

Research Article

Related Results to Hybrid Pair of Mappings and Applications in Bipolar Metric Spaces

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In this paper, we introduce the concept of multivalued contraction mappings in partially ordered bipolar metric spaces and establish the existence of unique coupled fixed point results for multivalued contractive mapping by using mixed monotone property in partially ordered bipolar metric spaces. Some interesting consequences of our results are obtained.

1. Introduction and Preliminaries

Fixed point theory has been playing a vital role in the study of nonlinear phenomena. The Banach fixed point theorem or contraction mapping principle was proved by Banach [1] in 1922. Turinici [2] extended the Banach contraction principle in the setting of partially ordered sets and laid the foundation of a new trend in fixed point theory.

The theory of mixed monotone multivalued mappings in ordered Banach spaces was extensively investigated by Y. Wu [3]. Existence of fixed points in ordered metric spaces was initiated by Ran and Reurings [4], and later on several authors studied the problem of existence and uniqueness of a fixed point for mappings satisfying different contractive conditions in the framework of partially ordered metric spaces ([1–27] and references therein).

In [19], Bhaskar and Lakshmikantham introduced the concept of coupled fixed point and proved some coupled fixed point theorems in partially ordered metric spaces (see also [1–27] for more works). The study of fixed points for multivalued contraction mappings using the Hausdorff metric was initiated by Markin [20]. Later, many authors established hybrid fixed point theorems and gave applications of their results (see also [21–24]).

Very recently, in 2016 Mutlu and Gürdal [25] introduced the notion of bipolar metric spaces. Also they investigated some fixed point and coupled fixed point results on this space (see [25, 26]).

This paper aims to introduce some coupled fixed point theorems for a multivalued mappings satisfying various contractive conditions defined on partially ordered bipolar metric spaces. We have illustrated the validity of the hypotheses of our results.

First we recall some basic definitions and results.

Definition 1 ([25]). Let A and B be two nonempty sets. Suppose that $d : A \times B \rightarrow [0, \infty)$ is a mapping satisfying the following properties:

- (B_0) $d(a, b) = 0$ if and only if $a = b$ for all $(a, b) \in A \times B$,
- (B_1) $d(a, b) = d(b, a)$, for all $a, b \in A \cap B$,
- (B_2) $d(a_1, b_2) \leq d(a_1, b_1) + d(a_2, b_1) + d(a_2, b_2)$, for all $a_1, a_2 \in A, b_1, b_2 \in B$.

Then the mapping d is called a bipolar metric on the pair (A, B) and the triple (A, B, d) is called a bipolar metric space.

Definition 2 ([25]). Assume (A_1, B_1) and (A_2, B_2) as two pairs of sets.

The function $F : A_1 \cup B_1 \longrightarrow A_2 \cup B_2$ is said to be a covariant map, if $F(A_1) \subseteq A_2$ and $F(B_1) \subseteq B_2$ and denote this as $F : (A_1, B_1) \rightrightarrows (A_2, B_2)$.

The mapping $F : A_1 \cup B_1 \longrightarrow A_2 \cup B_2$ is said to be a contravariant map, if $F(A_1) \subseteq B_2$ and $F(B_1) \subseteq A_2$ and denote this as $F : (A_1, B_1) \leftrightsquigarrow (A_2, B_2)$.

In particular, d_1 and d_2 are bipolar metrics in (A_1, B_1) and (A_2, B_2) , respectively. Sometimes we use the notations $F : (A_1, B_1, d_1) \rightrightarrows (A_2, B_2, d_2)$ and $F : (A_1, B_1, d_1) \leftrightsquigarrow (A_2, B_2, d_2)$.

Definition 3 ([25]). Let (A, B, d) be a bipolar metric space. A point $v \in A \cup B$ is said to be left point if $v \in A$, a right point if $v \in B$, and a central point if both.

Similarly, a sequence $\{a_n\}$ on the set A and a sequence $\{b_n\}$ on the set B are called a left and right sequence, respectively.

In a bipolar metric space, sequence is the simple term for a left or right sequence.

A sequence $\{v_n\}$ is convergent to a point v if and only if $\{v_n\}$ is a left sequence, v is a right point, and $\lim_{n \rightarrow \infty} d(v_n, v) = 0$; or $\{v_n\}$ is a right sequence, v is a left point, and $\lim_{n \rightarrow \infty} d(v, v_n) = 0$.

A bisequence $(\{a_n\}, \{b_n\})$ on (A, B, d) is sequence on the set $A \times B$. If the sequences $\{a_n\}$ and $\{b_n\}$ are convergent, then the bisequence $(\{a_n\}, \{b_n\})$ is said to be convergent. $(\{a_n\}, \{b_n\})$ is Cauchy sequence, if $\lim_{n, m \rightarrow \infty} d(a_n, b_m) = 0$. In a bipolar metric space, every convergent Cauchy bisequence is biconvergent.

A bipolar metric space is called complete, if every Cauchy bisequence is convergent, hence biconvergent.

Now we give our main results.

2. Main Results

The following definitions and results will be needed in the sequel.

Let (A, B, d) be a bipolar metric space. For points $a \in A, b \in B$ and the subsets $X \subseteq A, Y \subseteq B$, consider the bipolar metric $d(a, Y) = \inf\{d(a, y) / y \in Y\}$ and $d(X, b) = \inf\{d(x, b) / x \in X\}$. We denote by $CB(A)$ and $CB(B)$ a class of all nonempty closed and bounded subsets of A and B , respectively. Also denote $A^2 = A \times A$ and $B^2 = B \times B$. Let H be the Hausdorff bipolar metric induced by the bipolar metric d on (A, B) ; that is,

$$H(X, Y) = \max \left\{ \sup_{x \in X} d(x, B), \sup_{y \in Y} d(A, y) \right\}, \quad (1)$$

for every $X \in CB(A)$ and $Y \in CB(B)$.

Definition 4. Let $F : A^2 \cup B^2 \longrightarrow CB(A \cup B)$ be given the mapping; an element $(a, b) \in A^2 \cup B^2$ is called a coupled fixed point of a set valued mapping F if $a \in F(a, b)$ and $b \in F(b, a)$.

Lemma 5 ([21]). Let $\kappa \geq 0$. If $X \in CB(A), Y \in CB(B)$ with $H(X, Y) \leq \kappa$, then, for each $x \in X$, there exists an element $y \in Y$ such that $d(x, y) \leq \kappa$.

Definition 6. Let (A, B, \leq) be a partially ordered set and let $F : (A^2, B^2) \rightrightarrows CB(A, B)$ be covariant map. We say that F has the mixed monotone property if F is monotone-nondecreasing in its first argument a and is monotone-nonincreasing in its second argument b , that is, for any $(a, b) \in A^2 \cup B^2$.

$$\begin{aligned} (a_1, a_2) \in A^2, \quad a_1 \leq a_2 &\implies F(a_1, b) \leq F(a_2, b) \\ (b_1, b_2) \in B^2, \quad b_1 \leq b_2 &\implies F(a, b_1) \geq F(a, b_2) \end{aligned} \quad (2)$$

Note that if $a_1 \leq a_2, b_1 \geq b_2$, and F has mixed monotone property, by Definition 6, we obtain $F(a_1, b_1) \leq F(a_2, b_2)$ and $F(b_1, a_1) \geq F(b_2, a_2)$.

Theorem 7. Let (A, B, \leq) be a partially ordered set such that there exists a bipolar metric d on (A, B) with (A, B, d) being complete bipolar metric spaces. Consider the covariant mapping $F : (A^2, B^2) \rightrightarrows CB(A, B)$ satisfying the following condition:

$$H(F(a, b), F(p, q)) \leq \mu d(a, p) + \kappa d(b, q) \quad (3)$$

for all $a, b \in A, p, q \in B$, and μ, κ are nonnegative constants with $\mu + \kappa < 1$.

(7.1) F has a mixed monotone property

(7.2) There exists $(a_0, b_0) \in A^2 \cup B^2$ and, for some $a_1 \in F(a_0, b_0), b_1 \in F(b_0, a_0)$, we have $a_0 \leq a_1$ and $b_0 \geq b_1$

(7.3) If a nondecreasing sequence $(\{a_n\}, \{p_n\})$ is convergent to (p, a) for $a \in A, p \in B$, then $a_n \leq p, p_n \leq a$ for all n and if a nonincreasing sequence $(\{b_n\}, \{q_n\})$ is convergent to (q, b) for $b \in A, q \in B$, then $b_n \geq q, q_n \geq b$ for all n

Then F has a coupled fixed point.

Proof. Let $a_0, b_0 \in A$ and $p_0, q_0 \in B$. Consider the sequences $\{a_n\}, \{b_n\}, \{p_n\}$ and $\{q_n\}$ such that $a_1 \in F(a_0, b_0), b_1 \in F(b_0, a_0)$ and $p_1 \in F(p_0, q_0), q_1 \in F(q_0, p_0)$. By (7.2), we have that $a_0 \leq p_1$ and $b_0 \geq q_1$ and $p_0 \leq a_1$ and $q_0 \geq b_1$, where $a_1, b_1 \in A$ and $p_1, q_1 \in B$. \square

Applying this in inequality (3), we have

$$H(F(a_0, b_0), F(p_1, q_1)) \leq \mu d(a_0, p_1) + \kappa d(b_0, q_1) \quad (4)$$

and

$$H(F(b_0, a_0), F(q_1, p_1)) \leq \mu d(b_0, q_1) + \kappa d(a_0, p_1). \quad (5)$$

On adding (4) and (5), we get

$$\begin{aligned} H(F(a_0, b_0), F(p_1, q_1)) + H(F(b_0, a_0), F(q_1, p_1)) \\ \leq (\mu + \kappa)(d(a_0, p_1) + d(b_0, q_1)) \end{aligned} \quad (6)$$

On the other hand

$$H(F(a_1, b_1), F(p_0, q_0)) \leq \mu d(a_1, p_0) + \kappa d(b_1, q_0) \quad (7)$$

and

$$H(F(b_1, a_1), F(q_0, p_0)) \leq \mu d(b_1, q_0) + \kappa d(a_1, p_0) \quad (8)$$

On adding (7) and (8), we get

$$\begin{aligned} H(F(a_1, b_1), F(p_0, q_0)) + H(F(b_1, a_1), F(q_0, p_0)) \\ \leq (\mu + \kappa)(d(a_1, p_0) + d(b_1, q_0)) \end{aligned} \quad (9)$$

Moreover,

$$H(F(a_0, b_0), F(p_0, q_0)) \leq \mu d(a_0, p_0) + \kappa d(b_0, q_0) \quad (10)$$

and

$$H(F(b_0, a_0), F(q_0, p_0)) \leq \mu d(b_0, q_0) + \kappa d(a_0, p_0) \quad (11)$$

On adding (10) and (11), we get

$$\begin{aligned} H(F(a_0, b_0), F(p_0, q_0)) + H(F(b_0, a_0), F(q_0, p_0)) \\ \leq (\mu + \kappa)(d(a_0, p_0) + d(b_0, q_0)) \end{aligned} \quad (12)$$

Also, if $d(a_0, p_1) = d(b_0, q_1) = 0$, then $a_0 = p_1 \in F(p_0, q_0), b_0 = q_1 \in F(q_0, p_0)$. If $d(a_1, p_0) = d(b_1, q_0) = 0$, then $p_0 = a_1 \in F(a_0, b_0), q_0 = b_1 \in F(b_0, a_0)$. If $d(a_0, p_0) = d(b_0, q_0) = 0$, then $a_0 = p_0, b_0 = q_0$.

It follows that (a_0, b_0) is a coupled fixed point of F .

Assume that either $d(a_0, p_1) \neq 0$ or $d(b_0, q_1) \neq 0$ and $d(a_1, p_0) \neq 0$ or $d(b_1, q_0) \neq 0$; also $d(a_0, p_0) \neq 0$ or $d(b_0, q_0) \neq 0$.

Since $a_1 \in F(a_0, b_0), b_1 \in F(b_0, a_0)$, then from (6) and Lemma 5 there exist $p_2 \in F(p_1, q_1), q_2 \in F(q_1, p_1)$ such that

$$\begin{aligned} d(a_1, p_2) + d(b_1, q_2) \\ \leq (\mu + \kappa)(d(a_0, p_1) + d(b_0, q_1)) \end{aligned} \quad (13)$$

and since $p_1 \in F(p_0, q_0), q_1 \in F(q_0, p_0)$, then from (9) and Lemma 5 there exist $a_2 \in F(a_1, b_1), b_2 \in F(b_1, a_1)$ such that

$$\begin{aligned} d(a_2, p_1) + d(b_2, q_1) \\ \leq (\mu + \kappa)(d(a_1, p_0) + d(b_1, q_0)) \end{aligned} \quad (14)$$

Also since $a_1 \in F(a_0, b_0), b_1 \in F(b_0, a_0)$ and $p_1 \in F(p_0, q_0), q_1 \in F(q_0, p_0)$, from (12) and Lemma 5 then we have

$$\begin{aligned} d(a_1, p_1) + d(b_1, q_1) \\ \leq (\mu + \kappa)(d(a_0, p_0) + d(b_0, q_0)) \end{aligned} \quad (15)$$

Since, $a_0 \leq p_1, b_0 \geq q_1$ and $p_0 \leq a_1, q_0 \geq b_1, a_1 \in F(a_0, b_0), p_1 \in F(p_0, q_0), b_1 \in F(b_0, a_0), q_1 \in F(q_0, p_0)$ and $a_2 \in F(a_1, b_1), p_2 \in F(p_1, q_1), b_2 \in F(b_1, a_1), q_2 \in F(q_1, p_1)$, by assumption (7.1), we get

$$\begin{aligned} a_1 &\leq p_2, \\ b_1 &\geq q_2 \\ \text{and } p_1 &\leq a_2, \\ q_1 &\geq b_2. \end{aligned} \quad (16)$$

Similarly from (3) and above, we have

$$\begin{aligned} d(a_2, p_3) + d(b_2, q_3) \\ \leq (\mu + \kappa)(d(a_1, p_2) + d(b_1, q_2)) \end{aligned} \quad (17)$$

and

$$d(a_3, p_2) + d(b_3, q_2) \leq (\mu + \kappa)(d(a_2, p_1), d(b_2, q_1)) \quad (18)$$

and also

$$d(a_2, p_2) + d(b_2, q_2) \leq (\mu + \kappa)(d(a_1, p_1), d(b_1, q_1)) \quad (19)$$

Since, we have $a_1 \leq p_2, b_1 \geq q_2$ and $p_1 \leq a_2, q_1 \geq b_2, a_2 \in F(a_1, b_1), p_2 \in F(p_1, q_1), b_2 \in F(b_1, a_1), q_2 \in F(q_1, p_1)$ and $a_3 \in F(a_2, b_2), p_3 \in F(p_2, q_2), b_3 \in F(b_2, a_2), q_3 \in F(q_2, p_2)$. Again, applying our assumption (7.1), we get

$$\begin{aligned} a_2 &\leq p_3, \\ b_2 &\geq q_3 \\ \text{and } p_2 &\leq a_3, \\ q_2 &\geq b_3 \end{aligned} \quad (20)$$

Continuing similarly this process, we have $a_{n+1} \in F(a_n, b_n), p_{n+1} \in F(p_n, q_n), b_{n+1} \in F(b_n, a_n), q_{n+1} \in F(q_n, p_n)$ with

$$\begin{aligned} a_n &\leq p_{n+1}, \\ b_n &\geq q_{n+1} \\ \text{and } p_n &\leq a_{n+1}, \\ q_n &\geq b_{n+1} \end{aligned} \quad (21)$$

such that

$$\begin{aligned} d(a_n, p_{n+1}) + d(b_n, q_{n+1}) \\ \leq (\mu + \kappa)(d(a_{n-1}, p_n) + d(b_{n-1}, q_n)) \end{aligned} \quad (22)$$

and

$$\begin{aligned} d(a_{n+1}, p_n) + d(b_{n+1}, q_n) \\ \leq (\mu + \kappa)(d(a_n, p_{n-1}) + d(b_n, q_{n-1})) \end{aligned} \quad (23)$$

and also

$$\begin{aligned} d(a_n, p_n) + d(b_n, q_n) \\ \leq (\mu + \kappa)(d(a_{n-1}, p_{n-1}) + d(b_{n-1}, q_{n-1})) \end{aligned} \quad (24)$$

Put $t_n = d(a_n, p_{n+1}) + d(b_n, q_{n+1})$ for any $n \in N$; then

$$t_n \leq (\mu + \kappa)(t_{n-1}) \quad (25)$$

Put $s_n = d(a_{n+1}, p_n) + d(b_{n+1}, q_n)$ for any $n \in N$; then

$$s_n \leq (\mu + \kappa)(s_{n-1}) \quad (26)$$

Put $r_n = d(a_n, p_n) + d(b_n, q_n)$ for any $n \in N$; then

$$r_n \leq (\mu + \kappa)(r_{n-1}) \quad (27)$$

Therefore $\{t_n\}$, $\{s_n\}$, and $\{r_n\}$ are nonincreasing sequences. From (25), (26), and (27) we have that

$$\begin{aligned} \lim_{n \rightarrow \infty} t_n &= 0, \\ \lim_{n \rightarrow \infty} s_n &= 0 \\ \text{and } \lim_{n \rightarrow \infty} r_n &= 0 \end{aligned} \tag{28}$$

which implies that

$$\begin{aligned} \lim_{n \rightarrow \infty} d(a_n, p_{n+1}) &= \lim_{n \rightarrow \infty} d(b_n, q_{n+1}) = 0, \\ \lim_{n \rightarrow \infty} d(a_{n+1}, p_n) &= \lim_{n \rightarrow \infty} d(b_{n+1}, q_n) = 0, \\ \lim_{n \rightarrow \infty} d(a_n, p_n) &= \lim_{n \rightarrow \infty} d(b_n, q_n) = 0. \end{aligned} \tag{29}$$

Using the property (B_2) , we have

$$\begin{aligned} d(a_n, p_m) &\leq d(a_n, p_{n+1}) + d(a_{n+1}, p_{n+1}) + \dots \\ &\quad + d(a_{m-1}, p_m) \\ d(b_n, q_m) &\leq d(b_n, q_{n+1}) + d(b_{n+1}, q_{n+1}) + \dots \\ &\quad + d(b_{m-1}, q_m) \end{aligned} \tag{30}$$

and

$$\begin{aligned} d(a_m, p_n) &\leq d(a_m, p_{m-1}) + d(a_{m-1}, p_{m-1}) + \dots \\ &\quad + d(a_{n+1}, p_n) \\ d(b_m, q_n) &\leq d(b_m, q_{m-1}) + d(b_{m-1}, q_{m-1}) + \dots \\ &\quad + d(b_{n+1}, q_n) \end{aligned} \tag{31}$$

Next, we show that $(\{a_n\}, \{p_n\})$ and $(\{b_n\}, \{q_n\})$ are Cauchy bisequence in (A, B) for each $n; m \in N$ such that $n < m$. From (25), (26), (27), (30), and (31), we have

$$\begin{aligned} d(a_n, p_m) + d(b_n, q_m) &\leq (d(a_n, p_{n+1}) + d(b_n, q_{n+1})) \\ &\quad + (d(a_{n+1}, p_{n+1}) + d(b_{n+1}, q_{n+1})) + \dots \\ &\quad + (d(a_{m-1}, p_{m-1}) + d(b_{m-1}, q_{m-1})) \\ &\quad + (d(a_{m-1}, p_m) + d(b_{m-1}, q_m)) \\ &\leq t_n + r_{n+1} + \dots + t_{m-1} + r_{m-1} \\ &\leq (t_n + t_{n+1} + \dots + t_{m-1}) \\ &\quad + (r_{n+1} + r_{n+2} + \dots + r_{m-1}) \leq \sum_{k=n}^{m-1} t_k + \sum_{k=n+1}^{m-1} r_k \\ &\rightarrow 0 \text{ as } m, n \rightarrow \infty \end{aligned} \tag{32}$$

and

$$\begin{aligned} d(a_m, p_n) + d(b_m, q_n) &\leq (d(a_m, p_{m-1}) + d(b_m, q_{m-1})) \\ &\quad + (d(a_{m-1}, p_{m-1}) + d(b_{m-1}, q_{m-1})) + \dots \\ &\quad + (d(a_{n+1}, p_{n+1}) + d(b_{n+1}, q_{n+1})) \\ &\quad + (d(a_{n+1}, p_n) + d(b_{n+1}, q_n)) \\ &\leq s_{m-1} + r_{m-1} + \dots + r_{n+1} + s_n \\ &\leq (s_n + s_{n+1} + \dots + s_{m-1}) \\ &\quad + (r_{n+1} + r_{n+2} + \dots + r_{m-1}) \leq \sum_{k=n}^{m-1} s_k + \sum_{k=n+1}^{m-1} r_k \\ &\rightarrow 0 \text{ as } m, n \rightarrow \infty \end{aligned} \tag{33}$$

From above, it is clear that $(\{a_n\}, \{p_n\})$ and $(\{b_n\}, \{q_n\})$ are Cauchy bisequences in (A, B) . Since, (A, B, d) is complete, $a, b \in A$ and $p, q \in B$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} a_{n+1} &= p, \\ \lim_{n \rightarrow \infty} b_{n+1} &= q, \\ \lim_{n \rightarrow \infty} p_{n+1} &= a, \\ \lim_{n \rightarrow \infty} q_{n+1} &= b \end{aligned} \tag{34}$$

Now we will show that $a \in F(a, b), b \in F(b, a)$ and $p \in F(p, q), q \in F(q, p)$. As $(\{a_n\}, \{p_n\})$ is a nondecreasing bisequence and $(\{b_n\}, \{q_n\})$ is a nonincreasing bisequence in (A, B) ,

$$\begin{aligned} a_n &\rightarrow p, \\ b_n &\rightarrow q \\ \text{and } p_n &\rightarrow a, \\ q_n &\rightarrow b. \end{aligned} \tag{35}$$

By assumption (7.1), we get $a_n \leq p, p_n \leq a$ and $b_n \geq q, q_n \geq b$ for all n . If $a_n = p, p_n = a$ and $b_n = q, q_n = b$ for some $n \geq 0$, then $p = a_n \leq p_{n+1} \leq a = p_n = q = b_n \geq q_{n+1} \geq b = q_n$ and $a = p_n \leq a_{n+1} \leq p = a_n, b = q_n \geq b_{n+1} \geq q = b_n$ implies $a = p$ and $b = q$; therefore, $a_n = a_{n+1} \in F(a_n, b_n)$ and $b_n = b_{n+1} \in F(b_n, a_n)$.

So (a_n, b_n) is coupled fixed point of F .

Suppose that $(a_n, p_n) \neq (p, a)$ and $(b_n, q_n) \neq (q, b)$ for all $n \geq 0$.

From (3), we have

$$H(F(a, b), F(p_n, q_n)) \leq \mu d(a, p_n) + \kappa d(b, q_n) \tag{36}$$

and

$$H(F(b, a), F(q_n, p_n)) \leq \mu d(b, q_n) + \kappa d(a, p_n) \tag{37}$$

Therefore,

$$H(F(a, b), F(p_n, q_n)) + H(F(b, a), F(q_n, p_n)) \leq (\mu + \kappa)(d(a, p_n) + d(b, q_n)) \quad (38)$$

Letting $n \rightarrow \infty$, we have that

$$\lim_{n \rightarrow \infty} (H(F(a, b), F(p_n, q_n)) + H(F(b, a), F(q_n, p_n))) = 0 \quad (39)$$

Since $p_{n+1} \in F(p_n, q_n)$ and $\lim_{n \rightarrow \infty} d(a, p_{n+1}) = 0$, we have $a \in F(a, b)$ and since $q_{n+1} \in F(q_n, p_n)$ and $\lim_{n \rightarrow \infty} d(b, q_{n+1}) = 0$, we have $b \in F(b, a)$.

Similarly, we can prove $p \in F(p, q)$ and $q \in F(q, b)$.

On the other hand,

$$d(a, p) = d(\lim_{n \rightarrow \infty} p_n, \lim_{n \rightarrow \infty} a_n) = \lim_{n \rightarrow \infty} d(a_n, p_n) = 0 \quad (40)$$

and

$$d(b, q) = d(\lim_{n \rightarrow \infty} q_n, \lim_{n \rightarrow \infty} b_n) = \lim_{n \rightarrow \infty} d(b_n, q_n) = 0. \quad (41)$$

Therefore, $a = p$ and $b = q$ and hence F has a coupled fixed point.

Theorem 8. Let (A, B, \leq) be a partially ordered set such that there exists a bipolar metric d on (A, B) with (A, B, d) being complete bipolar metric spaces. Consider $F : (A^2, B^2) \rightrightarrows CB(A, B)$ a covariant set valued mapping, such that

$$H(F(a, b), F(p, q)) \leq \frac{\kappa}{2}(d(a, p) + d(b, q)) \quad (42)$$

for all $a, b \in A, p, q \in B$, and $\kappa \in (0, 1)$ with $a \geq p$ and $b \leq q$. Suppose also that

- (8.1) F has a mixed monotone property
- (8.2) there exist $(a_0, b_0) \in A^2 \cup B^2$ and for some $a_1 \in F(a_0, b_0), b_1 \in F(b_0, a_0)$ we have $a_0 \leq a_1$ and $b_0 \geq b_1$
- (8.3) if a nondecreasing sequence $(\{a_n\}, \{p_n\})$ is convergent to (p, a) for $a \in A, p \in B$, then $a_n \leq p, p_n \leq a$ for all n and if a nonincreasing sequence $(\{b_n\}, \{q_n\})$ is convergent to (q, b) for $b \in A, q \in B$, then $b_n \geq q, q_n \geq b$ for all n

Then F has a coupled fixed point; that is, there exist $(a, b) \in A^2 \cup B^2$ such that $a \in F(a, b)$ and $b \in F(b, a)$.

Example 9. Let $A = \{U_m(R)/U_m(R)$ be upper triangular matrices over $R\}$ and let $B = \{L_m(R)/L_m(R)$ be lower triangular matrices over $R\}$ with the bipolar metric $d(P, Q) = \sum_{i,j=1}^m |p_{ij} - q_{ij}|$ for all $P = (p_{ij})_{m \times m} \in U_m(R)$ and $Q = (q_{ij})_{m \times m} \in L_m(R)$. On the set (A, B) , consider the following relation:

$$(P, Q) \in A^2 \cup B^2, \quad P \leq Q \iff p_{ij} \leq q_{ij} \quad (43)$$

where \leq is usual ordering. Then, clearly, (A, B, d) is a complete bipolar metric space and (A, B, \leq) is a partially ordered set.

Let $F : (A^2, B^2) \rightrightarrows CB(A, B)$ be defined as $F(P, Q) = ((p_{ij} + q_{ij})/5)_{m \times m} + (3/5)(I_{ij})_{m \times m}$ for all $(P = (p_{ij})_{m \times m}, Q = (q_{ij})_{m \times m}) \in A^2 \cup B^2$.

Then obviously F has mixed monotone property; also there exist $P = (O_{ij})_{m \times m}$ and $Q = (I_{ij})_{m \times m}$ such that

$$F((O_{ij})_{m \times m}, (I_{ij})_{m \times m}) = \left(\frac{O_{ij} + I_{ij}}{5}\right)_{m \times m} + \frac{3}{5}(I_{ij})_{m \times m} \geq (O_{ij})_{m \times m} \quad (44)$$

and

$$F((I_{ij})_{m \times m}, (O_{ij})_{m \times m}) = \left(\frac{O_{ij} + I_{ij}}{5}\right)_{m \times m} + \frac{3}{5}(I_{ij})_{m \times m} \leq (I_{ij})_{m \times m} \quad (45)$$

Taking $(P = (p_{ij})_{m \times m}, Q = (q_{ij})_{m \times m}), (R = (r_{ij})_{m \times m}), (S = (s_{ij})_{m \times m}) \in A^2 \cup B^2$ with $P \geq R, Q \leq S$, that is, $p_{ij} \geq r_{ij}, q_{ij} \leq s_{ij}$, we have

$$\begin{aligned} d(F(P, Q), F(R, S)) &= d\left(\frac{p_{ij} + q_{ij}}{5} + \frac{3}{5}(I_{ij}), \frac{r_{ij} + s_{ij}}{5} + \frac{3}{5}(I_{ij})\right) \\ &= \frac{1}{5} \sum_{i,j=1}^m |(p_{ij} + q_{ij}) - (r_{ij} + s_{ij})| \\ &\leq \frac{1}{5} \left(\sum_{i,j=1}^m |p_{ij} - r_{ij}| + \sum_{i,j=1}^m |q_{ij} - s_{ij}| \right) \\ &\leq \frac{1}{5} (d(P, R) + d(Q, S)) \end{aligned} \quad (46)$$

Therefore, all the conditions of Theorem 8 hold and $((3/5)I_{ij})_{m \times m}, ((3/5)I_{ij})_{m \times m}$ is a coupled fixed point of F .

Definition 10. Let (A, B, d) be bipolar metric spaces, $a \in A, p \in B$, and let $F : (A \times B) \cup (B \times A) \rightarrow CB(A \cup B)$ be a covariant multivalued map. An element (a, p) is called a coupled fixed point of F if $a \in F(a, p)$ and $p \in F(p, a)$.

Theorem 11. Let (A, B, \leq) be a partially ordered set such that there exists a bipolar metric d on (A, B) with (A, B, d) being complete bipolar metric spaces. Consider $F : (A \times B, B \times A) \rightrightarrows CB(A, B)$ a covariant set valued mapping, such that

$$H(F(a, p), F(q, b)) \leq \mu d(a, q) + \kappa d(b, p) \quad (47)$$

for all $a, b \in A, p, q \in B$ and $\mu + \kappa < 1$ with $a \geq q$ and $b \leq p$. Suppose also that

- (11.1) F has a mixed monotone property
- (11.2) there exist $a_0 \in A, p_0 \in B$ and for some $a_1 \in F(a_0, p_0), p_1 \in F(p_0, a_0)$ we have $a_0 \leq a_1$ and $p_0 \geq p_1$
- (11.3) if a nondecreasing sequence $(\{a_n\}, \{q_n\})$ is convergent to (q, a) for $a \in A, q \in B$, then $a_n \leq q, q_n \leq a$ for all n and if a nonincreasing sequence $(\{b_n\}, \{p_n\})$ is convergent to (p, b) for $b \in A, p \in B$, then $b_n \geq p, p_n \geq b$ for all n

Then F has a coupled fixed point; that is, there exist $a \in A, p \in B$ such that $a \in F(a, p)$ and $p \in F(p, a)$.

Proof. The proof will follow when we replace $A \times B$ and $B \times A$ in place of A^2, B^2 , respectively, in Theorem 7. \square

Theorem 12. Let (A, B, \leq) be a partially ordered set such that there exists a bipolar metric d on (A, B) with (A, B, d) being complete bipolar metric spaces. Consider $F : (A \times B, B \times A) \rightrightarrows CB(A, B)$ a covariant set valued mapping, such that

$$H(F(a, p), F(q, b)) \leq \frac{\kappa}{2} (d(a, q) + d(b, p)) \quad (48)$$

for all $a, b \in A, p, q \in B$ and $\kappa \in (0, 1)$ with $a \geq q$ and $b \leq p$. Suppose also that

(12.1) F has a mixed monotone property

(12.2) there exist $a_0 \in A, p_0 \in B$ and for some $a_1 \in F(a_0, p_0), p_1 \in F(p_0, a_0)$ we have $a_0 \leq a_1$ and $p_0 \geq p_1$

(12.3) if a nondecreasing sequence $(\{a_n\}, \{q_n\})$ is convergent to (q, a) for $a \in A, q \in B$, then $a_n \leq q, q_n \leq a$ for all n and if a nonincreasing sequence $(\{b_n\}, \{p_n\})$ is convergent to (p, b) for $b \in A, p \in B$, then $b_n \geq p, p_n \geq b$ for all n

Then F has a coupled fixed point; that is, there exist $a \in A, p \in B$ such that $a \in F(a, p)$ and $p \in F(p, a)$.

3. Conclusions

In the present research, we introduced and proved a coupled fixed point theorem for a multivalued mapping, satisfying various contractive conditions, defined on a partially ordered bipolar metric space, and gave suitable example that supports our main result.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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