)$$

Evidently, each column in these matrices is composed by the coefficients of the monomials of the same degrees. By complete analogy for $B(x)$ we build the matrices:

$$M_B = \left\| \begin{matrix} \bar{b}_3 & \\ \bar{b}_2 & M_{0B} \end{matrix} \right\|,$$
$$M_{0B} = \left\| \begin{matrix} M_{3B} \\ M_{2B} \end{matrix} \right\|, \tag{26}$$
$$M_{3B} = \left\| \bar{b}_{32} \quad \bar{b}_{33} \quad \bar{0} \right\|,$$
$$M_{2B} = \left\| \bar{b}_{22} \quad \bar{b}_{23} \quad \bar{b}_{30} \right\|.$$

Theorem 2. *Let $A(x)$ be the reduced matrix of the form (1), in which $a_1(x) \equiv 0$, $a_2(x), a_3(x) \neq 0$, $a_2(x) = x^{k_1} a'_2(x)$, and $q_2 := co\deg a'_2$, $q_3 := co\deg a_3$. Then $A(x) \approx B(x)$, where in the reduced matrix $B(x)$ of the form (2) $b_1(x) \equiv 0$ and the elements $b_2(x), b_3(x) \neq 0$ satisfy one of the next conditions.*

(1) *In $b_3(x)$ the $(2q_3)$-monomial is absent, if $q_3 < q_2$, $q_3 < k_1$.*

(2) *In $b_3(x)$ the $(2q_3)$- and (q_2+q_3)-monomials are absent, if $q_3 < q_2$, $q_3 \geq k_1$.*

(3) *In $b_3(x)$ and $b'_2(x)$, where $b'_2(x) = b_2(x)/x^{k_1} \in \mathbf{C}[x]$, respectively, the $(2q_3)$- and q_3-monomials are absent, if $q_3 > q_2$, $q_3 < k_1$.*

(4) *In the first column of the matrix M_B (26) with the coefficients of the polynomials $b_3(x)$, $b'_2(x)$ are zero elements, which correspond to the maximal system of the first linearly independent rows of the submatrix M_{0B}, if $q_3 > q_2$, $q_3 \geq k_1$.*

The matrix $B(x)$ is uniquely defined.

Proof.

Existence. (1) Let $q_3 < q_2$, $q_3 < k_1$, and c_1 identify $(2q_3)$-coefficient of the polynomial $a_3(x)$ in the matrix $A(x)$. Let us apply to the matrix $A(x)$ ss.e.t.-III. In so doing, we set $s_{13} = c_1$ in the left transforming matrix S (see (9)). In the obtained reduced matrix $B(x)$ the element $b_3(x)$ satisfies congruence (19), where $r_{21}(x) \equiv 0$. From (19) we deduce that for $b_3(x)$ the condition (1) of the theorem is fulfilled.

(2) Let $q_3 < q_2$, $q_3 \geq k_1$. We may take that the element $a_3(x)$ of the matrix $A(x)$ does not contain $(2q_3)$-monomial. In the opposite case we apply to the matrix $A(x)$ ss.e.t.-III, as has been shown above in p. (1). Let us denote $(q_2 + q_3)$-coefficient of the polynomial $a_3(x)$ by c_2 and let us apply to the matrix $A(x)$ ss.e.t.-II. At that, we set in left transforming matrix (see (8)) $s_{23} = c_2$. For the elements $b_2(x), b_3(x)$ of the obtained reduced matrix $B(x)$ the congruence (16) holds true, where $a_1(x) = b_1(x) \equiv 0$. For this reason it may be written in the form

$$a_3(x) - b_3(x) - s_{23} a_3(x) b'_2(x) \equiv 0 \left(\mathrm{mod}\, x^{k_2}\right). \tag{27}$$

From the above it follows that in $b_3(x)$ the (q_2+q_3)-coefficient is zero. Since $q_3 < q_2$, then $2q_3 < q_2 + q_3$. Therefore, from the last congruence we see that in $b_3(x)$, as in $a_3(x)$, the $(2q_3)$-monomial is absent.

(3) Let $q_3 > q_2$, $q_3 < k_1$. We may take that in $a_3(x)$ the $(2q_3)$-monomial is absent. We can always do this as has been shown above in p. (1). Let us denote by c_3 the q_3-coefficient in $a'_2(x)$. We subject $A(x)$ to action of ss.e.t-I. In this case we set in the left transforming matrix (see (7)) $s_{12} = c_3$. As a result we obtain the reduced matrix $B(x)$ of the form (2), in which $b_3(x) = a_3(x)$ and the congruence

$$a'_2(x) \equiv b'_2(x) + s_{12} a_3(x) \left(\mathrm{mod}\, x^{k_2-k_1}\right). \tag{28}$$

holds true. The last congruence implies that in $b'_2(x)$ the q_3-monomial is absent. Then $B(x)$ is the matrix to be found.

(4) Let $q_3 > q_2$, $q_3 \geq k_1$. If $M_{0A} = 0$ in the matrix M_A (see (25)), then everything is proved; that is $A(x)$ is the matrix to be found. If the matrix M_{0A} is nonzero, but its block M_{3A} is zero, then the element $a_3(x)$ in $A(x)$ has the required property. In this case the second column of the matrix M_{0A} is zero. Let the first nonzero row $\| d_{11} \quad 0 \quad d_{13} \|$ of the matrix M_{0A} and the corresponding row $\| d_1 \quad d_{11} \quad 0 \quad d_{13} \|$, $d_1 \neq 0$, of the matrix M_A be formed of m_1-coefficients. Let us find some solution $\| x_{10} \quad x_{30} \|^t$ of the equation $\| d_{11} \quad d_{13} \| \, \| x_{10} \quad x_{30} \|^t = d_1$ and use ss.e.t.-I-II to $A(x)$. Herewith we set in left transforming matrix S of the form (20) $s_{12} = x_{30}$, $s_{23} = x_{10}$. The divisor $b'_2(x)$ of the element $b_2(x)$, $b_2(x) = x^{k_1} b'_2(x)$, of the obtained reduced matrix $B(x)$ satisfies the congruence

$$\begin{aligned} &a'_2(x) - s_{23} a'_2(x) b'_2(x) - s_{12} a_3(x) - b'_2(x) \\ &\quad \equiv 0 \left(\mathrm{mod}\, x^{k_2-k_1}\right). \end{aligned} \tag{29}$$

From the last one it follows that in $b'_2(x)$ the m_1-monomial is absent and $b'_2(x) \equiv a'_2(x) (\mathrm{mod}\, x^{m_1})$.

If $rank M_{0A} = 1$, then everything is proved; that is $B(x)$ is the matrix to be found. In the opposite case we take that already for $A(x)$ the polynomial $a'_2(x)$ does not contain m_1-monomial. Let the row $\| d_{21} \quad 0 \quad d_{23} \|$ of the matrix M_{0A} be the first linearly independent row of $\| d_{11} \quad 0 \quad d_{13} \|$ and $\| d_2 \quad d_{21} \quad 0 \quad d_{23} \|$ is the respective row of the matrix M_A. Let these rows be formed of the m_2-coefficients. We find the (unique) solution $\| y_{10} \quad y_{30} \|^t$ of the equation

$$\left\| \begin{matrix} d_{11} & d_{13} \\ d_{21} & d_{23} \end{matrix} \right\| \left\| \begin{matrix} y_1 \\ y_3 \end{matrix} \right\| = \left\| \begin{matrix} 0 \\ d_2 \end{matrix} \right\|. \tag{30}$$

By ss.e.t-I-II we pass from $A(x)$ to the reduced matrix $B(x)$. Here in the matrix S (see (20)) we set $s_{12} = y_{30}$, $s_{23} = y_{10}$. The divisor $b'_2(x)$ of the element $b_2(x)$ in $B(x)$, as is seen from the congruence (29), does not contain m_1- and m_2-coefficients.

If the matrix M_{3A} is nonzero, then the row $\bar{u}_1 = \| 1 \quad 0 \quad 1 \|$ of $(q_2 + q_3)$-coefficients is its first nonzero row. At the first step we apply to $A(x)$ ss.e.t-II. In addition we set in the left transforming matrix S of the form (8) in place of s_{23} the coefficient of the monomial of degree $q_2 + q_3$ in the polynomial $a_3(x)$ (see p. (2)). In the obtained reduced matrix the polynomial in the position $(3, 1)$ does not contain the $(q_2 + q_3)$-monomial. If $2q_3 < k_2$, then we consider the row of the form $\bar{u}_2 = \| * \quad 1 \quad 0 \|$ formed of the $(2q_3)$-coefficients in the matrix, analogous to M_{3A}. This row is the first linearly independent of $\bar{u}_1$. We do the second step. This is ss.e.t-III of the obtained matrix after the first step. For this purpose we set $(2q_3)$-coefficient of the polynomial in the position $(3,1)$ of the obtained matrix after the first step instead s_{13} in the left transforming matrix of the form (9) (see p. (1)). In the obtained matrix after the second step the polynomial in position $(3,1)$ does not contain (q_2+q_3)- and $(2q_3)$-monomials. In order not to introduce the new notations, we will consider that in the matrix $A(x)$ its element $a_3(x)$ possesses this property. If $q_3 < k_2 - k_1$, then in the submatrix M_{2A} of the matrix M_{0A} the row $\overline{u}_3 = \| * \ \ 0 \ \ 1\|$ of the q_3-coefficients is the first linearly independent row with the collection $\overline{u}_1, \overline{u}_2$. Then we make the third step. It is ss.e.t.-I of the matrix $A(x)$. For this purpose in the left transforming matrix of the form (7) instead of s_{12} we set q_3-coefficient in $a_2'(x)$ (see p. (3)). After this step we obtain the matrix in which the element in the position $(3,1)$ is not changing and the polynomial in the position $(3,2)$ does not contain (q_3+k_1)-monomial. We obtain the required matrix.

If after the first step it turns out that $2q_3 \ge k_2$, that is, $rank M_{3A} = 1$ but $q_3 < k_2 - k_1$, then we immediately take the third step. Next, if $q_2 + q_3 < k_2 - k_1$, then the row $\| * \ \ 1 \ \ * \|$ of the (q_2+q_3)-coefficients is the first linearly independent row with the collection $\overline{u}_1, \overline{u}_3$ in the submatrix, analogous to M_{0A}. Then we take the fourth step. It is ss.e.t.-III with a left transforming matrix (see (9)), in which instead of s_{13} is $(q_2+q_3+k_1)$-coefficients of the polynomial in the position $(3,2)$ of the matrix obtained after the third step. After that, the element in the position $(3,1)$ of the resulting matrix will not change, and in position $(3,2)$ the desired element will stand. The obtained matrix is the required one.

Uniqueness. (1) Let the matrices $A(x)$, $B(x)$ of the form (1), (2) satisfy the condition (1) and $A(x) \approx B(x)$. Then the congruence (19), where $r(x)_{21} \equiv 0$, and the congruence

$$a_2'(x) \equiv b_2'(x) + s_{13} a_2'(x) b_3(x) \equiv 0 \left(\operatorname{mod} x^{k_2-k_1}\right) \tag{31}$$

are fulfilled (see Corollary 1 and Remark [3]). If $2q_3 \ge k_2$, then $q_2 + q_3 \ge k_2 - k_1$ and from (19) we have $a_3(x) = b_3(x)$, and from (31) we will obtain $a_2'(x) = b_2'(x)$. If $2q_3 < k_2$, then comparing $(2q_3)$-coefficients in both sides (19), we get $s_{13} = 0$. Therefore, in each case $A(x)$ and $B(x)$ coincide.

(2) Let condition (2) for the matrices $A(x)$ and $B(x)$ of the form (1) and (2) be fulfilled. Then on the basis of Corollary 1 and Remark [3] on $A(x) \approx B(x)$ we can write the congruences:

$$\begin{aligned} a_2'(x) &\equiv b_2'(x) + s_{23} a_2'(x) b_2'(x) + s_{13} a_2'(x) b_3(x) \\ &\equiv 0 \left(\operatorname{mod} x^{k_2-k_1}\right), \end{aligned} \tag{32}$$

$$\begin{aligned} a_3(x) &\equiv b_3(x) + s_{23} a_3(x) b_2'(x) \\ &\quad + s_{13} a_3(x) b_3(x) \left(\operatorname{mod} x^{k_2}\right). \end{aligned} \tag{33}$$

It should be noted that $2q_3 < q_2 + q_3 < co\deg a_2 + q_3 < co\deg a_2 + q_2$. If $2q_3 \ge k_2$, then $q_2 + q_3 \ge k_2 - k_1$. Then from (32) and (33) we have $a_2'(x) = b_2'(x)$ and $a_3(x) = b_3(x)$, respectively. In the case of $2q_3 < k_2$ by comparison, in both parts (33) of the $(2q_3)$-coefficients we arrive at $s_{13} = 0$. If $q_2 + q_3 \ge k_2$, then from (32) and (33), as before, we have $a_2'(x) = b_2'(x)$ and $a_3(x) = b_3(x)$, respectively. If $q_2 + q_3 < k_2$, then by comparing the (q_2+q_3)-coefficients in both parts (33), we have $s_{23} = 0$. Thus, in this case $A(x) = B(x)$ too.

(3) For the reduced matrices $A(x)$, $B(x)$ of the form (1), (2) such that $A(x) \approx B(x)$, in the case of satisfying the condition (3) we have congruence (19) where $r(x)_{21} \equiv 0$, and congruence

$$\begin{aligned} a_2'(x) &\equiv b_2'(x) + s_{12} b_3(x) \\ &\quad + s_{13} a_2'(x) b_3(x) \left(\operatorname{mod} x^{k_2-k_1}\right). \end{aligned} \tag{34}$$

First let us note that $2q_3 < q_3 + k_1 < co\deg a_2 + q_3$. If $2q_3 \ge k_2$, then $q_3 \ge k_2 - k_1$. Then from (34) and (19) we have $a_2'(x) = b_2'(x)$ and $a_3(x) = b_3(x)$, respectively. If $2q_3 < k_2$, then by comparing the $(2q_3)$-coefficients from (19) we have $s_{13} = 0$ and $a_3(x) = b_3(x)$. If further $q_3 \ge k_2 - k_1$, then from (34) $a_2'(x) = b_2'(x)$ immediately follows. In the case $q_3 < k_2 - k_1$ we compare q_3-coefficients in both parts (34) and come to $s_{12} = 0$. So $A(x) = B(x)$.

(4) Suppose that for the reduced matrices $A(x)$, $B(x)$ of the form (1), (2) which satisfy condition (4) we have $A(x) \approx B(x)$. For their elements, taking into account Remark [3], we can write the congruence (33) and congruence

$$\begin{aligned} a_2'(x) &\equiv b_2'(x) + s_{12} b_3(x) + s_{13} a_2'(x) b_3(x) \\ &\quad + s_{23} a_2'(x) b_2'(x) \left(\operatorname{mod} x^{k_2-k_1}\right). \end{aligned} \tag{35}$$

If $q_2 + q_3 \ge k_2$ and $m := \min(q_3, 2q_2) \ge k_2 - k_1$, then $2q_3 \ge k_2$ and $q_2 + q_3 \ge k_2 - k_1$. Then the matrices M_{0A}, M_{0B} are zero and, as can be seen, from (33) and (35), we have $a_3(x) = b_3(x)$ and $a_2'(x) = b_2'(x)$. Let $q_2 + q_3 \ge k_2$ but $m < k_2 - k_1$. Then, in M_{0A}, M_{0B} the submatrices M_{3A}, M_{3B} and the second columns are zero, and from (33), (35) we have $a_3(x) = b_3(x)$, $a_2'(x) \equiv b_2'(x) (\operatorname{mod} x^m)$, respectively. Thus, in the columns $\overline{a}_2$, $\overline{b}_2$ of the matrices M_A, M_B the corresponding first $l = m - q_2$ elements coincide and the first l rows in the matrices M_{2A}, M_{2B} are zero. The first nonzero rows in these matrices are their $(l+1)$-th rows. They are equal among themselves. We will denote them by $\|u_{11} \ \ 0 \ \ u_{13}\|$. The corresponding elements in the columns $\overline{a}_2$, $\overline{b}_2$ are zero (see (8)). Therefore, we have $a_2'(x) \equiv b_2'(x) (\operatorname{mod} x^{m+1})$. This means that the following $(l+2)$-th rows in M_{2A}, M_{2B} coincide. We will denote them by $\|u_{21} \ \ 0 \ \ u_{23}\|$. From (35) it is clear that $\|u_{11} \ \ u_{13}\| \, \|s_{23} \ \ s_{12}\|^t = 0$. If $\|u_{11} \ \ u_{13}\|$, $\|u_{21} \ \ u_{23}\|$ are linearly independent, then the corresponding elements in $\overline{a}_2$, $\overline{b}_2$ are zero. On this basis, from (35) we have $\|u_{21} \ \ u_{23}\| \, \|s_{23} \ \ s_{12}\|^t = 0$. Therefore, $s_{23} = s_{12} = 0$ and all is proved. If $\|u_{11} \ \ u_{13}\|$, $\|u_{21} \ \ u_{23}\|$ are linearly dependent, then $\|u_{21} \ \ u_{23}\| \, \|s_{23} \ \ s_{12}\|^t = 0$, and this, as can be seen from (35), means that in the columns $\overline{a}_2$, $\overline{b}_2$ their $(l+2)$-th elements coincide. Therefore, the following $(l+3)$-th rows in the matrices M_{2A}, M_{2B} coincide. Let us denote them by $\|u_{31} \ \ 0 \ \ u_{33}\|$ and consider the cases where the rows $\|u_{11} \ \ u_{13}\|$, $\|u_{31} \ \ u_{33}\|$ are linearly independent and when they are linearly dependent. In the first case, as above, we will get $s_{23} = s_{12} = 0$. Then everything is proved. In the second case we obtain that the $(l+3)$-th elements in $\overline{a}_2$, $\overline{b}_2$ coincide. Therefore, the $(l+4)$-th rows in M_{2A}, M_{2B} coincide. Continuing similar consideration, at some step we will get $s_{23} = s_{12} = 0$. This on the basis of (35) means $a_2'(x) \equiv b_2'(x) (\operatorname{mod} x^{k_2-k_1})$, that is $a_2'(x) = b_2'(x)$, or we will come to $\overline{a}_2 = \overline{b}_2$. In any case we will have $A(x) = B(x)$.

Let $q_2+q_3 < k_2$. Then in the columns $\overline{a}_3, \overline{b}_3$, as is seen from (33), the first $l_1 = q_2$ elements coincide, and in the matrices M_{3A}, M_{3B} the first l_1 rows are zero. The first nonzero row in each of the matrices M_{3A}, M_{3B} is their (l_1+1)-th row, which has the form $\overline{u}_1 = \|1 \ \ 0 \ \ 0\|$. Since (l_1+1)-th elements in the columns $\overline{a}_3, \overline{b}_3$ are zero, then as is clear from (33), $s_{23} = 0$. This means, that the corresponding first $l_2 = q_3$, $l_2 > l_1$, elements in the columns $\overline{a}_3$, $\overline{b}_3$ coincide. If $2q_3 < k_2$, then the first linearly independent with the row $\overline{u}_1$ in each of the matrices M_{3A}, M_{3B} is the same (l_2+1)-th row $\overline{u}_2 = \|* \ \ 1 \ \ 0\|$. Since (l_2+1)-th elements in $\overline{a}_3, \overline{b}_3$ are zero, then from (33) we have $s_{13} = 0$. That means that $a_3(x) = b_3(x)$. As can be seen from (35), in the columns $\overline{a}_2, \overline{b}_2$ the corresponding first $l_3 = q_3 - q_2$ elements coincide. Then in the case $q_3 \geq k_2 - k_1$ we have $a_2(x) = b_2(x)$; that is, everything has already been proved. If $q_3 < k_2 - k_1$, then in each of the matrices M_{2A}, M_{2B}, the first l_3 rows are linearly dependent on the collection $\overline{u}_1, \overline{u}_2$, and their (l_3+1)-th row $\overline{u}_3 = \|* \ \ 0 \ \ 1\|$ is the first linearly independent of $\overline{u}_1$, $\overline{u}_2$ in these matrices. Since in the columns $\overline{a}_2$, $\overline{b}_2$ their (l_3+1)-th elements are zero, then from (35) we obtain $s_{12} = 0$. Then $\overline{a}_2 = \overline{b}_2$, $a_2(x) = b_2(x)$ and everything is proved.

If $2q_3 \geq k_2$, then from (33) we have $a_3(x) = b_3(x)$. Recall that in (33), (35) $s_{23} = 0$. Also $s_{12} = 0$, if $q_3 < k_2 - k_1$. Therefore, if $q_2 + q_3 \geq k_2 - k_1$, then everything has been proved. Otherwise, if $q_2 + q_3 < k_2 - k_1$, then in the columns $\overline{a}_2, \overline{b}_2$ the corresponding first $l_2 = q_3$ elements coincide and in each of the matrices M_{2A}, M_{2B} their (l_2+1)-th row $\|* \ \ 1 \ \ *\|$ is first linearly independent of the collection $\overline{u}_1$, $\overline{u}_3$. Since (l_2+1)-th elements in $\overline{a}_2$, $\overline{b}_2$ are zero, then from (35) we get $s_{13} = 0$. That is, every time $A(x) = B(x)$. The theorem is proved. □

Let in reduced matrices $A(x), B(x)$ of the form (1), (2) $a_2(x) = b_2(x) \equiv 0$, $a_1(x), a_3(x), b_1(x), b_3(x) \neq 0$. Define the polynomials $a_{31}(x)$ and $a_{33}(x)$ as follows:

$$\begin{aligned} a_{31}(x) &:\equiv a_3(x)\, a_1(x) \left(\operatorname{mod} x^{k_2}\right), \\ a_{33}(x) &:\equiv \left(a_3(x)\right)^2 \left(\operatorname{mod} x^{k_2}\right). \end{aligned} \tag{36}$$

We will construct columns $\overline{a}_3$, $\overline{a}_{31}$, and $\overline{a}_{33}$ of height $k_2 - co\deg a_3$ from the coefficients of the polynomials $a_3(x)$, $a_{31}(x)$, and $a_{33}(x)$, respectively. In these columns, in the first place, we put the coefficients of the monomials of degree $co\deg a_3$. Then, in the order of increasing powers of monomials, we place the rest of the coefficients, together with zero ones up to (k_2-1)-coefficients. We create columns $\overline{a}_1$, $\overline{a}_{03}$ and $\overline{a}_{13}$ $\overline{a}_{11}$ of height $k_1 - co\deg a_1$ from the coefficients of polynomials $a_1(x)$, $a_3(x)$ and from the coefficients of polynomials:

$$\begin{aligned} a_{13}(x) &:\equiv a_1(x)\, a_3(x) \left(\operatorname{mod} x^{k_1}\right), \\ a_{11}(x) &:\equiv \left(a_1(x)\right)^2 \left(\operatorname{mod} x^{k_1}\right), \end{aligned} \tag{37}$$

respectively. Here, in the first place, we put the coefficients of the monomials of degree $co\deg a_1$, and then in the order of increasing powers of monomials we place the remaining coefficients up to the monomials of degree $k_1 - 1$ inclusive. For $A(x)$ we construct matrices of the form

$$\begin{aligned} N_A &= \left\| \begin{matrix} \overline{a}_3 & \multirow{2}{*}{N_{0A}} \\ \overline{a}_1 & \end{matrix} \right\|, \\ N_{0A} &= \left\| \begin{matrix} N_{3A} \\ N_{1A} \end{matrix} \right\|, \\ N_{3A} &= \left\| \overline{0} \ \ \overline{a}_{33} \ \ \overline{a}_{31} \right\|, \\ N_{1A} &= \left\| \overline{a}_{03} \ \ \overline{a}_{13} \ \ \overline{a}_{11} \right\|. \end{aligned} \tag{38}$$

Obviously, each row in these matrices consists of the coefficient of the monomials of the same degree.

Quite similarly for $B(x)$ we construct matrices:

$$\begin{aligned} N_B &= \left\| \begin{matrix} \overline{b}_3 & \multirow{2}{*}{N_{0B}} \\ \overline{b}_1 & \end{matrix} \right\|, \\ N_{0B} &= \left\| \begin{matrix} N_{3B} \\ N_{1B} \end{matrix} \right\|, \\ N_{3B} &= \left\| \overline{0} \ \ \overline{b}_{33} \ \ \overline{b}_{31} \right\|, \\ N_{1B} &= \left\| \overline{b}_{03} \ \ \overline{b}_{13} \ \ \overline{b}_{11} \right\|. \end{aligned} \tag{39}$$

Theorem 3. *Let in the reduced matrix $A(x)$ of the form (1) $a_2(x) \equiv 0$, $a_1(x), a_3(x) \neq 0$ and $q_1 := co\deg a_1$, $q_3 := co\deg a_3$. Then $A(x) \approx B(x)$, where in the reduced matrix $B(x)$ of the form (2) $b_2(x) \equiv 0$ and the elements $b_1(x), b_3(x) \neq 0$ satisfy one of the following conditions.*

(1) *In $b_3(x)$ the $(2q_3)$-monomial is absent, if $q_3 < q_1$, $q_3 < k_2 - k_1$.*

(2) *In $b_3(x)$ the $(2q_3)$- and the (q_1+q_3)-monomials are absent, if $q_3 < q_1$, $q_3 \geq k_2 - k_1$.*

(3) *In $b_1(x)$ the q_3-monomial is absent and in the first of the polynomials $b_i(x)$, $i = 1, 3$, for which $q_i + q_3 < k_i$, the (q_i+q_3)-monomial is absent, if $q_3 > q_1$, $q_3 < k_2 - k_1$.*

(4) *In the first column of the matrix N_B (39) with the coefficients of the polynomials $b_3(x)$, $b_1(x)$ there are zero elements corresponding to the maximal system of the first linearly independent rows of the submatrix N_{0B}, if $q_3 > q_1$, $q_3 \geq k_2 - k_1$.*

The matrix $B(x)$ is uniquely defined.

Proof.

Existence. (1) The proof is completely analogous to the proof of condition (1) in Theorem 2.

(2) We can assume that in the matrix $A(x)$ element $a_3(x)$ does not contain $(2q_3)$-monomial. Otherwise, we act as in p. (1) of Theorem 2. If $q_1 + q_3 \geq k_2$, then everything is proved. Otherwise, we denote the (nonzero) coefficient of the $(q_1 + q_3)$-monomial in $a_3(x)$ by c_1 and apply to $A(x)$ ss.e.t.-I. In this case, in the left transforming matrix (see (7)) we put $s_{12} = c_1$. Elements $b_1(x)$, $b_3(x)$ of the resulting reduced matrix $B(x)$ satisfy the congruence

$$a_3(x) \equiv b_3(x) + s_{12}a_1(x)b_3(x)\left(\operatorname{mod} x^{k_2}\right), \quad (40)$$

from which it is seen that in $b_3(x)$, there is no $(q_1 + q_3)$-monomial. Since $q_3 < q_1$, then in $b_3(x)$, as in $a_3(x)$, there is no $(2q_3)$-monomial.

(3) Let $q_3 > q_1$ and $q_3 < k_2 - k_1$. If $q_3 \geq k_1$, then in $a_1(x)$ there is no q_3-monomial, since $\deg a_1 < k_1$. Otherwise, we denote the (nonzero) coefficient of the q_3-monomial in $a_1(x)$ by c_2 and apply to $A(x)$ ss.e.t.-II. Thus, in the left transforming matrix of the form (8) we put $s_{23} = -c_2$. In the resulting reduced matrix $B(x)$ element $b_1(x)$ satisfy the congruence (14) and $b_3(x) = a_3(x)$. From (14) it is clear that in $b_1(x)$ there is no q_3-monomial. In order not to introduce new notation, we assume that already in $A(x)$ the polynomial $a_1(x)$ does not contain a q_3-monomial. If $q_i + q_3 \geq k_i$, $i = 1, 3$, then everything has already been proved. Then the matrix $A(x)$ is the desired one. Otherwise, we find the first of two values $i = 1, i = 3$, for what $q_i + q_3 < k_i$. Denote by c_3 the coefficient of the $(q_i + q_3)$-monomial in the polynomial $a_i(x)$ for this value i (fixed now). Apply to $A(x)$ ss.e.t.-III. Here in the left transforming matrix of the form (9) we put $s_{13} = c_3$. As a result in the reduced matrix $B(x)$ the element $b_i(x)$ (for the above-defined index i) satisfies the congruence

$$a_i(x) \equiv b_i(x) + s_{13}a_3(x)b_i(x)\left(\operatorname{mod} x^{k_i}\right). \quad (41)$$

From this congruence it follows that in $b_i(x)$ there is no $(q_i + q_3)$-monomial. If $i = 1$, then from the same congruence it is also evident that in $b_1(x)$, as in $a_1(x)$, there is no q_3-monomial. The same thing is proved by the congruence

$$a_1(x) \equiv b_1(x) + s_{13}a_3(x)b_1(x)\left(\operatorname{mod} x^{k_1}\right), \quad (42)$$

if $i = 3$.

(4) If $N_{0A} = 0$, then everything has already been proved. Then the matrix $A(x)$ is the desired one. If $N_{0A} \neq 0$, but $N_{3A} = 0$, then the element $a_3(x)$ in $A(x)$ satisfies condition (4). This element will not change with all subsequent ss.e.t. Then $q_1 + q_3 \geq k_2$. This means that in N_{1A} the second column is zero. Let $\|d_{11} \;\; 0 \;\; d_{13}\|$ be the first nonzero row of the matrix N_{1A} and $\|d_1 \;\; d_{11} \;\; 0 \;\; d_{13}\|$ is the corresponding row of the matrix N_A. Let these rows be composed of the coefficients of the monomials of degree m_1. We find some solution $\|x_{10} \;\; x_{30}\|^t$ of the equation $\|-d_{11} \;\; d_{13}\| \, \|x_1 \;\; x_3\|^t = d_1$ and apply to $A(x)$ ss.e.t.-I-II. In this case, we put $s_{12} = x_{30}$, $s_{23} = x_{10}$, in the left transforming matrix S (see (20)). The element $b_3(x)$ of the resulting reduced matrix $B(x)$ coincides with $a_3(x)$, and the element $b_1(x)$ satisfies the congruence

$$\begin{aligned} a_1(x) + s_{23}a_3(x) &\equiv b_1(x) \\ &\quad + s_{12}a_1(x)b_1(x)\left(\operatorname{mod} x^{k_1}\right). \end{aligned} \quad (43)$$

It follows from this that in $b_1(x)$ there is no m_1-monomial and $a_1(x) \equiv b_1(x)(\operatorname{mod} x^{m_1})$. If $rank N_{0A} = 1$, then all subsequent rows in N_{1A} are linearly dependent on $\|d_{11} \;\; 0 \;\; d_{13}\|$, then the matrix $B(x)$ is to be sought. In order not to introduce new notations, we assume that the element $a_1(x)$ in $A(x)$ does not contain m_1-monomial. Let $\|d_{21} \;\; 0 \;\; d_{23}\|$ be the first linearly independent row from the row $\|d_{11} \;\; 0 \;\; d_{13}\|$ of the matrix N_{0A} and let $\|d_2 \;\; d_{21} \;\; 0 \;\; d_{23}\|$ be the corresponding row of the matrix N_A. Let these rows be composed of the coefficients of the monomials of degree $m_2 > m_1$. From equation

$$\left\| \begin{matrix} -d_{11} & d_{13} \\ -d_{21} & d_{31} \end{matrix} \right\| \left\| \begin{matrix} y_1 \\ y_3 \end{matrix} \right\| = \left\| \begin{matrix} 0 \\ d_2 \end{matrix} \right\| \quad (44)$$

we find its (unique) solution $\|y_{10} \;\; y_{30}\|^t \neq 0$. Using ss.e.t.-I-II we pass from $A(x)$ to the reduced matrix $B(x)$. Here in the left transforming matrix S of the form (20) we assume $s_{12} = y_{30}$, $s_{23} = y_{10}$. The element $b_1(x)$ in $B(x)$, as is shown in (43), does not contain m_1- and m_2-monomials and $b_3(x) = a_3(x)$. Therefore, $B(x)$ is the desired matrix.

Next let us consider the situation when $N_{3A} \neq 0$. Then, $n_1 := q_1 + q_3 < k_2$ and the row $\overline{u}_1 = \|0 \;\; 0 \;\; 1\|$ of the n_1-coefficients is the first nonzero row of the matrix N_{3A}. We do the first step. It is ss.e.t.-I of the matrix $A(x)$, in which we put the n_1-coefficient of polynomial $a_3(x)$ instead of s_{12} in the left transforming matrix (see (7)). In the obtained reduced matrix $B(x)$, element $b_3(x)$ does not contain a n_1-monomial (see p. (2)). If $n_2 := 2q_3 < k_2$, then the first linearly independent row with row $\overline{u}_1$ in N_{3B} is a row $\overline{u}_2 = \|0 \;\; 1 \;\; *\|$ of n_2-coefficients, $n_2 > n_1$. We do the second step. This is ss.e.t.-III of the matrix $B(x)$, in which in the left transforming matrix of the form (9) we assume the n_2-coefficient of polynomial $b_3(x)$ instead of s_{13}. In the matrix obtained after the second step, the element in position $(3, 1)$ does not contain n_1- and n_2-monomials. In order not to introduce new notations for the matrices resulting from transformations, we assume that the element $a_3(x)$ in $A(x)$ has this property. If $n_3 := q_3 < k_1$, then in the matrix N_{0A} the row $\overline{u}_3 = \|1 \;\; 0 \;\; *\|$ of the n_3-coefficients is the first linearly independent row with the collection $\overline{u}_1, \overline{u}_2$. To $A(x)$, apply ss.e.t.-II. In this case, in place of s_{23} in the left transforming matrix (see (8)) we put the n_3-coefficient of the polynomial $a_1(x)$ taken with the opposite sign. This will be the third step. In the resulting matrix, the element in position $(2, 1)$ does not contain a n_3-coefficient. Since the element in position $(3, 1)$ does not change at the same time, such a matrix is the desired one.

If after the first step it turns out that $n_2 \geq k_2$, but $n_3 < k_1$, then we do the third step. Further, if $n_4 := q_1 + q_3 < k_1$, then the row $\|* \;\; 1 \;\; *\|$ of the n_4-coefficients in the matrix analogous to N_{0A} is the first linearly independent with the collection $\overline{u}_1, \overline{u}_3$. We do the fourth step. It will be the ss.e.t-III of the matrix obtained after the third step. In this case, in the left transforming matrix (see (9)) instead of s_{13}, we put the n_4-coefficient of the polynomial in the position $(2, 1)$. As a result, the element in the position $(3, 1)$ will not change, and $(2, 1)$-element will be free from n_3- and n_4-monomials. The resulting matrix is the required one.

Uniqueness. (1) Let $A(x) \approx B(x)$ and the elements of the reduced matrices $A(x)$, $B(x)$ of the form (1), (2) satisfy condition (1) of the theorem. Then, taking into account Corollary 1 and Remark [3], we can write the congruences:

$$a_1(x) - b_1(x) - s_{12}a_1(x)b_1(x) - s_{13}a_3(x)b_1(x) \equiv 0 \left(\operatorname{mod} x^{k_1}\right), \tag{45}$$

$$a_3(x) - b_3(x) - s_{12}a_1(x)b_3(x) - s_{13}a_3(x)b_3(x) \equiv 0 \left(\operatorname{mod} x^{k_2}\right). \tag{46}$$

Since $s_{12}x^{k_1}b_3(x) \equiv 0 \ (\operatorname{mod} x^{k_2})$, then $s_{12} = 0$. If $2q_3 \geq k_2$, then $q_1 + q_3 > k_2$. Then from (45) and (46) we have $a_1(x) = b_1(x)$ and $a_3(x) = b_3(x)$, respectively. Otherwise, when $2q_3 < k_2$, comparing the $(2q_3)$-coefficients in both parts of the congruence (46), we arrive at $s_{13} = 0$. Therefore, in each case, $A(x) = B(x)$.

(2) Let condition (2) of the theorem be satisfied for the reduced matrices $A(x)$, $B(x)$ of the form (1), (2) and $A(x) \approx B(x)$. Then we can write (45) and (46). If $2q_3 \geq k_2$, then $q_1 + q_3 > k_2$ and $2q_1 \geq k_1$. Then from (45) and (46) we have $a_1(x) = b_1(x)$ and $a_3(x) = b_3(x)$, respectively. Otherwise, when $2q_3 < k_2$, from (46) we obtain $s_{13} = 0$, since $q_1 + q_3 > 2q_3$ and in $a_3(x)$, $b_3(x)$ there are no $(2q_3)$-monomials. If $q_1 + q_3 \geq k_2$, then again from (45) and (46) we obtain the coincidence $a_1(x)$, $a_3(x)$ from $b_1(x)$, $b_3(x)$, respectively. If $q_1 + q_3 < k_2$, then comparing the $(q_1 + q_3)$-coefficients in both parts (46) we obtain $s_{12} = 0$. It means $A(x) = B(x)$.

(3) Let the condition (3) of the theorem be satisfied in the reduced ss.e. matrices $A(x)$, $B(x)$ of the form (1), (2). Then for the elements of these matrices we have the fulfillment of the congruence (46), where $s_{12} = 0$, and the congruence

$$a_1(x) + s_{23}a_3(x) - b_1(x) - s_{13}a_3(x)b_1(x) \equiv 0 \left(\operatorname{mod} x^{k_1}\right). \tag{47}$$

If $q_3 < k_1$, then from (47) we have $s_{23} = 0$. In any case, the second term in (47) can be omitted. Then (46) (where $s_{12} = 0$) and (47) can be presented in the form

$$a_i(x) - b_i(x) - s_{13}a_3(x)b_i(x) \equiv 0 \left(\operatorname{mod} x^{k_i}\right), \quad i = 1, 3 \tag{48}$$

If $q_i + q_3 \geq k_i$ for each $i = 1, 3$, then immediately we have $a_i(x) = b_i(x)$. Otherwise, if we compare the $(q_i + q_3)$-coefficients in both parts (48) for the first of two values of the index i such that $q_i + q_3 < k_i$, then we obtain $s_{13} = 0$. In any case $A(x) = B(x)$.

(4) Let the reduced matrices $A(x)$, $B(x)$ of the form (1), (2) be ss.e. and let them satisfy condition (4) of the theorem. Then for their elements it is possible to write the congruence (46) and congruence

$$a_1(x) + s_{23}a_3(x) - b_1(x) - s_{12}a_1(x)b_1(x) - s_{13}a_3(x)b_1(x) \equiv 0 \left(\operatorname{mod} x^{k_1}\right). \tag{49}$$

If $q_1 + q_3 \geq k_2$ and $m = \min(q_3, 2q_1) \geq k_1$, then in matrices N_A, N_B we have $N_{0A} = N_{0B} = 0$ and, as can be seen from (46), (49), $a_3(x) = b_3(x)$, $a_1(x) = b_1(x)$.

If $q_1 + q_3 \geq k_2$ and $m < k_1$, then in matrices N_{0A}, N_{0B} we have $N_{3A} = N_{3B} = 0$ and the second columns in N_{1A}, N_{1B} are zero. In this case, from (46) and (49) we have $a_3(x) = b_3(x)$ and $a_1(x) \equiv b_1(x) \ (\operatorname{mod} x^m)$, respectively. Therefore, in subcolumns $\overline{a}_1, \overline{b}_1$ of the matrices N_A, N_B the first $l = m - q_1$ elements coincide and the corresponding l rows in the matrices N_{1A}, N_{1B} are zero. In these matrices their $(l+1)$-th rows are the first nonzero rows. They are the same. We will denote them by $\overline{v}_1 = \|v_{11} \ \ 0 \ \ v_{13}\|$. The elements in $\overline{a}_1, \overline{b}_1$ corresponding to these rows are zero (see (8)). So we really have $a_1(x) \equiv b_1(x) \ (\operatorname{mod} x^{m+1})$. This means that the $(l+2)$-th rows in the matrices N_{1A}, N_{1B} coincide. We will denote them by $\overline{v}_2 = \|v_{21} \ \ 0 \ \ v_{23}\|$. From (49) it is clear that $\|v_{11} \ \ v_{13}\| \, \|s_{23} \ \ s_{13}\|^t = 0$. If $\overline{v}_1, \overline{v}_2$ are linearly independent, then in $\overline{a}_1, \overline{b}_1$ there are zero elements corresponding to $\overline{v}_2$. Therefore, on the basis of (49) we have $\|v_{21} \ \ v_{23}\| \, \|s_{23} \ \ s_{13}\|^t = 0$. Hence $s_{23} = s_{12} = 0$ and everything is proved. If $\overline{v}_1, \overline{v}_2$ are linearly dependent, then $\|v_{21} \ \ v_{23}\| \, \|s_{23} \ \ s_{13}\|^t = 0$, and this on the basis of (49) means that the $(l+2)$-th elements in $\overline{a}_1, \overline{b}_1$ coincide. Therefore, the following $(l+3)$-th rows in N_{1A}, N_{1B} coincide. Denote them as $\overline{v}_3 = \|v_{31} \ \ 0 \ \ v_{33}\|$ and again consider two situations: when $\overline{v}_1, \overline{v}_3$ are linearly independent and when they are linearly dependent. In the first case, we will have that $s_{23} = s_{12} = 0$. Then the proof is finished. In the second case we have that the $(l+3)$-th elements in $\overline{a}_1, \overline{b}_1$ coincide. This means that in N_{1A}, N_{1B} the $(l+4)$-th rows coincide. Continuing similarly to our considerations, at some point we will obtain $s_{23} = s_{12} = 0$ or eventually we will have $\overline{a}_1 = \overline{b}_1$. In any case $A(x) = B(x)$.

Let us now consider $q_1 + q_3 < k_2$. Then in columns $\overline{a}_3, \overline{b}_3$, as can be seen from (46), their first $l_1 = q_1$ corresponding elements coincide, and in matrices N_{3A}, N_{3B} their first l_1 rows are zero. The first nonzero row in these matrices N_{3A}, N_{3B} is a row $\overline{u}_1 = \|0 \ \ 0 \ \ 1\|$. Since the $(l_1 + 1)$-th elements in $\overline{a}_3, \overline{b}_3$ are zero (see (8)), then from (46) $s_{12} = 0$ follows. This means that $a_3(x) \equiv b_3(x) \ (\operatorname{mod} x^{2l_2})$, where $l_2 = q_3$. If $2l_2 < k_2$, then in $\overline{a}_3, \overline{b}_3$ the first l_2 corresponding elements coincide. Therefore, in N_{3A}, N_{3B} the first l_2 corresponding rows coincide, and each of them is linearly dependent with $\overline{u}_1$. The first linearly independent row with $\overline{u}_1$ in each of the matrix N_{3A}, N_{3B} is their (l_2+1)-th row $\overline{u}_2 = \|0 \ \ 1 \ \ *\|$. Since (l_2+1)-th corresponding elements in $\overline{a}_3, \overline{b}_3$ are zero, it follows from (46) that $s_{13} = 0$ and $a_3(x) = b_3(x)$. If $q_3 \geq k_1$, then $a_1(x) = b_1(x)$ and all is proved. In another case, as can be seen from (49), in $\overline{a}_1, \overline{b}_1$, the first $l_3 = q_3 - q_1$ corresponding elements coincide, and in each of the matrix N_{1A}, N_{1B} each of their first l_3 rows is linearly dependent on the collection $\overline{u}_1, \overline{u}_2$. The first linearly independent row with the collection $\overline{u}_1, \overline{u}_2$ in each of the matrices N_{1A}, N_{1B} is one and the same (l_3+1)-th row $\overline{u}_3 = \|1 \ \ 0 \ \ *\|$. Since (l_3+1)-th elements in $\overline{a}_1, \overline{b}_1$ are zero, then from (49) $s_{23} = 0$ follows. Then $a_1(x) = b_1(x)$.

Let $2l_2 \geq k_2$, and, as before, $q_1 + q_3 < k_2$. Then from (46), where $s_{12} = 0$, we have at once $a_3(x) = b_3(x)$. If $q_1 + q_3 \geq k_1$, then from (49) $a_1(x) = b_1(x)$ follows. If $q_1 + q_3 < k_1$, then in $\overline{a}_1, \overline{b}_1$ the corresponding first l_2 elements and in N_{1A}, N_{1B} the corresponding first l_2 rows coincide. Each of the these rows is linearly dependent on the collection $\overline{u}_1, \overline{u}_3$. In each of the matrices N_{1A}, N_{1B} their (l_2+1)-th rows $\|* \quad 1 \quad *\|$ are the first linearly independent rows with the collection $\overline{u}_1, \overline{u}_3$. Since $(l_2 + 1)$-th elements in $\overline{a}_1, \overline{b}_1$ are zero, then from (49), where $s_{23} = 0$, we will have $s_{13} = 0$. In any case $A(x) = B(x)$. The theorem is proved. □

Theorem 4. *Let the elements of the reduced matrix $A(x)$ of the form (1) satisfy conditions: $a_1(x), a_2(x) \neq 0$, $a_3(x) \equiv 0$, $a_2(x) = x^{k_1} a_2'(x)$, and $q_1 := co\deg a_1$, $q_2 := co\deg a_2'$. Then, $A(x) \approx B(x)$, where the elements of the reduced matrix $B(x)$ of the form (2) satisfy conditions: $b_1(x), b_2(x) \neq 0$, $b_3(x) \equiv 0$, $b_2(x) = x^{k_1} b_2'(x)$, and in $b_2(x)$ there are no $(2q_2 + k_1)$- and $(2q_2 + q_1 + k_1)$-monomials. The matrix $B(x)$ is uniquely determined.*

Proof.

Existence. If $2q_2 - k_1 \geq k_2$, then everything has been already proved. Then $B(x) = A(x)$. Otherwise, apply to $A(x)$ ss.e.t.-II. In this case, instead of s_{23} in the left transforming matrix (see (8)) we assume the $(2q_2 + k_1)$-coefficient in $a_2(x)$. Then we obtain a matrix $B(x)$ of the form (2), which will also be reduced, and its elements $b_1(x) = a_1(x)$, $b_2(x) \neq 0$, $b_3(x) \equiv 0$, satisfy the congruence (15). From the latter it can be seen that in $b_2(x)$ there is no $(2q_2 + k_1)$-monomial and $co\deg a_2 = co\deg b_2$. If in the obtained matrix $B(x)$ we have $2q_2 + q_1 + k_1 \geq k_2$, then everything has been already proved. Then the matrix $B(x)$ is the desired one. Otherwise apply to $B(x)$ ss.e.t.-III. To do this, in the left transforming matrix (see (9)) instead of s_{13} we put the $(2q_2 + q_1 + k_1)$-coefficient in $b_2(x)$, taken with opposite sign. The resulting matrix will be the sought one.

Uniqueness. Let $A(x) \approx B(x)$, where $A(x)$, $B(x)$ are reduced matrices of the form (1), (2), in which $a_3(x) = b_3(x) = 0$, $a_1(x), a_2(x), b_1(x), b_2(x) \neq 0$, and in each of $a_2(x)$, $b_2(x)$ there are no $(2q_2+k_1)$- and $(2q_2+q_1+k_1)$-monomials. For elements of the matrices $A(x)$, $B(x)$ a congruence (11) and congruences

$$b_2'(x)\left(a_1(x) - b_1(x) - s_{12}a_1(x)b_1(x)\right) \equiv 0 \left(\operatorname{mod} x^{k_2}\right), \tag{50}$$

$$a_2(x) - b_2(x) - s_{23}a_2'(x)b_2(x) + s_{12}b_1(x)b_2(x) + s_{13}b_1(x)b_2(x)a_2'(x) \equiv 0 \left(\operatorname{mod} x^{k_2}\right). \tag{51}$$

are fulfilled. We recall that according to the definition of the reduced matrix in $a_1(x), b_1(x)$ there are no $(2q_1)$-monomials. If $2q_1 < k_1$, then from (11) we obtain $s_{12} = 0$. The same is true from (50), if $2q_1 \geq k_1$, but $2q_1 + q_2 < k_2$. Then $a_1(x) = b_1(x)$. If $2q_2 + k_1 < k_2$, then from (51) we have $s_{23} = 0$. Similarly, if $2q_2 + q_1 + k_1 < k_2$, then from (51) we obtain $s_{13} = 0$. Then $a_2(x) = b_2(x)$. Note that from $2q_1 + q_2 \geq k_2$ follows $q_1 + q_2 + k_1 \geq k_2$. Therefore, regardless of whether $2q_2 + k_1 \geq k_2$ or $2q_2 + q_1 + k_1 \geq k_2$, the coincidence $a_2(x)$ and $b_2(x)$ will be obtained from (51). The theorem is proved. □

4. Conclusions

The matrix $B(x)$, established by each of Theorems 1–4, can be considered canonical for the class $\{PF(x)Q(x)\}$ of ss.e. matrices. It can be applied to the classification of sets of numerical matrices (over the field $\mathbf{C}$) with respect to simultaneous similarity. In this context, the work of [12–15] should be noted. From the proof of Theorems 1–4 we can construct an algorithm for finding the transforming matrices (left nonsingular numerical and right invertible polynomial) that reduce $A(x)$ to the canonical matrix $B(x)$.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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