Hindawi
Journal of Mathematics
Volume 2019, Article ID 8603878, 7 pages
https://doi.org/10.1155/2019/8603878

Research Article

Approximate Controllability of Fractional Nonlinear Hybrid Differential Systems via Resolvent Operators

Mohammed M. Matar

Mathematics Department, Al-Azhar University-Gaza, State of Palestine

Correspondence should be addressed to Mohammed M. Matar; mohammed_mattar@hotmail.com

Received 26 November 2018; Revised 17 February 2019; Accepted 20 February 2019; Published 20 March 2019

Guest Editor: Thabet Abdeljawad

We obtain sufficient conditions for the approximate controllability of a fractional nonlinear hybrid differential system. The results are obtained by using resolvent and sectorial operators technique via Dhage fixed point theorem.

1. Introduction

Fractional differential systems have described many practical dynamical phenomena more efficiently than the corresponding integer-order systems; hence they have attracted the attention of many researchers in such fields (see [1–10] and references cited therein).

One of these systems is the fractional control system with all its branches such as stability, controllability, and observability. In the recent years, many investigations on the controllability problems of fractional behaviour have extensively appeared with various applications on linear and nonlinear systems. Particularly, the researches have focused on exact (complete) and approximate controllability (see the articles [11–17] and the references therein).

The fractional control systems involving a linear closed (unbounded) operator which generates resolvent operators were considered recently by many authors [15, 18–20]. The lack of the semigroup property of the generated resolvent operator was the most popular difficulty that has been faced by the interested researchers. However, some authors used the idea of analytic sectorial operators to overcome this problem. For more details, we refer the reader to the papers [20–23] and references therein. To the best of our knowledge, there is not any investigation in the controllability problem via resolvent operators applied on hybrid systems such as the system that has been discussed in the article [24].

In this article, we study the approximate controllability for a fractional hybrid differential system of the form

$$\begin{aligned} &\left({}^{c}D_{0}^{\alpha} + A\right) x(t) \\ &\quad = g(t, x(t))\, I_{0}^{1-\alpha}\left(Bu(t) + f(t, x(t))\right), \quad 0 < t \leq b, \qquad (1) \\ &x(0) = x_{0}, \end{aligned}$$

where ${}^{c}D_{0}^{\alpha}$ is the Caputo fractional derivative of order α such that $0 < \alpha < 1$, X and U are two real Hilbert spaces, $A : D(A) \subseteq X \longrightarrow X$ is the infinitesimal generator of a resolvent operator $S_{\alpha}(t)$, $t \in J = [0, b]$, $B : U \longrightarrow X$ is a bounded linear operator, $u \in L^{2}(J, U)$, $I_{0}^{1-\alpha}$ denotes the $(1-\alpha)$-order fractional integral, and $f, g : J \times X \longrightarrow X$ are given functions such that g does not vanish on $J \times X$.

2. Preliminaries

Let $C(J, X)$ be the space of all X-valued continuous functions defined on J with the norm $\|x\| = \sup\{\|x(t)\|, t \in J\}$, and let $L^{p}(J, X)$ be the space of X-valued Bochner integrable functions defined on J with the norm $\|f\|_{L^{p}} = (\int_{0}^{b} \|f(t)\|^{p} dt)^{1/p}$, where $1 \leq p \leq \infty$.

Now, let us recall some basic preliminaries on fractional calculus [25] and operator theory [26].

Definition 1. The fractional-order integral of a function $f \in C(J, X)$(or $L^1(J, X)$) of order $\alpha > 0$ is defined by

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds. \tag{2}$$

Definition 2. The Caputo fractional derivative of order $\alpha \in (0, 1)$ of a function $f \in C^1(J, X)$ (or $L^p(J, X)$) is defined by

$$\begin{aligned} {}^cD_0^\alpha f(t) &= I_0^{1-\alpha} f'(t) \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)\, ds. \end{aligned} \tag{3}$$

Definition 3. The Laplace transform of a function f is given by

$$\mathscr{L}(f)(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\, dt, \quad \lambda \in \mathbb{C}. \tag{4}$$

The Laplace transform of the Caputo fractional derivative is given by

$$\mathscr{L}\left\{{}^cD_0^\alpha f(t)\right\}(\lambda) = \lambda^\alpha \mathscr{L}\{f(t)\} - \lambda^{\alpha-1} f(0), \quad 0 < \alpha < 1. \tag{5}$$

The Laplace transform of the fractional integral $I_0^{1-\alpha}$ is given by

$$\mathscr{L}\left\{I_0^{1-\alpha} f(t)\right\} = \lambda^{\alpha-1} \mathscr{L}\{f(t)\}, \quad \alpha > 0. \tag{6}$$

The inverse Laplace transform of a function $F = \mathscr{L}\{f\}$ is given by

$$\mathscr{L}^{-1} F(\lambda) = \frac{1}{2\pi i} \int_c e^{\lambda t} F(\lambda)\, d\lambda, \tag{7}$$

for some suitable path c to ensure the existence of the integral.

The resolvent operator of an operator A is defined as

$$R(\lambda, A) = \left(\lambda^\alpha I + A\right)^{-1}, \quad \lambda \in \mathbb{C}. \tag{8}$$

The resolvent set $\rho(A)$ is the set of all regular values of $\lambda \in \mathbb{C}$ such that $R(\lambda, A)$ is injective, bounded linear operator.

The following fixed point theorem, which is due to Dhage [27], is essential tool for the proof of the main result.

Theorem 4. *Let Ω be a nonempty bounded closed convex subset of a Banach algebra X. Let $\Phi : \Omega \longrightarrow X$ and $\Theta : X \longrightarrow X$ be continuous operators satisfying the following:*

(a) *Φ is completely continuous,*

(b) *Θ is Lipschitzian with a Lipschitz constant k_Θ,*

(c) *$x = \Phi y \Theta x$ implies $x \in \Omega$ for all $y \in \Omega$, and*

(d) *$Mk_\Theta < 1$, where $M = \sup\{\|\Phi x\| : x \in \Omega\}$.*

Then the operator equation $x = \Phi x \Theta x$ has a solution in Ω.

3. Fractional Control Systems via Resolvent Operators

Let $A : D(A) \longrightarrow X$ be a linear operator defined on the subspace $D(A) \subseteq X$, the domain of A to the space X. An operator A is said to be closed if and only if its domain $D(A)$ is a complete space with respect to the norm $\|x\|_{D(A)} = \|x\| + \|Ax\|$. An operator A is said to be densely defined if its domain is dense in X. The denseness of the domain is necessary and sufficient for the existence of the adjoint. The adjoint operator of unbounded operators can be defined as bounded operators. For more details on these topics, the reader may refer to [26, 28].

Next, we introduce some information about solution operators [29].

Consider system

$$\begin{aligned} {}^cD_0^\alpha x(t) &= -Ax(t), \\ x(0) &= x \in D(A), \end{aligned} \tag{9}$$

which has an integral solution given by

$$x(t) = x - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(a)} Ax(s)\, ds \in D(A), \quad t \in J. \tag{10}$$

Definition 5. Let A be a closed and densely defined operator on X. A family $\{S_\alpha(t)\}_{t\geq 0}$ of bounded linear operators in X is called a solution operator (or α-resolvent) generated by A if the following conditions are satisfied:

(S1) $S_\alpha(t)$ is strong continuous on $\mathbb{R}_+$ and $S_\alpha(0) = I$, where I is the identity operator.

(S2) $S_\alpha(t)D(A) \subseteq D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in D(A)$ and $t \geq 0$.

(S3) $S_\alpha(t)x$ is a solution of the integral equation (10).

Moreover, a solution operator $S_\alpha(t)$ is called compact if for every $t > 0$, $S_\alpha(t)$ is a compact operator. If $S_\alpha(t)$ is a solution operator of system (9), then by (S3), we deduce that

$$Ax = \Gamma(\alpha+1) \lim_{t\longrightarrow 0^+} \frac{x - S_\alpha(t)x}{t^\alpha}, \tag{11}$$

where $D(A)$ consists of all x for which the limit exists. We call A as the infinitesimal generator of $S_\alpha(t)$ or simply we say that A generates the solution operator $S_\alpha(t)$.

Definition 6. Let $A : D(A) \subseteq X \longrightarrow X$ be a closed linear operator. A is said to be a sectorial operator of type (M, θ, α, μ) if there exist $\mu \leq 0$, $0 < \theta < \pi/2$, and $M > 0$ such that the solution operator S_α of A exists throughout the sector

$$\begin{aligned} \gamma &= \left\{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > (-\mu)^{1/\alpha};\ |\arg(\lambda)| < \theta\right\} \\ &\subseteq \rho(A), \end{aligned} \tag{12}$$

and $\|R(\lambda^\alpha, A)\| \leq M/|\lambda^\alpha + \mu|$, $\lambda \in \gamma$.

Hereafter, we assume that A is a sectorial operator of type (M, θ, α, μ) that generates the solution operator $S_\alpha(t)$. In this case, we can write the solution operator $S_\alpha(t)$ of system (9) as

$$S_\alpha(t) = \mathscr{L}^{-1}\left(\lambda^{\alpha-1} R(\lambda^\alpha, A)\right) = \frac{1}{2\pi i}\int_c e^{\lambda t}\lambda^{\alpha-1} R(\lambda^\alpha, A)\, d\lambda, \tag{13}$$

with c being a suitable path in a sector γ.

Lemma 7. *The linear fractional system*

$$\left({}^cD_0^\alpha + A\right)x(t) = g(t) I_0^{1-\alpha}[Bu(t) + f(t)], \quad x(0) = x_0, \tag{14}$$

has an integral solution given by

$$x(t) = g(t)\left(S_\alpha(t)x_0(g(0))^{-1} + \int_0^t S_\alpha(t-s)(Bu(s) + f(s))\, ds\right). \tag{15}$$

Proof. Letting $g(t) \neq 0$, for any $t \in J$, then system (14) is equivalent to system

$${}^cD_0^\alpha\left(\frac{x(t)}{g(t)}\right) + A\left(\frac{x(t)}{g(t)}\right) = I_0^{1-\alpha}[Bu(t) + f(t)], \quad x(0) = x_0, \tag{16}$$

Applying the Laplace transform to system (16), we have

$$\mathscr{L}\left\{{}^cD_0^\alpha \frac{x(t)}{g(t)}\right\} + A\mathscr{L}\left\{\frac{x(t)}{g(t)}\right\} = \mathscr{L}\left\{I_0^{1-\alpha}(Bu(t) + f(t))\right\}, \tag{17}$$

and that implies

$$\lambda^\alpha\mathscr{L}\left\{\frac{x(t)}{g(t)}\right\} - \lambda^{\alpha-1}\frac{x(0)}{g(0)} + A\mathscr{L}\left\{\frac{x(t)}{g(t)}\right\} = \lambda^{\alpha-1}\mathscr{L}\{Bu(t) + f(t)\}. \tag{18}$$

Therefore,

$$\mathscr{L}\left\{\frac{x(t)}{g(t)}\right\} = \lambda^{\alpha-1}(\lambda^\alpha I + A)^{-1}x_0(g(0))^{-1} + \lambda^{\alpha-1}(\lambda^\alpha I + A)^{-1}\mathscr{L}\{Bu(t) + f(t)\}. \tag{19}$$

Now, taking the inverse Laplace transform, we get the solution (15). This finishes the proof. □

We define a mild solution for system (1).

Definition 8. A function $x(t) \in C(J, X)$ is called a mild solution of system (1) if it satisfies

$$x(t) = g(t, x(t))\left(S_\alpha(t)x_0(g(0, x_0))^{-1} + \int_0^t S_\alpha(t-s)(Bu(s) + f(s, x(s)))\, ds\right), \tag{20}$$

for any $t \in J$.

We introduce some preliminaries about controllability (see [3, 11–13, 18–20, 27]). We assume that x is a mild solution (we call it now as state function) of the fractional differential system (1) corresponding to a control u.

Definition 9. System (1) is said to be approximately controllable on J if for every desired final state $x_b \in X$ and $\varepsilon > 0$, there exists a control $u \in L^2(J, U)$ such that x satisfies $\|x(b; u) - x_b\| < \varepsilon$.

The set

$$\boldsymbol{R} = \left\{x(b; u) \in X : u \in L^2(J, U),\ x \text{ is the mild solution of (1) with control } u\right\}, \tag{21}$$

is called the reachability set of system (1). Therefore, the fractional system (1) is said to be approximately controllable on J if $\overline{\boldsymbol{R}} = X$, where $\overline{\boldsymbol{R}}$ denotes the closure of $\boldsymbol{R}$. If the used control function is fixed, the symbol $x(b)$ is used instead of $x(b; u)$.

We define the controllability operator $L_t : L^2(J, U) \longrightarrow X$ as

$$L_t u = \int_0^t S_\alpha(t-s)Bu(s)\, ds, \quad t \in J. \tag{22}$$

Then L_t is a bounded linear operator defined on $L^2(J, U)$. The adjoint operator

$$L_b^* : X \longrightarrow L^2(J, U) \tag{23}$$

of L_b is given by

$$L_b^* = B^* S_\alpha^*(b - \cdot). \tag{24}$$

The controllability Gramnian $W : X \longrightarrow X$ is defined by

$$W = L_b L_b^* = \int_0^b S_\alpha(b-s)BB^*S_\alpha^*(b-s)\, ds. \tag{25}$$

Following the idea, as in [20], the suggested control function u for system (1) can be written in the form.

$$u(t) = B^*S_\alpha^*(b-t)(\lambda I + W)^{-1}\Lambda x, \quad t \in J, \tag{26}$$

where

$$\Lambda x = x_b(g(b, x(b)))^{-1} - S_\alpha(b)x_0(g(0, x_0))^{-1} - \int_0^b S_\alpha(b-s)f(s, x(s))\, ds. \tag{27}$$

4. Approximate Controllability

We prove the approximate controllability of the fractional control system (1) by using the mild solution (20) and the control defined by (26). More precisely, we prove the existence of at least one state $x \in C(J, X)$ satisfying (20) and (26) following the same arguments presented in [20], but using Dhage fixed point theorem. For this lets

$$\Theta x(t) = g(t, x(t)), \tag{28}$$

and

$$\begin{aligned}\Phi x(t) &= S_\alpha(t)\, x_0 \left(g(0,x_0)\right)^{-1} \\ &\quad + \int_0^t S_\alpha(t-s)\left(Bu(s) + f(s,x(s))\right) ds,\end{aligned} \tag{29}$$

where u is given by (26).

If S_α is compact C_0-semigroup, then the Cauchy operator $\Psi : C(J,X) \longrightarrow C(J,X)$ defined as

$$\Psi h(t) = \int_0^t S_\alpha(t-s)\, h(s)\, ds, \quad t \in J, \tag{30}$$

is also compact. Unfortunately, the resolvent operator does not have the property of semigroups which leads to the impossibility of obtaining the compactness of the Cauchy operator Ψ. However, we can prove the continuity of the solution operator in the case of analytic operators by which we can prove the compactness of the Cauchy operator Ψ.

Let r be a fixed positive real number such that $B_r = \{x \in C(J,X) : \|x\| \le r\}$. Clearly, B_r is a bounded closed and convex set. We need the following assumptions:

(H1) $S_\alpha(t)$ is compact analytic operator such that $M_S = \sup\{\|S_\alpha(t)\| : t \in J\} < \infty$.

(H2) $f : J \times X \longrightarrow X$ is continuous and there exists a positive constant K_f such that $\|f(t,x)\| < K_f$, for all $(t,x) \in J \times X$.

(H3) $g : J \times X \longrightarrow X$ is continuous and there exists a function $\rho \in L^1(J, \mathbb{R}_+)$ such that $\|g(t,x) - g(t,y)\| < \|\rho\|_{L^1}\|x - y\|$, for all $(t,x) \in J \times X$.

(H4) $B : U \longrightarrow X$ is a linear bounded operator and there exists $N > 0$ such that $\|B\| = N$.

(H5) Let $M := (N^2 M_s^2/\lambda)(\|x_b(g(b,x(b)))^{-1}\| + M_s\|x_0(g(0,x_0))^{-1}\| + M_s K_f b) + M_s\|x_0(g(0,x_0))^{-1}\| + M_s^2 K_f b$, such that $M\|\rho\|_{L^1} < 1$.

(H6) $\|(\lambda I + W)^{-1}\| \le 1/\lambda$, $\lambda > 0$.

Theorem 10. *Assume that conditions (H1)-(H6) are satisfied. Then system (1) has a mild solution on J.*

Proof. We show the operators Θ and Φ satisfying the hypotheses of Dhage fixed point theorem. For the sake of clarity, we split the proof into two main steps.

Step 1. Firstly, we prove the continuity of Θ and Φ. Let $(x_n)_{n\ge1}$ be a sequence in $C(J,X)$ with $\lim_{n\longrightarrow\infty} x_n = x$ in $C(J,X)$. By the hypotheses (H2) and (H3), we obtain the convergence of $f(t,x_n(t))$ and $g(t,x_n(t))$ to $f(t,x(t))$ and $g(t,x(t))$, respectively, for any $t \in J$. Hence

$$\begin{aligned}&\|\Lambda x_n - \Lambda x\| \\ &\quad \le \|x_b\| \left\|\left(g(b,x_n(b))\right)^{-1} - \left(g(b,x(b))\right)^{-1}\right\| \\ &\quad\quad + M_S \int_0^b \left\|f(s,x_n(s)) - f(s,x(s))\right\| ds.\end{aligned} \tag{31}$$

Thus, by Lebsegue Dominated Convergence Theorem and the fact that $\|(g(b,x_n(b)))^{-1}\| \ge \delta$, for some $\delta > 0$ and for any $n \in \mathbb{N}$, we have $\lim_{n\longrightarrow\infty}\|\Lambda x_n - \Lambda x\| = 0$. Then

$$\begin{aligned}&\left\|\Phi x_n(t) - \Phi x(t)\right\| \\ &\quad \le \frac{M_S^2 N^2}{\lambda} \int_0^b \left\|\Lambda x_n(s) - \Lambda x(s)\right\| ds \\ &\quad\quad + M_S \int_0^b \left\|f(s,x_n(s)) - f(s,x(s))\right\| ds.\end{aligned} \tag{32}$$

As $n \longrightarrow \infty$ and using again dominated convergence theorem we have $\|\Phi x_n(t) - \Phi x(t)\| \longrightarrow 0$. Thus Θ and Φ are continuous on $C(J,X)$. Next, we show that Φ is bounded on $C(J,X)$. In fact, for all $x \in C(J,X)$, we have

$$\begin{aligned}\|\Phi x\| &\le M_s \left\|x_0\left(g(0,x_0)\right)^{-1}\right\| \\ &\quad + \frac{N^2 M_s^2}{\lambda}\left(\left\|x_b\left(g(b,x(b))\right)^{-1}\right\|\right. \\ &\quad \left. + M_s\left\|x_0\left(g(0,x_0)\right)^{-1}\right\| + M_s K_f b\right) + M_s^2 K_f b.\end{aligned} \tag{33}$$

Then, the inequality $\|\Phi x\| \le M$ holds for all $x \in B_r$.

The last thing in this step, we show that $\Phi : C(J,X) \longrightarrow C(J,X)$ is a compact operator. It is sufficient to prove that

$$\int_0^t S_\alpha(t-s)\left(Bu(s) + f(s,x(s))\right) ds \tag{34}$$

is compact. But this has been proved in many articles see, for example, ([20]: Theorem 3.3) by using the Ascoli-Arzela theorem. Hence we conclude that Φ is compact. Therefore, Φ is completely continuous. By following the same arguments presented in [13], we can prove.....

Step 2. The hypothesis (H3) shows that the operator Θ is Lipschitz with Lipschitzian constant $k_\Theta = \|\rho\|_{L^1}$. Next, we show that $x \in B_r$ whenever $x = \Phi y \Theta x$ for all $y \in B_r$. For this, letting $y \in B_r$ and $k_0 = \sup_{t\in J}\|g(t,0)\|$, we have

$$\begin{aligned}\|x\| &\le \left(\left\|g(t,x) - g(t,0)\right\| + \left\|g(t,0)\right\|\right) M \\ &\le \left(\|\rho\|_{L^1}\|x\| + k_0\right) M.\end{aligned} \tag{35}$$

In consequence, this implies that

$$\|x\| \le \frac{k_0 M}{1 - M\|\rho\|_{L^1}}. \tag{36}$$

□

If r is chosen large enough such that $r > M\max\{1, k_0/(1 - M\|\rho\|_{L^1})\}$, then we ensure that $x \in B_r$. Therefore, all hypotheses of Dhage Theorem are satisfied; then there exists a fixed point $x \in B_r$ satisfying the operator equation $x = \Theta x \Phi x$, which is a solution of system (1).

Next result, we investigate the approximate controllability of the fractional control system (1). We introduce the following extra conditions:

(H7) $\lambda(\lambda I+W)^{-1} \longrightarrow 0$ as $\lambda \longrightarrow 0^{+}$ in the strong operator topology.

(H8) The sequence $\{g(\cdot, x_{\lambda}(\cdot)) : \lambda > 0\}$ is bounded in $L^{2}(J, X)$.

Theorem 11. *Assume that conditions (H1)-(H8) are satisfied. Then, the fractional system (1) is approximately controllable on J.*

Proof. In virtue of Theorem 10, there exists a mild solution $x_{\lambda} \in C(J, X)$ such that

$$\begin{aligned} x_{\lambda}(t) = g(t, x_{\lambda}(t)) \Big(& S_{\alpha}(t) x_{0} (g(0, x_{0}))^{-1} \\ & + \int_{0}^{t} S_{\alpha}(t-s)(Bu(s) + f(s, x_{\lambda}(s))) ds \Big), \end{aligned} \quad (37)$$

where

$$\begin{aligned} u(t) = {} & B^{*} S_{\alpha}^{*}(b-t)(\lambda I+W)^{-1} \Big(x_{b}(g(b, x_{\lambda}(b)))^{-1} \\ & - S_{\alpha}(b) x_{0}(g(0, x_{0}))^{-1} \\ & - \int_{0}^{b} S_{\alpha}(b-s) f(s, x(s)) ds \Big). \end{aligned} \quad (38)$$

Therefore,

$$\begin{aligned} x_{\lambda}(b) = {} & g(b, x_{\lambda}(b)) \Big(S_{\alpha}(b) x_{0}(g(0, x_{0}))^{-1} \\ & + \int_{0}^{b} S_{\alpha}(t-s)(Bu(s) + f(s, x_{\lambda}(s))) ds \Big) = g(b, \\ & x_{\lambda}(b)) \Big[S_{\alpha}(b) x_{0}(g(0, x_{0}))^{-1} + (\lambda I + W - \lambda I)(\lambda I \\ & + W)^{-1} \times \Big(x_{b}(g(b, x_{\lambda}(b)))^{-1} \\ & - S_{\alpha}(b) x_{0}(g(0, x_{0}))^{-1} \\ & - \int_{0}^{b} S_{\alpha}(b-s) f(s, x_{\lambda}(s)) ds \Big) \\ & + \int_{0}^{b} S_{\alpha}(b-s) f(s, x_{\lambda}(s)) ds \Big]. \end{aligned} \quad (39)$$

Hence,

$$\begin{aligned} x_{\lambda}(b) - x_{b} = {} & -g(b, x_{\lambda}(b)) \lambda (\lambda I + W)^{-1} \\ & \cdot \Big(x_{b}(g(b, x_{\lambda}(b)))^{-1} - S_{\alpha}(b) x_{0}(g(0, x_{0}))^{-1} \\ & - \int_{0}^{b} S_{\alpha}(b-s) f(s, x_{\lambda}(s)) ds \Big). \end{aligned} \quad (40)$$

Now, by condition (H2), we have

$$\int_{0}^{b} \| f(s, x_{\lambda}(s)) \|^{2} ds \leq K_{f}^{2} b, \quad (41)$$

which implies that the sequence $\{f(\cdot, x_{\lambda}(\cdot)) : \lambda > 0\}$ is bounded in the Hilbert space $L^{2}(J, X)$. Together with (H8), there exist subsequences of $\{f(\cdot, x_{\lambda}(\cdot)) : \lambda > 0\}$ and $\{g(\cdot, x_{\lambda}(\cdot)) : \lambda > 0\}$ in $L^{2}(J, X)$ converging weakly to some points $\omega, \nu \in L^{2}(J, X)$, respectively. Let

$$\begin{aligned} \eta = {} & x_{b}(\nu(b))^{-1} - S_{\alpha}(b) x_{0}(g(0, x_{0}))^{-1} \\ & - \int_{0}^{b} S_{\alpha}(b-s) \omega(s) ds. \end{aligned} \quad (42)$$

Thus,

$$\begin{aligned} & \Big\| x_{b}(g(b, x_{\lambda}(b)))^{-1} - S_{\alpha}(b) x_{0}(g(0, x_{0}))^{-1} \\ & \quad - \int_{0}^{b} S_{\alpha}(b-s) f(s, x_{\lambda}(s)) ds - \eta \Big\| \\ & \leq \left\| (g(b, x_{\lambda}(b)))^{-1} - (\nu(b))^{-1} \right\| \| x_{b} \| \\ & \quad + \left\| \int_{0}^{b} S_{\alpha}(b-s) [f(s, x_{\lambda}(s)) - \omega(s)] ds \right\|. \end{aligned} \quad (43)$$

Using the compactness of $S_{\alpha}(t)$, we can deduce that the mapping

$$x(t) \longrightarrow \int_{0}^{t} S_{\alpha}(t-s) x(s) ds, \quad (44)$$

from $L^{2}(J, X)$ to $C(J, X)$ is compact. So, we obtain that

$$\begin{aligned} & \left\| (g(b, x_{\lambda}(b)))^{-1} - (\nu(b))^{-1} \right\| \| x_{b} \| \\ & \quad + \int_{0}^{b} S_{\alpha}(b-s) [f(s, x_{\lambda}(s)) - \omega(s)] ds \longrightarrow 0, \end{aligned} \quad (45)$$

as $\lambda \longrightarrow 0^{+}$. This implies that

$$\begin{aligned} & \Big\| x_{b}(g(b, x_{\lambda}(b)))^{-1} - S_{\alpha}(b) x_{0}(g(0, x_{0}))^{-1} \\ & \quad - \int_{0}^{b} S_{\alpha}(b-s) f(s, x_{\lambda}(s)) ds - \eta \Big\| \longrightarrow 0, \end{aligned} \quad (46)$$

as $\lambda \longrightarrow 0^+$. In view of (40) and condition (H5), we obtain that

$$\begin{aligned}\left\|x_\lambda(b)-x_b\right\| &\leq \left\|g\left(b,x_\lambda(b)\right)\lambda R(\lambda,W)\right\| \\ &\cdot \left\|x_b\left(g\left(b,x_\lambda(b)\right)\right)^{-1}-S_\alpha(b)x_0\left(g\left(0,x_0\right)\right)^{-1} \right.\\ &\left.-\int_0^b S_\alpha(b-s)f\left(s,x_\lambda(s)\right)ds\right\| \leq \left\|g\left(b,x_\lambda(b)\right)\right\| \\ &\cdot \left\|\lambda R(\lambda,W)\right\| \left\|x_b\left(g\left(b,x_\lambda(b)\right)\right)^{-1} \right.\\ &-S_\alpha(b)x_0\left(g\left(0,x_0\right)\right)^{-1} \\ &\left.-\int_0^b S_\alpha(b-s)f\left(s,x_\lambda(s)\right)ds-\eta\right\| \\ &+\left\|g\left(b,x_\lambda(b)\right)\right\|\left\|\lambda R(\lambda,W)\eta\right\|.\end{aligned} \tag{47}$$

Hence $\|x_\lambda(b)-x_b\| \longrightarrow 0$ as $\lambda \longrightarrow 0^+$, which implies that the fractional system (1) is approximately controllable on J. This finishes the proof. □

Example 12. Consider the fractional control system

$$\begin{aligned}&{}^cD_0^{0.8}\frac{x(t,y)}{1+Lt^2\sin x(t,y)} \\ &\quad=-\frac{\partial^2}{\partial y^2}\left(\frac{x(t,y)}{1+Lt^2\sin x(t,y)}\right) \\ &\quad+I_0^{1-\alpha}\left(u\left(t,x(t,y)\right)+\frac{\left|x(t,y)\right|}{1+\left|x(t,y)\right|}\right), \\ &\qquad\qquad y\in(0,\pi),\ t\in(0,1], \\ &x(t,0)=x(t,\pi)=0,\quad t\in[0,1], \\ &x(0,y)=x_0(y),\quad y\in(0,\pi).\end{aligned} \tag{48}$$

Let $X=U=L^2[0,\pi]$ and $A=\partial^2/\partial y^2$, where $D(A)=\{\varphi\in X:\varphi$ and φ' are absolutely continuous, $\varphi''\in X, \varphi(0)=\varphi(\pi)=0\}$. Then A is an infinitesimal generator of an analytic semigroup (t), $t>0$, which can be written in the form

$$S(t)\varphi=\sum_{n=1}^{\infty}e^{-n^2\pi^2 t}\left\langle\varphi,e_n\right\rangle e_n,\quad t>0,\ x\in X, \tag{49}$$

where $e_n(y)=\sqrt{2/\pi}\sin ny, n=1,2,\ldots$ is the orthonormal basis for X. It can be shown that A is also a generator of a compact analytic operator $S_\alpha(t)$, $t>0$, given by

$$S_\alpha(t)=\int_0^\infty S\left(st^\alpha\right)\psi_\alpha(s)\,ds,\quad t>0, \tag{50}$$

where

$$\psi_\alpha(s)=\sum_{n=0}^{\infty}\frac{(-s)^n}{n!\Gamma(1-\alpha-n\alpha)},\quad 0<\alpha<1 \tag{51}$$

is the Wright function (see [19, 21]). Setting $x(t)y=x(t,y)$, $(Bu)(t)(y)=u(t,y)$, $g(t,x(t))(y)=1+Lt^2\sin x(t,y)$, and $f(t,x(t))(y)=|x(t,y)|/(1+|x(t,y)|)$, then system (48) is equivalent to system (1) for any $t\in[0,1]$. The operator $S_\alpha(t)$ satisfies the hypothesis (H1) such that $M_S=1$. Simple calculations lead to $K_f=1$, $\|\rho\|_{L^1}=L/3$, and $N=1$. Therefore, if we choose L such that $ML<3$ and that (H6) and (H7) are both satisfied, then, using Theorems 10, and 11, we ensure that system (48) is approximately controllable on $[0,1]$.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that they have no conflicts of interest.

References

[1] T. Abdeljawad, F. Jarad, and J. Alzabut, "Fractional proportional differences with memory," *The European Physical Journal Special Topics*, vol. 226, no. 16-18, pp. 3333–3354, 2017.

[2] J. Alzabut, S. Tyagi, and S. Abbas, "Discrete fractional-order BAM neural networks with leakage delay: existence and stability results," *Asian Journal of Control*, 2018.

[3] D. Boyadzhiev, H. Kiskinov, and A. Zahariev, "Integral representation of solutions of fractional system with distributed delays," *Integral Transforms and Special Functions*, vol. 29, no. 9, pp. 725–744, 2018.

[4] H. Kiskinov and A. Zahariev, "Asymptotic stability of delayed fractional system with nonlinear perturbation," *AIP Conference Proceedings*, vol. 2048, no. 1, Article ID 050014, 2018.

[5] R. L. Magin, *Fractional Calculus in Bioengineering*, Begell, House Publisher, Conn, USA, 2006.

[6] M. M. Matar and E. S. Abu Skhail, "On stability of nonautonomous perturbed semilinear fractional differential systems of order α ∈(1,2)," *Journal of Mathematics*, vol. 2018, Article ID 1723481, 10 pages, 2018.

[7] I. M. Stamova, G. T. Stamov, and J. O. Alzabut, "Global exponential stability for a class of impulsive BAM neural networks with distributed delays," *Applied Mathematics & Information Sciences*, vol. 7, no. 4, pp. 1539–1546, 2013.

[8] V. E. Tarasov, "Fractional hydrodynamic equations for fractal media," *Annals of Physics*, vol. 318, no. 2, pp. 286–307, 2005.

[9] G. M. Zaslavsky, *Hamiltonian Chaos and Fractional Dynamics*, Oxford University Press, Oxford, UK, 2005.

[10] H. Zhou, J. Alzabut, and L. Yang, "On fractional Langevin differential equations with anti-periodic boundary conditions," *The European Physical Journal Special Topics*, vol. 226, no. 16-18, pp. 3577–3590, 2017.

[11] K. Balachandran and J. Kokila, "On the controllability of fractional dynamical systems," *International Journal of Applied Mathematics and Computer Science*, vol. 22, no. 3, pp. 523–531, 2012.

[12] K. Balachandran, M. Matar, and J. J. Trujillo, "Note on controllability of linear fractional dynamical systems," *Journal of Control and Decision*, vol. 3, no. 4, pp. 267–279, 2016.

[13] K. Balachandran, J. Y. Park, and J. J. Trujillo, "Controllability of nonlinear fractional dynamical systems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 75, no. 4, pp. 1919–1926, 2012.

[14] M. Matar, "Controllability of fractional semilinear mixed Volterra-Fredholm integrodifferential equations with nonlocal conditions," *International Journal of Mathematical Analysis*, vol. 4, no. 21-24, pp. 1105–1116, 2010.

[15] M. M. Matar and H. N. Abu Ghalwa, "Approximate controllability of nonlocal fractional integrodifferential control systems of order $1< \alpha <2$," *Acta Mathematica Universitatis Comenianae*, vol. 88, no. 1, pp. 131–144, 2019.

[16] M. M. Matar, "On controllability of linear and nonlinear fractional integrodifferential systems," *Fractional Differential Calculus*, vol. 9, no. 1, pp. 19–32, 2019.

[17] H. Yang and E. Ibrahim, "Approximate controllability of fractional nonlocal evolution equations with multiple delays," *Advances in Difference Equations*, vol. 2017, article 272, 15 pages, 2017.

[18] Y.-K. Chang, A. Pereira, and R. Ponce, "Approximate controllability for fractional differential equations of Sobolev type via properties on resolvent operators," *Fractional Calculus and Applied Analysis*, vol. 20, no. 4, pp. 963–987, 2017.

[19] L. Chen, Z. Fan, and G. Li, "On a nonlocal problem for fractional differential equations via resolvent operators," *Advances in Difference Equations*, vol. 2014, article 251, pp. 1–12, 2014.

[20] Z. Fan, "Approximate controllability of fractional differential equations via resolvent operators," *Advances in Difference Equations*, vol. 2014, article 54, 2014.

[21] Z. Fan and G. Mophou, "Nonlocal problems for fractional differential equations via resolvent operators," *International Journal of Differential Equations*, vol. 2013, Article ID 490673, 9 pages, 2013.

[22] Z. Fan and G. Mophou, "Existence and optimal controls for fractional evolution equations," *Nonlinear Studies. The International Journal*, vol. 20, no. 2, pp. 163–172, 2013.

[23] T. Lian, Z. Fan, and G. Li, "Approximate controllability of semilinear fractional differential system of order $1<q<2$ via resolvent operators," *Filomat*, vol. 31, no. 18, pp. 5769–5781, 2017.

[24] N. Mahmudov and M. M. Matar, "Existence of mild solution for hybrid differential equations with arbitrary fractional order," *TWMS Journal of Pure and Applied Mathematics*, vol. 8, no. 2, pp. 160–169, 2017.

[25] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, New York, NY, USA, Elsevier, 2006.

[26] D. R. Smart, *Fixed Point Theorems*, Cambridge University Press, London, UK, 1980.

[27] B. C. Dhage, "On a fixed point theorem in Banach algebras with applications," *Applied Mathematics Letters*, vol. 18, no. 3, pp. 273–280, 2005.

[28] E. Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley & Sons, New York, NY, USA, 1978.

[29] J. Prüss, *Evolutionary Integral Equations and Applications*, vol. 87 of *Monographs in Mathematics*, Birkhäuser, Basel, Switzerland, 1993.