Hindawi
Journal of Mathematics
Volume 2019, Article ID 9456167, 7 pages
https://doi.org/10.1155/2019/9456167

Research Article

Strong Law of Large Numbers of Pettis-Integrable Multifunctions

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Received 11 February 2019; Accepted 26 March 2019; Published 15 April 2019

Academic Editor: Tepper L Gill

Using reversed martingale techniques, we prove the strong law of large numbres for independent Pettis-integrable multifunctions with convex weakly compact values in a Banach space. The Mosco convergence of reversed Pettis-integrable martingale of the form $(E^{\mathfrak{B}_n}X)_{n\geq 1}$, where $(\mathfrak{B}_n)_{n\geq 1}$ is a decreasing sequence of the sub σ-algebra of $\mathfrak{F}$ is provided.

1. Introduction

The strong law of large numbers (SLLN) is used in a variety of fields including statistics, probability theory, and areas of economics and insurance. In recent years, SLLN has been extensively studied by several researchers. Let us mention Artstein and Hart [1], Castaing and Ezzaki [2], Etemadi [3], Ezzaki [4], Hess [5], and Hiai [6].

In the theory of integration in infinite-dimensional spaces, Pettis-integrability is a more general concept than that of Bochner-integrability. The purpose of this paper is to prove the SLLN for measurable and Pettis-integrable multifunctions by using the techniques of reversed martingale. The proof is based on the recent properties of Pettis-integrable multifunctions. See for example Akhiat et al. [7], Chowdhury [8], El Amri and Hess [9], Geitz et al. [10], Thobie and Satco [11], and Musial [12].

The paper is organized as follows.

In Section 2, we recall some definitions and results that will be used after. In Section 3, we prove the SLLN for Pettis-integrable multifunctions with convex weakly compact values.

2. Notations and Definitions

Throughout this paper, we assume that $(\Omega, \mathfrak{F}, P)$ is a complete probability space and $(\mathfrak{B}_n)_{n\geq 1}$ is a decreasing sequence of sub-σ-algebra of $\mathfrak{F}$, such that $\mathfrak{B}_\infty = \bigcap_{n\geq 1} \mathfrak{B}_n$. E is a separable Banach space with the dual space E^*.

Let $(e_k^*)_{k\in\mathbb{N}}$ be a dense sequence in E^* with respect to the Mackey topology $\tau(E^*, E)$, $\overline{B_E}$(resp., $\overline{B_{E^*}}$), and the closed unit ball of E (resp., E^*). Let $cc(E)$(resp., $cwk(E)$) be the family of all nonempty convex and closed (resp., convex weakly compact) subsets of E.

Given B in $cc(E)$, the distance function and the support function associated with B are defined by

$$\begin{aligned} d(x, B) &= \inf\{\|x-y\|,\ y \in B\}, \quad (x \in E), \\ \delta^*(x^*, B) &= \sup\{\langle x^*, y\rangle,\ y \in B\}, \quad (x^* \in E^*). \end{aligned} \tag{1}$$

For any C in $cwk(E)$, we get

$$|C| = \sup\{\|x\|,\ x \in C\}. \tag{2}$$

A $cc(E)$-valued sequence $(Y_n)_{n\geq 1}$ is called Mosco convergent to a closed convex set Y_∞ if $Y_\infty = s - li\, Y_n = w - ls\, Y_n$, where

$$\begin{aligned} s - li\, Y_n &= \{y \in E,\ \exists y_n \longrightarrow y \text{ a.s. } y_n \in Y_n,\ \forall n \geq 1\}, \\ w - ls\, Y_n & \\ &= \left\{y \in E,\ \exists y_k \longrightarrow y \text{ weakly},\ y_k \in Y_{n_k},\ \forall k \geq 1\right\}. \end{aligned} \tag{3}$$

If $(Y_n)_{n\geq 1}$ Mosco converges to Y_∞ in $cc(E)$, we write $M - \lim Y_n = Y_\infty$.

A measurable function $g : \Omega \longrightarrow E$ is Pettis-integrable, if g is scalarly integrable

(i. e. $\langle x^*, g\rangle$ is integrable), and, for each $A \in \mathfrak{F}$, there exists x_A in E, such that

$$\int_A \langle x^*, g\rangle \, dP = \langle x^*, x_A\rangle, \quad \forall x^* \in E^*. \tag{4}$$

x_A is called the Pettis-integral of g over A. We will denote by $P_E^1(\mathfrak{F})$ the space of all measurable and Pettis-integrable E-valued function defined on $(\Omega, \mathfrak{F}, P)$. We consider the space $P_E^1(\mathfrak{F})$ provided with the following topologies.

(i) The topology of the usual Pettis norm is as follows:

$$\|g\|_{Pe} = \sup_{x^* \in B_{E^*}} \int_\Omega |\langle x^*, g\rangle| \, dP. \tag{5}$$

(ii) The topology is induced by the duality $(P_E^1(X), L^\infty \otimes E^*)$. Recall that a sequence $(g_n)_{n\geq 1}$ in $P_E^1(X)$ converges to g in this topology, if, for each $h \in L^\infty(\mathfrak{F})$ and for all $x^* \in E^*$, one has

$$\begin{aligned} \lim_{n\longrightarrow +\infty} & \int_\Omega h(\omega) \langle x^*, g_n(\omega)\rangle \, dP \\ &= \int_\Omega h(\omega) \langle x^*, g(\omega)\rangle \, dP. \end{aligned} \tag{6}$$

This topology is known as the weak topology and is denoted by w-Pe.

A multifunction $X : \Omega \longrightarrow cc(E)$ is said to be measurable, if for every open set U of E the subset $X^-(U) = \{\omega \in \Omega / X(\omega) \cap U \neq \emptyset\}$ is an element of $\mathfrak{F}$. The Effros σ-field $\mathfrak{B}$ on $cc(E)$ is generated by the subsets $U^- = \{F \in cc(E)/F \cap U \neq \emptyset\}$, so X is measurable if, for any $B \in \mathfrak{B}$, one has $X^{-1}(B) \in \mathfrak{F}$.

Two measurable multifunctions X and Y are said to be equal scalarly almost surely, if the following equality holds:

$$\delta^*(x^*, X(.)) = \delta^*(x^*, Y(.)) \quad a.s.\ \forall x^* \in E^*. \tag{7}$$

For any convex and weakly compact measurable multifunction X, the measurability of X is equivalent to that of its support functional $\delta^*(x^*, X)$. (See [13].)

The tribe trace of $\mathfrak{B}$ on $cwk(E)$ is defined by the set $\{cwk(E) \cap B / B \in \mathfrak{B}\}$ and is denoted by ξ.

A measurable function $f : \Omega \longrightarrow E$ is said to be a selection of X, if, for any $\omega \in \Omega$, $f(\omega) \in X(\omega)$. We denote by $S_X^1(\mathfrak{F})$ the set of all measurable selection of X. It is known that a convex and closed valued multifunction X in separable Banach space is measurable if $\mathrm{dom}(X) \in \mathfrak{F}$ and X has a Castaing representation (i.e., there exists a sequence $(f_n)_{n\geq 1}$ of measurable selections of X such that for all $\omega \in \Omega$, $X(\omega) = cl\{f_n(\omega),\ n \geq 1\}$).

The distribution P_Γ of the measurable multifunction $\Gamma : \Omega \longrightarrow cc(E)$ on the measurable space $(cc(E), \mathfrak{B})$ is defined by

$$P_\Gamma(B) = P\left(\Gamma^{-1}(B)\right), \quad \forall B \in \mathfrak{B}. \tag{8}$$

Two measurable multifunctions Γ and Δ are said to be independent, if the following equality holds:

$$P_{(\Gamma,\Delta)} = P_\Gamma \otimes P_\Delta. \tag{9}$$

The measurable multifunction $X : \Omega \longrightarrow cwk(E)$ is scalarly integrable, if, for any $x^* \in E^*$, the real function $\delta^*(x^*, X(.))$ is integrable.

We say that the measurable multifunction X is Pettis-integrable, if it is scalarly integrable and, for each $A \in \mathfrak{F}$, there exists $K_A \in cwk(E)$ such that

$$\int_A \delta^*(x^*, X) \, dP = \delta^*(x^*, K_A), \quad \forall x^* \in E^*. \tag{10}$$

K_A is called the Pettis-integral of X over A. Let $S_X^{Pe}(\mathfrak{F})$ be the set of all $\mathfrak{F}$-measurable and Pettis-integrable selections of X.

The multivalued Pettis-integral of a $cwk(E)$-valued multifunction X is defined by

$$\int_\Omega X dP = \left\{\int_\Omega f dP,\ f \in S_X^{Pe}(\mathfrak{F})\right\}, \tag{11}$$

and $\int_A X dP$ is convex and $\sigma(E, E^*)$ compact and, for each $A \in \mathfrak{F}$, we have

$$\int_A \delta^*(x^*, X) \, dP = \delta^*\left(x^*, \int_A X dP\right), \quad \forall x^* \in E^*. \tag{12}$$

We will denote by $P^1_{cwk(E)}(\mathfrak{F})$ the set of all Pettis-integrable $cwk(E)$-valued multifunctions.

Given a sub σ-algebra $\mathscr{B}$ and a $cwk(E)$-valued Pettis-integrable multifunction, the Pettis conditional expectation of X with respect to $\mathscr{B}$ is a $\mathscr{B}$-measurable $cwk(E)$-valued Pettis-integrable multifunction denoted by $E^{\mathscr{B}}X$ which satisfies

$$\int_A E^{\mathscr{B}} X dP = \int_A X dP, \quad \forall A \in \mathscr{B}. \tag{13}$$

The two following propositions (see [7]) give a sufficient condition of the existence of the conditional expectation for a Pettis integrable E-valued function and for a $cwk(E)$-valued Pettis-integrable multifunction.

Proposition 1. *Assume that E is separable. Let $\mathscr{B}$ be a sub σ-algebra of $\mathfrak{F}$ and let X be a Pettis-integrable E-valued function such that $E^{\mathscr{B}}|X| \in [0, +\infty[$. Then, there exists a unique $\mathscr{B}$-measurable Pettis-integrable E-valued function denote by $E^{\mathscr{B}}X$, which enjoys the following property; for every $h \in L^\infty(\mathscr{B})$, one has*

$$\int_\Omega h E^{\mathscr{B}} X dP = \int_\Omega h X dP. \tag{14}$$

Proposition 2. *Assume that E^* is separable. Let $\mathscr{B}$ be a sub-σ-algebra of $\mathfrak{F}$ and let X be a $cwk(E)$-valued Pettis-integrable multifunction such that $E^{\mathscr{B}}|X| \in [0, +\infty[$. Then, there exists a unique $\mathscr{B}$-measurable $cwk(E)$-valued Pettis-integrable multifunction, which enjoys the following property; for every $h \in L^\infty(\mathscr{B})$, one has*

$$\int_\Omega h E^{\mathscr{B}} X dP = \int_\Omega h X dP. \tag{15}$$

Akhiat et al. [14] extended the previous theorem in a $cc(E)$-valued Pettis-integrable multifunction.

We close this section by the following useful corollaries.

Corollary 3 (see [15]). *Let $(\mathfrak{B}_n)_{n\geq 1}$ be a decreasing sequence of a sub-σ-algebra of $\mathfrak{F}$ and let $f \in L^1_E(\mathfrak{F})$; set $\mathfrak{B}_\infty = \bigcap_{n\geq 1} \mathfrak{B}_n$. Then,*

$$\lim_{n\longrightarrow+\infty} E^{\mathfrak{B}_n} f = E^{\mathfrak{B}_\infty} f \ a.s. \tag{16}$$

Corollary 4. *Let $\mathscr{B}$ be a sub-σ-algebra of $\mathfrak{F}$ and let X be a $cwk(E)$-valued Pettis-integrable multifunction such that $E^{\mathscr{B}}|X| \in [0,+\infty[$. Then, for all $f \in S^{Pe}_X(\mathfrak{F})$, the following properties hold:*

(1) $E^{\mathscr{B}} f(.) \in E^{\mathscr{B}} X(.)$ *almost surely.*

(2) $\delta^*(x^*, E^{\mathscr{B}} X(.)) = E^{\mathscr{B}} \delta^*(x^*, X(.))$ *almost surely.*

Proof. (1) Let $f \in S^{Pe}_X(\mathfrak{F})$ and $(e^*_k)_{k\geq 1}$ be a dense sequence in E^* with respect to the Mackey topology $\tau(E^*, E)$,

then

$$\int_A \left\langle e^*_k, f\right\rangle dP \leq \int_A \delta^*\left(e^*_k, X\right) dP, \quad \forall A \in \mathscr{B}, \tag{17}$$

so

$$\begin{aligned} \left\langle e^*_k, \int_A f dP\right\rangle &\leq \delta^*\left(e^*_k, \int_A X dP\right), \\ \left\langle e^*_k, \int_A E^{\mathscr{B}} f dP\right\rangle &\leq \delta^*\left(e^*_k, \int_A E^{\mathscr{B}} X\right) dP, \end{aligned} \tag{18}$$

and hence

$$\left\langle e^*_k, E^{\mathscr{B}} f(.)\right\rangle \leq \delta^*\left(e^*_k, E^{\mathscr{B}} X(.)\right) \ a.s. \tag{19}$$

We conclude that

$$E^{\mathscr{B}} f(.) \in E^{\mathscr{B}} X(.) \ a.s. \tag{20}$$

(2) For any $A \in \mathscr{B}$, we have $\delta^*(x^*, \int_A X dP) = \int_A \delta^*(x^*, X) dP$, then, by Proposition 2, we have $\delta^*(x^*, \int_A E^{\mathscr{B}} X dP) = \int_A \delta^*(x^*, X) dP$. Since $E^{\mathscr{B}} X$ is a Pettis-integrable multifunction,

$$\begin{aligned} \int_A \delta^*\left(x^*, E^{\mathscr{B}} X\right) dP &= \int_A \delta^*\left(x^*, X\right) dP \\ &= \int_A E^{\mathscr{B}} \delta^*\left(x^*, X\right) dP. \end{aligned} \tag{21}$$

And hence by uniqueness of the conditional expectation of $\delta^*(x^*, X)$, we obtain

$$\delta^*\left(x^*, E^{\mathscr{B}} X(.)\right) = E^{\mathscr{B}} \delta^*\left(x^*, X(.)\right) \ a.s. \tag{22}$$

□

3. Strong Law of Large Numbers for Pettis-Integrable Multifunctions

Our first result is the following theorem.

Theorem 5. *Let $(\mathfrak{B}_n)_{n\geq 1}$ be a decreasing sequence of sub-σ-algebras of $\mathfrak{F}$ and set $\mathfrak{B}_\infty = \bigcap_{n\geq 1} \mathfrak{B}_n$. Let $\mathbf{E}$ be a separable Banach space and $f \in P^1_E(\mathfrak{F})$ such that $E^{\mathfrak{B}_\infty}|f| \in [0,+\infty[$. Then*

$$\lim_{n\longrightarrow+\infty} E^{\mathfrak{B}_n} f(.) = E^{\mathfrak{B}_\infty} f(.) \ a.s. \tag{23}$$

Proof. Since $E^{\mathfrak{B}_\infty}|f| \in [0,+\infty[$, so, by Proposition 1, we have that $E^{\mathfrak{B}_n} f$ exists and is in $P^1_E(\mathfrak{B}_n)$ and provides a $\mathfrak{B}_\infty$-measurable partition $(\mathscr{B}_k)_{k\geq 1}$ of Ω such that $f_k = f 1_{\mathscr{B}_k} \in L^1_{\mathbf{E}}(\mathfrak{F})$ (see [14]). Using Corollary 3, we obtain

$$\lim_{n\longrightarrow+\infty} E^{\mathfrak{B}_n} f_k = E^{\mathfrak{B}_\infty} f_k \ a.s. \tag{24}$$

As $\mathscr{B}_k \subset \mathfrak{B}_\infty \subset \mathfrak{B}_n$, for every $n \geq 1$ and for every $k \geq 1$, then

$$E^{\mathfrak{B}_n} f_k = E^{\mathfrak{B}_n} 1_{\mathscr{B}_k} f = 1_{\mathscr{B}_k} E^{\mathfrak{B}_n} f. \tag{25}$$

On the other hand,

$$\begin{aligned} E^{\mathfrak{B}_\infty} f &= \sum_{k=1}^{+\infty} E^{\mathfrak{B}_\infty} f_k = \sum_{k=1}^{+\infty} \lim_{n\longrightarrow+\infty} E^{\mathfrak{B}_n} f_k = \\ &= \sum_{k=1}^{+\infty} 1_{\mathscr{B}_k} \lim_{n\longrightarrow+\infty} E^{\mathfrak{B}_n} f = \lim_{n\longrightarrow+\infty} E^{\mathfrak{B}_n} f. \end{aligned} \tag{26}$$

Therefore,

$$\lim_{n\longrightarrow+\infty} E^{\mathfrak{B}_n} f(.) = E^{\mathfrak{B}_\infty} f(.) \ a.s. \tag{27}$$

□

Let us prove the following results which will be used after.

Proposition 6. *Assume that E^* is separable. Let X and Y be two $cwk(E)$-valued Pettis-integrable multifunctions and $\mathfrak{B}$ be a sub-σ-algebra of $\mathfrak{F}$ such that $E^{\mathfrak{B}}|X| \in [0,+\infty[$ and $E^{\mathfrak{B}}|Y| \in [0,+\infty[$. Then*

$$E^{\mathfrak{B}}(X+Y) = E^{\mathfrak{B}} X + E^{\mathfrak{B}} Y \ a.s. \tag{28}$$

Proof. Let $(e^*_k)_{k\geq 1}$ be a dense sequence in E^* with respect to the Mackey topology $\tau(E^*, E)$. Since X and Y are $cwk(E)$-valued Pettis-integrable multifunctions, then, for any A in $\mathfrak{B}$, there exist two sets $K_A(X)$ and $K_A(Y)$ in $cwk(E)$, such that for all $k \geq 1$ $\delta^*(e^*_k, K_A(X)) = \int_A \delta^*(e^*_k, X) dP$ and $\delta^*(e^*_k, k_A(Y)) = \int_A \delta^*(e^*_k, Y) dP$.

By Theorem 5.1.6 in [8], $X + Y$ is $cwk(E)$-valued Pettis-integrable multifunction, then, for any A in $\mathfrak{B}$, there exists a set $K_A(X+Y)$ in $cwk(E)$ such that

$$\delta^*\left(e^*_k, K_A(X+Y)\right) = \int_A \delta^*\left(e^*_k, X+Y\right) dP. \tag{29}$$

On the other hand,

$$\begin{aligned} &\delta^*\left(e^*_k, K_A(X) + K_A(Y)\right) \\ &\quad = \delta^*\left(e^*_k, K_A(X)\right) + \delta^*\left(e^*_k, K_A(Y)\right) = \end{aligned}$$

$$
\begin{aligned}
&= \int_A \delta^*\left(e_k^*, X\right) dP + \int_A \delta^*\left(e_k^*, Y\right) dP \\
&= \int_A \delta^*\left(e_k^*, X\right) + \delta^*\left(e_k^*, Y\right) dP = \\
&= \int_A \delta^*\left(e_k^*, X + Y\right) dP = \delta^*\left(e_k^*, K_A\left(X + Y\right)\right).
\end{aligned}
\tag{30}
$$

Then $K_A(X+Y) = K_A(X) + K_A(Y)$ *a.s.*

Therefore,

$$
\int_A (X+Y)\,dP = \int_A X dP + \int_A Y dP \ \ a.s. \tag{31}
$$

On the other hand,

$$
\begin{aligned}
&\int_A \delta^*\left(e_k^*, E^{\mathfrak{B}}\left(X + Y\right)\right) dP \\
&\quad = \delta^*\left(e_k^*, \int_A E^{\mathfrak{B}}\left(X + Y\right) dP\right) \\
&\quad = \delta^*\left(e_k^*, \int_A X + Y dP\right)
\end{aligned}
\tag{32}
$$

By (31),

$$
\begin{aligned}
&\delta^*\left(e_k^*, \int_A X + Y dP\right) = \delta^*\left(e_k^*, \int_A X dP + \int_A Y dP\right) \\
&\quad = \delta^*\left(e_k^*, \int_A E^{\mathfrak{B}} X dP + \int_A E^{\mathfrak{B}} Y dP\right) = \\
&\quad = \int_A \delta^*\left(e_k^*, E^{\mathfrak{B}} X + E^{\mathfrak{B}} Y\right) dP,
\end{aligned}
\tag{33}
$$

then

$$
\begin{aligned}
&\int_A \delta^*\left(e_k^*, E^{\mathfrak{B}}\left(X + Y\right)\right) dP \\
&\quad = \int_A \delta^*\left(e_k^*, E^{\mathfrak{B}} X + E^{\mathfrak{B}} Y\right) dP,
\end{aligned}
\tag{34}
$$

and therefore

$$
E^{\mathfrak{B}}\left(X + Y\right) = E^{\mathfrak{B}} X + E^{\mathfrak{B}} Y \ \ a.s. \tag{35}
$$

□

Theorem 7. *Let X and Y be in $P^1_{cwk(E)}(\mathfrak{F})$; let Z be a random variable with values in a measurable space (Σ, Γ) such that (Z, X) and (Z, Y) have the same distribution. If $E^{\sigma(Z)}|X| \in [0, +\infty[$ and $E^{\sigma(Z)}|Y| \in [0, +\infty[$. Then*

$$
E^{\sigma(Z)} X\left(.\right) = E^{\sigma(Z)} Y\left(.\right) \ \ a.s. \tag{36}
$$

Proof. Since X is Pettis-integrable with values in $cwk(E)$, $S_X^{Pe}(\sigma(X)) \neq \emptyset$. So by [11], there exists $(f_n)_{n\geq 1}$ selection Pettis-integrable of X, $\sigma(X)$-measurable such that

$$
X\left(w\right) = cl\left\{f_n\left(\omega\right),\ n \geq 1\right\}. \tag{37}
$$

Let $f : \Omega \longrightarrow E$ be a fixed element in $S_X^{Pe}(\sigma(X))$, $h \in L^\infty(\sigma(Z))$, and $x^* \in E^*$. Then, by Doob's factorisation lemma, we can find a ξ-measurable mapping u from $cwk(E)$ into E (i.e., $(cwk(E), \xi) \xrightarrow{u} (E, \mathfrak{B}(E))$) and a Γ-measurable function v from Σ into $\mathbb{R}$ (i.e., $(\Sigma, \Gamma) \xrightarrow{v} (\mathbb{R}, \mathfrak{B}(\mathbb{R}))$), which satisfy

$$
\begin{aligned}
&f : (\Omega, \mathfrak{F}, P) \xrightarrow{X} \left(cwk\left(E\right), \xi\right) \xrightarrow{u} \left(E, \mathfrak{B}\left(E\right)\right), \\
&\qquad f\left(\omega\right) = \left(u \circ X\right)\left(\omega\right). \\
&h : (\Omega, \mathfrak{F}, P) \xrightarrow{Z} \left(\Sigma, \Gamma\right) \xrightarrow{v} \left(\mathbb{R}, \mathfrak{B}\left(\mathbb{R}\right)\right), \\
&\qquad h\left(\omega\right) = \left(v \circ Z\right)\left(\omega\right).
\end{aligned}
\tag{38}
$$

On the other hand,

$$
\begin{aligned}
&\delta^*\left(h \otimes x^*, S_X^{Pe}\left(\sigma\left(X\right)\right)\right) = \int_\Omega \delta^*\left(h \otimes x^*, X\right) dP \\
&\quad = \sup\left\{\int_\Omega h\left(\omega\right)\left\langle x^*, f\right\rangle dP,\ f \in S_X^{Pe}\left(\sigma\left(X\right)\right)\right\}.
\end{aligned}
\tag{39}
$$

And

$$
\begin{aligned}
&\int_\Omega h\left(\omega\right)\left\langle x^*, f\right\rangle dP \\
&\quad = \int_\Omega v \circ Z\left(\omega\right)\left\langle x^*, u \circ X\left(\omega\right)\right\rangle dP.
\end{aligned}
\tag{40}
$$

By using the application

$$
\begin{aligned}
&(\Omega, \mathfrak{F}, P) \xrightarrow{g} \\
&\left(\Sigma \times cwk\left(E\right), \Gamma \otimes \xi\right) \xrightarrow{\varphi} \\
&\left(\mathbb{R}, \mathfrak{B}\left(\mathbb{R}\right)\right) \omega \longrightarrow \\
&\left(Z, X\right) \longrightarrow \\
&voZ\left(\omega\right)\left\langle x^*, uoX\left(\omega\right)\right\rangle = h\left(\omega\right)\left\langle x^*, f\left(\omega\right)\right\rangle.
\end{aligned}
\tag{41}
$$

And by the classical transfer theorem, we have

$$
\begin{aligned}
&\int_\Omega \varphi \circ g\left(\omega\right) dP = \int_\Omega h\left(\omega\right)\left\langle x^*, f\left(\omega\right)\right\rangle dP \\
&\quad = \int_\Omega v \circ Z\left(\omega\right)\left\langle x^*, u \circ X\left(\omega\right)\right\rangle dP = \\
&\quad = \int_{\Sigma \otimes cwk(E)} \varphi\left(z, B\right) dP_{(Z,X)}\left(z, B\right) \\
&\quad = \int_{\Sigma \otimes cwk(E)} v\left(z\right)\left\langle x^*, u\left(B\right)\right\rangle dP_{(Z,X)}\left(z, B\right) = \\
&\quad = \int_{\Sigma \otimes cwk(E)} v\left(z\right)\left\langle x^*, u\left(B\right)\right\rangle dP_{(Z,Y)}\left(z, B\right)
\end{aligned}
$$

$$= \int_{\Sigma\otimes cwk(E)} \varphi(z,B)\, dP_{(Z,Y)}(z,B) =$$
$$= \int_{\Omega} v\circ Z(\omega) \langle x^*, u\circ Y(\omega)\rangle\, dP$$
$$= \int_{\Omega} h(\omega) \langle x^*, u\circ Y(\omega)\rangle\, dP. \tag{42}$$

Then

$$\int_{\Omega} h(\omega)\langle x^*, f(\omega)\rangle\, dP = \int_{\Omega} h(\omega)\langle x^*, u\circ Y(\omega)\rangle\, dP. \tag{43}$$

By combining relation (39), (43) and the fact that $\forall B \in cwk(E), u(B) \in B, P_Y$ *a.s.*, we obtain

$$\int_{\Omega} \delta^*(h\otimes x^*, X)\, dP = \sup\left\{\int_{\Omega} h(\omega)\langle x^*, f(\omega)\rangle\, dP,\ f \in S_Y^{Pe}(\sigma(Y))\right\} = \int_{\Omega} \delta^*(h\otimes x^*, Y)\, dP. \tag{44}$$

Then

$$\delta^*\left(h\otimes x^*, S_X^{Pe}(\sigma(X))\right) = \int_{\Omega} \delta^*(h\otimes x^*, X)\, dP = \int_{\Omega} \delta^*(h\otimes x^*, Y) = \delta^*\left(h\otimes x^*, S_Y^{Pe}(\sigma(Y))\right). \tag{45}$$

In particular, if $h = 1_A$, for any $A \in \sigma(Z)$, we have

$$\int_{\Omega} 1_A \delta^*(x^*, X)\, dP = \int_{\Omega} 1_A \delta^*(x^*, Y)\, dP, \tag{46}$$

then

$$\delta^*\left(x^*, \int_A X dP\right) = \delta^*\left(x^*, \int_A Y dP\right), \tag{47}$$

and therefore

$$\delta^*\left(x^*, \int_A E^{\sigma(Z)} X\right) dP = \delta^*\left(x^*, \int_A E^{\sigma(Z)} Y\right) dP, \tag{48}$$

And hence, by uniqueness of the conditional expectation of X relative to $\sigma(Z)$,

$$E^{\sigma(Z)} X(.) = E^{\sigma(Z)} Y(.) \quad a.s. \tag{49}$$

□

Before giving the principal result, we also need the following classical theorem (see p. 52 in [16]).

Theorem 8. *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be two sub-σ-algebras of $\mathfrak{F}$, $\Phi = \sigma(\mathfrak{B}_1, \mathfrak{B}_2)$ is a σ-algebra generated by $\mathfrak{B}_1$ and $\mathfrak{B}_2$, and let f_1 an integrable real measurable function. If f_1 and $\mathfrak{B}_1$ are independent of $\mathfrak{B}_2$, then*

$$E^{\Phi} f_1(.) = E^{\mathfrak{B}_1} f_1(.) \quad a.s. \tag{50}$$

Now, we give the main result of this work.

Theorem 9. *Assume that E^* is separable. Let $(X_n)_{n\geq 1}$ be a sequence of independent measurable multifunctions in $P^1_{cwk(E)}(\mathfrak{F})$.*

Let $S_n = X_1 + X_2 + X_3 + \cdots + X_n$, $\mathfrak{B}_n = \sigma(S_n, X_{n+1}, X_{n+2}, \ldots)$, $\mathfrak{B}_\infty = \bigcap_{n\geq 1}\mathfrak{B}_n$ and assume that

(i) *$\forall n, \forall j \in \{1, 2, \ldots, n\}$, (S_n, X_1) and (S_n, X_j) have the same distribution.*

(ii) *$\forall n, \forall j \in \{1, 2, \ldots, n\}$, $E^{\mathfrak{B}_n}|X_j| \in [0, +\infty[$.*

Then, we have the following assertions:

(1) *$\forall n, \forall j \in \{1, 2, \ldots, n\}$, $E^{\sigma(S_n)} X_j(.) = E^{\sigma(S_n)} X_1(.)$ a.s.*

(2) *$E^{\mathfrak{B}_n} X_1(.) = E^{\sigma(S_n)} X_1(.)$ a.s.*

(3) *$M - \lim_{n\to\infty}(S_n/n) = M - \lim_{n\to\infty} E^{\mathfrak{B}_n} X_1(.) = E^{\mathfrak{B}_\infty} X_1(.)$ a.s.*

Proof. The first equality follows from the Theorem 7.

Now let us prove the second equality. Let $(e_k^*)_{k\geq 1}$ be a dense sequence in E^* for the Mackey topology. Set $\mathfrak{B}_1 = \sigma(S_n)$, $\mathfrak{B}_2 = \sigma(X_{n+1}, X_{n+2}, \ldots)$ and $f_1 = \delta^*(e_k^*, X_1)$.

Since X_1 and S_n are independent of $\mathfrak{B}_2$, f_1 and S_n are independent of $\mathfrak{B}_2$, then, by applying the Theorem 8, we have

$$E^{\mathfrak{B}_n} f_1(.) = E^{\mathfrak{B}_1} f_1(.) \quad a.s. \tag{51}$$

So, $E^{\mathfrak{B}_n}\delta^*(e_k^*, X_1) = E^{\mathfrak{B}_1}\delta^*(e_k^*, X_1)$, then, by using Corollary 4, we obtain

$$\delta^*\left(e_k^*, E^{\mathfrak{B}_n} X_1\right) = \delta^*\left(e_k^*, E^{\mathfrak{B}_1} X_1\right), \tag{52}$$

and since $E^{\mathfrak{B}_n} X_1(.)$ and $E^{\sigma(S_n)} X_1(.)$ are convex and weakly compact, then

$$E^{\mathfrak{B}_n} X_1(.) = E^{\sigma(S_n)} X_1(.) \quad a.s. \tag{53}$$

(i) Now, we show the last assertion.

Step 1. We claim that $S_n/n = E^{\mathfrak{B}_n} X_1(.)$ *a.s.*

We have $S_n = \sum_{i=1}^n X_i = E^{\sigma(S_n)}(\sum_{i=1}^n X_i)$, so, by Proposition 6 and the first and the second assertions of the theorem, we obtain

$$E^{\sigma(S_n)}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n E^{\sigma(S_n)} X_i = \sum_{i=1}^n E^{\sigma(S_n)} X_1 = nE^{\sigma(S_n)} X_1 = nE^{\mathfrak{B}_n} X_1. \tag{54}$$

Then

$$\frac{S_n}{n} = E^{\mathfrak{B}_n} X_1(.) \quad a.s. \tag{55}$$

Step 2. We show that $M - \lim_{n\to\infty} E^{\mathfrak{B}_n} X_1 = E^{\mathfrak{B}_\infty} X_1(.)$ *a.s.*

(ii) We begin by proving that $E^{\mathfrak{B}_\infty} X_1(.) \subset s - li\, E^{\mathfrak{B}_n} X_1(.)$ *a.s.*

Since X_1 is Pettis-integrable, then there exists $(g_k)_{k\geq 1}$ Pettis-integrable selection of X_1 such that

$$X_1 = cl\{g_k,\ k \geq 1\}. \tag{56}$$

On the other hand $E^{\mathfrak{B}_n}|X_1| \in [0, +\infty[$, then $E^{\mathfrak{B}_n}X_1$ exists and is in $P^1_{cwk(E)}(\mathfrak{B}_n)$. Let $g \in S^{Pe}_{X_1}(\mathfrak{B}_n)$; by Corollary 4, $E^{\mathfrak{B}_n}g$ exists and $E^{\mathfrak{B}_n}g(.) \in E^{\mathfrak{B}_n}X_1(.)$ a. s.

Using Theorem 5, we have $\lim_{n\longrightarrow\infty} E^{\mathfrak{B}_n}g = E^{\mathfrak{B}_\infty}g$ a.s.

Since by [17]

$$S^{Pe}_{E^{\mathfrak{B}_\infty}X} = \left\{E^{\mathfrak{B}_\infty}f, f \in S^{Pe}_X(\mathfrak{F})\right\} \tag{57}$$

Then $E^{\mathfrak{B}_\infty}g \in S^{Pe}_{s-li\,E^{\mathfrak{B}_n}X_1}$ and by (56), $S^{Pe}_{E^{\mathfrak{B}_\infty}X_1(.)} \subset S^{Pe}_{s-li\,E^{\mathfrak{B}_n}X_1(.)}$ *a.s.*

Then

$$E^{\mathfrak{B}_\infty}X_1(.) \subset s-li\,E^{\mathfrak{B}_n}X_1(.) \;\; a.s. \tag{58}$$

(iii) Now, we show that $w-ls\,E^{\mathfrak{B}_n}X_1(.) \subset E^{\mathfrak{B}_\infty}X_1(.)$ *a.s.*

Let $(e^*_k)_{k\ge1}$ be a dense sequence in E^* for the Mackey topology; we have

$$\delta^*\left(e^*_k, E^{\mathfrak{B}_n}X_1\right) = E^{\mathfrak{B}_n}\delta^*\left(e^*_k, X_1\right), \tag{59}$$

then, by Corollary 3,

$$\lim_{n\longrightarrow+\infty} E^{\mathfrak{B}_n}\delta^*\left(e^*_k, X_1(.)\right) = E^{\mathfrak{B}_\infty}\delta^*\left(e^*_k, X_1(.)\right) \;\; a.s. \tag{60}$$

Hence, there exists a negligible set $N = \bigcup_{k\ge1} N_k$; for all $k \ge 1$ and for every $\omega \in \Omega \setminus N$, we have

$$\begin{aligned}\lim_{n\longrightarrow+\infty} E^{\mathfrak{B}_n}\delta^*\left(e^*_k, X_1(\omega)\right) &= E^{\mathfrak{B}_\infty}\delta^*\left(e^*_k, X_1(\omega)\right)\\ &= \delta^*\left(e^*_k, E^{\mathfrak{B}_\infty}X_1(\omega)\right) \;\; a.s.\end{aligned} \tag{61}$$

So, for all $x^* \in E^*$ and for all $\omega \in \Omega \setminus N$,

$$\begin{aligned}&\lim_{n\longrightarrow+\infty} \delta^*\left(x^*, E^{\mathfrak{B}_n}X_1(\omega)\right)\\ &\quad = \delta^*\left(x^*, E^{\mathfrak{B}_\infty}X_1(\omega)\right) \;\; a.s.\end{aligned} \tag{62}$$

Let $\omega \in \Omega \setminus N$ and $x \in w-ls\,E^{\mathfrak{B}_n}X_1(\omega)$, then there exists $(x_m)_{m\ge1}$ in $E^{\mathfrak{B}_{n_m}}X_1(\omega)$ such that $x_m \xrightarrow{w} x$, which implies $\lim_{m\longrightarrow+\infty}\langle x^*, x_m\rangle = \langle x^*, x\rangle$.

Then

$$\begin{aligned}\langle x^*, x\rangle &= \lim_{m\longrightarrow+\infty}\langle x^*, x_m\rangle\\ &\le \limsup_{m\longrightarrow\infty}\delta^*\left(x^*, E^{\mathfrak{B}_{n_m}}X_1\right) =\\ &= \limsup_{m\longrightarrow\infty} E^{\mathfrak{B}_{n_m}}\delta^*\left(x^*, X_1\right)\\ &= \delta^*\left(x^*, E^{\mathfrak{B}_\infty}X_1(\omega)\right) \;\; a.s.\end{aligned} \tag{63}$$

Consequently, $x \in E^{\mathfrak{B}_\infty}X_1(\omega)$, then

$$w-ls\,E^{\mathfrak{B}_n}X_1(.) \subset E^{\mathfrak{B}_\infty}X_1(.) \;\; a.s. \tag{64}$$

This yields

$$\begin{aligned}M-\lim_{n\longrightarrow\infty}\frac{S_n}{n} &= M-\lim_{n\longrightarrow\infty}E^{\mathfrak{B}_n}X_1(.)\\ &= E^{\mathfrak{B}_\infty}X_1(.) \;\; a.s.\end{aligned} \tag{65}$$

□

Corollary 10. *Under the same hypothesis of Theorem 9, we have*

$$M-\lim_{n\longrightarrow\infty}\frac{S_n}{n} = E^{\mathfrak{B}_\infty}X_1(.) = \int X_1(\omega)\,dP(\omega) \;\; a.s. \tag{66}$$

Proof. By the previous theorem, we need only to check that

$$E^{\mathfrak{B}_\infty}X_1(.) = \int_\Omega X_1(\omega)\,dP(\omega) \;\; a.s. \tag{67}$$

Since $\int_\Omega X_1(\omega)dP(\omega)$ is convex and weakly compact. Now let $(e^*_k)_{k\ge1}$ be a dense sequence in E^* for the Mackey topology; we have

$$\begin{aligned}\lim_{n\longrightarrow\infty}\delta^*\left(e^*_k, E^{\mathfrak{B}_n}X_1\right) &= \lim_{n\longrightarrow\infty}E^{\mathfrak{B}_n}\delta^*\left(e^*_k, X_1\right)\\ &= E^{\mathfrak{B}_\infty}\delta^*\left(e^*_k, X_1\right)\\ &= \delta^*\left(e^*_k, E^{\mathfrak{B}_\infty}X_1\right) \;\; a.s.\end{aligned} \tag{68}$$

On the other hand, $(E^{\mathfrak{B}_n}\delta^*(e^*_k, X_1))_n$ is a Pettis reversed martingale, so, for each positive integer m, set, $\mathfrak{F}_m = \sigma(X_m, X_{m+1}, X_{m+2}, \dots)$ and $\mathfrak{F}_\infty = \bigcap_m \mathfrak{F}_m$. Hence, for all any $n \ge m$, the multifunction $\sum_{j=m}^n X_j$ is $\mathfrak{F}_m$-measurable. Moreover, by the previous theorem, we have

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^n\delta^*\left(e^*_k, X_j\right) &= \frac{1}{n}\sum_{j=1}^{m-1}\delta^*\left(e^*_k, X_j\right)\\ &\quad + \frac{1}{n}\sum_{j=m}^n\delta^*\left(e^*_k, X_j\right) =\\ &= \frac{1}{n}\delta^*\left(e^*_k, \sum_{j=1}^{m-1}X_j\right)\\ &\quad + \frac{1}{n}\sum_{j=m}^n\delta^*\left(e^*_k, X_j\right)\\ &= \frac{m-1}{n}E^{\mathfrak{B}_{m-1}}\delta^*\left(e^*_k, X_1\right)\\ &\quad + \frac{1}{n}\sum_{j=m}^n\delta^*\left(e^*_k, X_j\right) \;\; a.s.\end{aligned} \tag{69}$$

Then

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^n\delta^*\left(e^*_k, X_j\right) &= \frac{m-1}{n}E^{\mathfrak{B}_{m-1}}\delta^*\left(e^*_k, X_1\right)\\ &\quad + \frac{1}{n}\sum_{j=m}^n\delta^*\left(e^*_k, X_j\right) \;\; a.s.\end{aligned} \tag{70}$$

Since, for every fixed positive integer m, we have $\lim_{n\longrightarrow\infty}((m-1)/n)E^{\mathfrak{B}_{m-1}}\delta^*(e^*_k, X_1) = 0$ *a.s.*

Then by (70)

$$\begin{aligned}\lim_{n\longrightarrow\infty}\frac{1}{n}\sum_{j=m}^n\delta^*\left(e^*_k, X_j\right) &= \lim_{n\longrightarrow\infty}\frac{1}{n}\sum_{j=1}^n\delta^*\left(e^*_k, X_j\right)\\ = \lim_{n\longrightarrow\infty}\delta^*\left(e^*_k, E^{\mathfrak{B}_n}X_1\right) &= E^{\mathfrak{B}_\infty}\delta^*\left(e^*_k, X_1\right) \;\; a.s.\end{aligned} \tag{71}$$

Since, for all fixed integer m, the multifunction $(1/n)\sum_{j=m}^n X_j(.)$ is $\mathfrak{F}_m$-measurable and by (71) $E^{\mathfrak{B}_\infty}X_1$

is $\mathfrak{F}_m$-measurable and so is $\mathfrak{F}_\infty$-measurable. Then by the independence of $(\sigma(X_n))_{n\in\mathbb{N}^*}$ and the Kolmogorov's Zero-One law (see [18]), we conclude that for all $k \in \mathbb{N}^*$

$$\begin{aligned}\delta^*\left(e_k^*, E^{\mathfrak{B}_\infty} X_1(.)\right) &= E^{\mathfrak{B}_\infty}\delta^*\left(e_k^*, X_1(.)\right)\\ &= \int_\Omega \delta^*\left(e_k^*, X_1(\omega)\right) dP(\omega)\\ &= \delta^*\left(e_k^*, \int_\Omega X_1(\omega)\, dP(\omega)\right) \ a.s.\end{aligned} \tag{72}$$

Since $E^{\mathfrak{B}_\infty}X_1$ and $\int_\Omega X_1(.)dP$ are $cwk(E)$-valued multifunctions and (72) is true for all $k \geq 1$, then $E^{\mathfrak{B}_\infty}X_1(.) = \int_\Omega X_1(\omega)dP(\omega)$ $a.s.$

Therefore,

$$M - \lim_{n\longrightarrow\infty} \frac{S_n}{n} = E^{\mathfrak{B}_\infty} X_1(.) = \int_\Omega X_1(\omega)\, dP(\omega) \ a.s. \tag{73}$$

□

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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