

Research Article

The K-Size Edge Metric Dimension of Graphs

Tanveer Iqbal, Muhammad Naeem Azhar, and Syed Ahtsham Ul Haq Bokhary

Centre for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University, Multan, Pakistan

Correspondence should be addressed to Syed Ahtsham Ul Haq Bokhary; sihtsham@gmail.com

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In this paper, a new concept k-size edge resolving set for a connected graph G in the context of resolvability of graphs is defined. Some properties and realizable results on k-size edge resolvability of graphs are studied. The existence of this new parameter in different graphs is investigated, and the k-size edge metric dimension of path, cycle, and complete bipartite graph is computed. It is shown that these families have unbounded k-size edge metric dimension. Furthermore, the k-size edge metric dimension of the graphs $P_m \square P_n$, $P_m \square C_n$ for $m, n \geq 3$ and the generalized Petersen graph is determined. It is shown that these families of graphs have constant k-size edge metric dimension.

1. Introduction

Kelenc et al. [1] recently defined the concept of edge resolvability in graphs and initiated the study of its mathematical properties. The edge metric dimension of graph $G$ is the minimum cardinality of edge resolving set, say $X$, and is denoted as $\beta_e(G)$. An edge metric generator for $G$ of cardinality $\beta_e(G)$ is an edge metric basis for $G$ [1]. This concept of an edge metric generator may have a weakness with respect to possible uniqueness of the edge identifying a pair of different vertices of the graph. Consider, for example, in a network, a vertex $x$ is identified by a unique edge $e$ in a metric basis $X$, but if at some point the communication between the vertex $x$ and edge $e$ is blocked, then the vertex $x$ cannot be accessed by the edge metric basis $X$. To avoid this situation, one can think of defining a metric edge basis in which every vertex can be identified by at least two edges. Inspired by the motivation of idea of $k$-size resolving sets in graphs by Naeem et al. [2], we present a new concept in the context of edge resolvability, called the $k$-size edge resolving set in graphs.

For an undirected, simple, and connected graph $G$, the vertex set is $V(G)$ and edge set is $E(G)$. The distance parameter in graphs has been used to distinguish (resolve or determine) the vertices or edges of $G$. The distance between the vertex $α$ and the edge $β = a_1a_2$ in a graph $G$ is given by $d_G(β, α) = \min\{d_G(a_1, α), d_G(a_2, α)\}$. Any two edges $β_1$ and $β_2$ are resolved by a vertex $α$ of a graph $G$, whenever $d_G(α, β_1) \neq d_G(α, β_2)$. A set of vertices $X$ is an edge metric generator for a graph $G$, whenever every two edges of $G$ are resolved by some vertex of $X$. The edge metric dimension of graph $G$ is the minimum cardinality of set $X$ and is denoted as $β_e(G)$. An edge metric generator for $G$ of cardinality $β_e(G)$ is an edge metric basis for $G$ [1].

Definition 1. A set of vertices $W$ is said to be a $k$-size edge resolving set of a graph $G$ of order $n \geq 2$ if $W$ is an edge resolving set and the size of subgraph $\langle W \rangle$ induced by $W$ is equal to $k$. The $k$-size edge metric dimension of $G$, denoted by $β_{kse}(G)$, is the minimum cardinality of a $k$-size edge resolving set of $G$. Moreover, the $k$-size edge resolving set of cardinality $β_{kse}(G)$ is represented as $kse$-set, where $k \geq 1$ and belongs to a set of natural number.

Now, we discuss the existence of this new parameter in some simple, nontrivial connected graphs.

Let $H$ be a connected graph having the vertex set $V(H) = \{t_1, t_2, \ldots, t_p, s_1, s_2, \ldots, s_q\}$ and edge set $E(H) = \{t_it_j, t_is_j, 1 \leq i \leq p, 1 \leq j \leq q\}$, as shown in
Figure 1: The graphs $G$ and $H$.

Figure 1. A set $W = \{l_1, l_2, \ldots, l_p, s_2, s_3, \ldots, s_q\}$ is a 1-seq-set for $H$. It can be seen that, for $3 \leq p \leq q$, the $k$-size edge resolving set for graph $H$ exist only for $k = 1$.

We observe that the set $S = \{q_1, q_2, q_3\}$ is a minimum 2-seq-set for $G$ of Figure 1. Moreover, for the graph $G$, 1-seq-set does not exist.

We have the following remark from these two examples.

**Remark 1**

(i) The existence of $\beta_{kse}(G)$ does not imply the existence of $\beta_{(k+1)se}(G)$ for $t \geq 1$ and vice versa in any nontrivial connected graph $G$.

(ii) $2 \leq \beta_{kse}(G) \leq n$, for any simple graph $G$, where $1 \leq k \leq \left(\frac{n}{2}\right)$.

In this paper, we compute the $k$-size edge metric dimension in several well known families of graphs, Cartesian product graphs ($P_m \square P_n$, $P_m \square C_n$), and generalized Petersen graphs $GP(q, 1)$. Moreover, we present some realizable result on $k$-size edge metric dimension in graphs for $k = 1, 2$.

2. Applications

Resolvability in graphs has diverse applications related to the navigation of robots in networks [3], pattern identification, and image processing. It has also many applications in pharmaceutical chemistry and drugs [4–6]. Few interesting connections between metric generators in graphs and the mastermind game or coin weighing problem have been presented in [7]. The other important results about the metric and edge metric dimension can be found in [8–12].

3. Existence of K-Size Edge Resolving Sets in Well-Known Classes of Graphs

Now, we firstly initiate the study of existence of this new parameter in some basic families of graphs and compute their $k$-size edge metric dimension.

**Lemma 1.** For a path graph $G = P_n (1 \leq k \leq n-1)$, $\beta_{kse}(G) = k + 1$ if $n \geq 2$.

**Proof.** Consider a path graph $P_n$ with vertex set $V(P_n) = \{l_1, l_2, \ldots, l_n\}$ and edge set $E(P_n) = \{l_i, l_{i+1} : 1 \leq i \leq n-1\}$. Let $W = \{l_1, l_2, l_3, \ldots, l_k, 1\}$ be a subset of vertex set of $P_n$. The code of each edge $l_i l_{i+1} (1 \leq s \leq k)$ with respect to $W$ is distinct because each edge has the 0 entry at its $s^{th}$ and $(s + 1)^{th}$ place. The code of each edge $l_i l_{i+1}$ is $c_W (l_i l_{i+1}) = (l_i - 1, l_i + 1, l_i - 2, l_i - 3, \ldots, l_i - (k + 1))$, for $k \geq 2$.

Thus, $W$ is an edge resolving set for $P_n$. Since the size of $P_n$ is $n - 1$, it implies that the subgraph $\langle W \rangle$ induced by $W$ has $k$ edges. Therefore, $W$ is a $k$-size edge resolving set for $P_n$. Hence, $\beta_{kse}(P_n) = k + 1$ for $1 \leq k \leq n - 1$.

**Lemma 2.** For a simple and connected graph $G$ of order $n \geq 6$, $\beta_{kse}(G) = k + 1$ if $G = C_n (1 \leq k \leq n - 2)$.

**Proof.** Let $C_n: t_1, t_2, t_3, \ldots, t_{n-1}$ be a cycle graph of order $n \geq 6$, and let $W = \{t_1, t_2, t_3, \ldots, t_{n-1}\}$. We define $l = [n/2]$.

The code of each edge $t_l t_{l+1}$ for $1 \leq p \leq k (1 \leq k \leq n-1)$ and for $1 \leq p \leq l (l + 1 \leq k \leq n - 2)$ with respect to $W$ has the 0 entry at its $p^{th}$ and $(p + 1)^{th}$ place. The code of each edge $t_l t_{l+1}$, for $k + 1 \leq p \leq l (1 \leq k \leq n - 1)$, the $q^{th}$ and $(q + 1)^{th}$ entries of edges $t_l t_{l+1}$ are equal to $l$ when it is even, $1 \leq q < k, p = l + q, s \leq q, s \leq l, s \leq q < l, l + 1 \leq k \leq n - 2)$ and equal to $l - 2$ and $l - 1$, respectively, when it is odd, $1 < q \leq k, p = l + q, s \leq q, s < l, s \leq q < l, l + 1 \leq k \leq n - 2)$. The code of remaining edges $t_l t_{l+1}$ is $(n - p, n - p + 1, n - p + 2, \ldots, n + k - p)$, for $l + 1 \leq k \leq \frac{n}{2}$, $s \leq q < l - 2$, when it is even, and, for $l + 1 \leq k \leq p$, $s \leq q \leq l - 1$, when it is odd. We note that the codes of all the edges are distinct. Therefore, $W$ is a $k$-size edge resolving set for $C_n$. Hence, $\beta_{kse}(C_n) = k + 1$ for $1 \leq k \leq n - 2$.

The $k$-size edge resolving sets of a complete bipartite graph $K_{s_1, s_2}$ exist only for the values of $k$ given in the following result.

**Lemma 3.** For the complete bipartite graph $K_{s_1, s_2} (s_1 = s_2)$ and $s_1, s_2 \geq 4$,

$$\beta_{kse}(K_{s_1, s_2}) = \begin{cases} 2s_1 - 2, & k = (s_1 - 1) (s_2 - 1) \\ s_1, & s_1 \leq k \leq s_2 - s_1 \\
\end{cases}$$

while, for $s_1 \geq s_2 \geq 1$, we have

$$\beta_{kse}(K_{s_1, s_2}) = \begin{cases} s_1 + k_1 - 1, & k = k_1 (s_1 - 1) - 1 \leq k_1 \leq s_1 - 1 \\
\end{cases}$$

**Observation.** It cannot be necessary that a $k$-size edge resolving set has at least $k$ vertices in it.

To justify our above observation, we consider a graph $G = K_4$. A set of vertices $W = \{f_1, f_2, f_3, f_4\}$ is the vertex set of $G = K_4$. One can observe that the set $V(K_4)$ is a 6-size edge resolving set for $K_4$. Therefore, $\beta_{kse}(K_4) = 4$. 
4. K-Size Edge Metric Dimension of Cartesian Product of Graphs

Let $G = P_m \square P_n$ be the Cartesian product of two path graphs $P_m$ and $P_n$ for $m,n \geq 4$. Let $E_1(G) = \{r_{gh}r_{g(h+1)}: 1 \leq g \leq m, 1 \leq h \leq n\}$ be the set of horizontal edges and $E_2(G) = \{s_{gh}r_{s(h+1)}: 1 \leq g \leq m-1, 1 \leq h \leq n\}$ be the set of vertical edges of $P_m \square P_n$. The graph of $P_m \square P_n$ is shown in Figure 2. To find distances, we embed $G$ into $xy$ plane in such a way that each vertex is in an ordered pair form. Let the vertices $v_1, v_2, \ldots, v_{mn}$ be the corner vertices of $G$. In the next two lemmas, we shall discuss size 1, size 2, and size 3 edge metric dimension of $G = P_m \square P_n$ and $G = P_m \square C_n$, for $m,n \geq 5$.

Lemma 4. Let $G$ be the cartesian product graph $P_m \square P_n$; then, we have

$$
\beta_{k_{ee}}(P_m \square P_n) = \begin{cases} 
3, & k = 1, \\
4, & k = 2, \\
5, & k = 3.
\end{cases}
$$

Proof. Here, we will prove this result for $k = 3$ (only). For this we consider $W = \{a_1, a_2, a_3, a_4, a_5\}$, where $a_1 = (0,0)$, $a_2 = (1,0), a_3 = (m-1,0), a_4 = (0,1)$, and $a_5 = (2,0)$, and prove that $W$ is a 3er-set for $P_m \square P_n$. Note that $d((r_1, s_1), (r_2, s_2)) = |r_1 - r_2| + |s_1 - s_2|$ is the distance between any two vertices of $P_m \square P_n$. Let $a = (r_1, s_1)(r_2, s_2)$ be an edge. The distances of the edge $a$ from the vertices of $W$ are calculated as follows.

\[d(a, a) = r_1 + s_1, d(a, a) = m - 1 - r_2 + s_1, d(a_2, a) = r_1 + s_2 - 1, d(a_3, a) = r_1 + s_1 - 1 + 2m, d(a_4, a) = r_1 + s_1 + 1, d(a_5, a) = r_1 + s_2 - 1,\]

whenever $a \in \{r_{gh}r_{g(h+1)}, s_{gh}r_{s(h+1)}: 1 \leq g \leq m, 2 \leq h \leq n\}$, $d(a, a) = r_1 + s_1 - 1, d(a_2, a) = r_1 + s_1 + 2, d(a_3, a) = r_1 + s_2 + 2e, d(a_4, a) = r_1 + s_1 + 1, d(a_5, a) = r_1 + s_2 - 1,$

whenever $a \in \{r_{gh}r_{g(h+1)}, s_{gh}r_{s(h+1)}: 1 \leq g \leq m, 1 \leq h \leq n\}$, $d(a, a) = r_1 + s_1 - 1, d(a_2, a) = r_1 + s_1 + 2e, d(a_3, a) = r_1 + s_2 + 2e, d(a_4, a) = r_1 + s_1 + 1, d(a_5, a) = r_1 + s_2 - 1.$

Hence, we have the following equalities:

\[r_1 + s_1 = f_1 + u_1, \]
\[m - 1 - r_2 + s_1 = m - 1 - t_2 + u_1. \]

The above equalities imply that $s_1 - u_1 = r_2 - t_2$. Thus, it follows that $r_1 + r_2 = t_1 + t_2$. In both the cases $r_1 = r_2$ or $r_1 = r_2 - 1$, and we get $r_1 = t_1$ and $r_2 = t_2$. The equality $r_1 = t_1$ together with $r_1 + s_1 = t_1 + u_1$ implies that $s_1 = u_1$. Both the vertices $s_2$ and $u_2$ can either equal to $s_1$ or equal $s_1 + 1$. One of the edges $a_1$ or $a_2$ does not represent an edge if they have distinct values. So, finally we have $a_1 = a_2$, which is a contradiction. Therefore, $W$ is an edge resolving set for $P_m \square P_n$. Moreover, the subgraph induced by $W$ has 3 edges. Hence, we conclude that $\beta_{k_{ee}}(P_m \square P_n) = 3$. Similarly, we can prove the result for the values of $k = 1, 2$. Hence, we conclude the result.

We present the following result on the $k$-size edge metric dimension of the Cartesian product graph $P_m \square C_n$ without proof.

Lemma 5. Let $G$ be the cartesian product graph $P_m \square C_n$; then, we have

$$
\beta_{k_{ee}}(P_m \square C_n) = \begin{cases} 
3, & k = 1,2, \\
4, & k = 3.
\end{cases}
$$

5. K-Size Edge Metric Dimension of Generalized Petersen Graphs

The generalized Petersen graph $GP(q,s)$ is a 3-regular graph containing $2q$ vertices and $3q$ edges. The vertex set of $GP(q,s)$ is $V(GP(q,s)) = \{c_1, d_1: 1 \leq t \leq q\}$, and the edge set is $E(GP(q,s)) = \{c_i c_{i+1}, c_i d_i, d_i d_{i+s}: 1 \leq t \leq q, 1 \leq s < [q/2]\}$. The edges $c_i d_i$ for $1 \leq t \leq q$ are said to be spokes of $GP(q,s)$. The outer cycle of $GP(q,s)$ is said to be the principal cycle $GP(q,s)$.

We will compute the $k$-size edge metric dimension of $GP(q,s)$ for $k = 1, 2, 3$ and $s = 1$ in the following two sections. Firstly, we will find upper bound for the $1$-size edge metric dimension of $GP(q,1)$.

Lemma 6. For all $q \geq 7$, we have $\beta_{k_{ee}}(GP(q,1)) \leq 3$.

Proof. Let $W = \{c_1, c_3, d_1\}$ be a set of vertices of $GP(q,1)$.

Case 1. When $q$ is odd.

Here, we define $t_1 = (q + 1)/2$. There is only one edge in the subgraph $\langle W \rangle$ induced by $W$ and the codes of all the edges of $GP(q,1)$ are given in Tables 1–3.

Case 2. When $q$ is even.

We define $t_2 = (q/2)$. The induced subgraph $\langle W \rangle$ by $W$ has only one edge, and the codes of all the edges of $GP(n,1)$ are given in Tables 4–6. We observe that codes of all the edges in both cases are distinct. So, $W$ is 1-size edge resolving set for $GP(q,1)$. Hence, we have $\beta_{k_{ee}}(GP(q,1)) \leq 3$. Now, we will compute upper bound for the size 2 edge metric dimension of generalized Petersen graphs $GP(q,1)$.
Table 1: Codes of outer edges of \(GP(q,1)\) when \(q\) is odd.

<table>
<thead>
<tr>
<th>(d(___))</th>
<th>(c_1)</th>
<th>(c_3)</th>
<th>(d_1)</th>
</tr>
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<tbody>
<tr>
<td>(c_1c_2)</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(c_1c_3); (2 \leq t \leq 3)</td>
<td>(t-1)</td>
<td>0</td>
<td>(t)</td>
</tr>
<tr>
<td>(c_1t_1; \quad 4 \leq t \leq t_1)</td>
<td>(t-1)</td>
<td>(t-3)</td>
<td>(t)</td>
</tr>
<tr>
<td>(c_1c_{t_1}; \quad t=1+t_1)</td>
<td>(q-t)</td>
<td>(q-t)</td>
<td>(1+q-t)</td>
</tr>
<tr>
<td>(c_1c_{t_1}; \quad 2+t_1 \leq t \leq q)</td>
<td>(q-t)</td>
<td>(2+q-t)</td>
<td>(1+q-t)</td>
</tr>
</tbody>
</table>

Table 2: Codes of spokes of \(GP(q,1)\) when \(q\) is odd.

<table>
<thead>
<tr>
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<th>(c_3)</th>
<th>(d_1)</th>
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<tr>
<td>(c_1d_1)</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>(c_1d_2)</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(c_1d_3)</td>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>(c_1d_t; \quad 4 \leq t \leq t_1)</td>
<td>(t-1)</td>
<td>(t-3)</td>
<td>(t)</td>
</tr>
<tr>
<td>(c_1d_{t_1}; \quad t=1+t_1)</td>
<td>(1+q-t)</td>
<td>(q-t)</td>
<td>(1+q-t)</td>
</tr>
<tr>
<td>(c_1d_{t_1}; \quad t=2+t_1)</td>
<td>(1+q-t)</td>
<td>(2+q-t)</td>
<td>(1+q-t)</td>
</tr>
<tr>
<td>(c_1d_{t_1}; \quad 3+t_1 \leq t \leq q)</td>
<td>(1+q-t)</td>
<td>(3+q-t)</td>
<td>(1+q-t)</td>
</tr>
</tbody>
</table>

Table 3: Codes of inner edges of \(GP(q,1)\) when \(q\) is odd.

<table>
<thead>
<tr>
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<th>(c_3)</th>
<th>(d_1)</th>
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<tr>
<td>(d_1d_2)</td>
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<tr>
<td>(d_1d_3)</td>
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<td>1</td>
</tr>
<tr>
<td>(d_1d_{t_1}; \quad 3 \leq t \leq t_1)</td>
<td>(t)</td>
<td>(t-2)</td>
<td>(t-1)</td>
</tr>
<tr>
<td>(d_1d_{t_1}; \quad t=1+t_1)</td>
<td>(1+q-t)</td>
<td>(q-t)</td>
<td>(q-t)</td>
</tr>
<tr>
<td>(d_1d_{t_1}; \quad k \leq t \leq q)</td>
<td>(q-t)</td>
<td>(1+q-t)</td>
<td>(1+q-t)</td>
</tr>
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</table>

Table 4: Codes of outer edges of \(GP(q,1)\) when \(q\) is even.

<table>
<thead>
<tr>
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<tr>
<td>(c_1c_2)</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(c_1c_3); (2 \leq t \leq 3)</td>
<td>(t-1)</td>
<td>0</td>
<td>(i)</td>
</tr>
<tr>
<td>(c_1t_1; \quad 4 \leq t \leq t_2)</td>
<td>(t-1)</td>
<td>(t-3)</td>
<td>(t)</td>
</tr>
<tr>
<td>(c_1c_{t_1}; \quad t=1+t_2)</td>
<td>(q-t)</td>
<td>(q-t)</td>
<td>(1+q-t)</td>
</tr>
<tr>
<td>(c_1c_{t_1}; \quad t=2+t_2)</td>
<td>(q-t)</td>
<td>(1+q-t)</td>
<td>(1+q-t)</td>
</tr>
<tr>
<td>(c_1c_{t_1}; \quad 3+t_2 \leq t \leq q)</td>
<td>(q-t)</td>
<td>(2+q-t)</td>
<td>(1+q-t)</td>
</tr>
</tbody>
</table>

Lemma 7. For all \(q \geq 5\), \(\beta_{2se}(GP(q,1)) \leq 3\).

Proof. Let \(W = \{c_1, c_2, d_1\}\) be an edge resolving set for \(GP(q,1)\). Define \(k = [q/2]\). The codes of all the edges of \(GP(q,1)\) with respect to \(W\) are given Table 7–9.

Table 5: Codes of spokes of \(GP(q,1)\) when \(q\) is even.

<table>
<thead>
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<tr>
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<td>0</td>
</tr>
<tr>
<td>(c_1d_2)</td>
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<td>1</td>
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<tr>
<td>(c_1d_3)</td>
<td>2</td>
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<td>2</td>
</tr>
<tr>
<td>(c_1d_t; \quad 4 \leq t \leq t_1)</td>
<td>(t-1)</td>
<td>(t-3)</td>
<td>(t)</td>
</tr>
<tr>
<td>(c_1d_{t_1}; \quad t=2+t_2)</td>
<td>(t_2-1)</td>
<td>(t_2-1)</td>
<td>(t_2-1)</td>
</tr>
<tr>
<td>(c_1d_{t_1}; \quad 3+t_2 \leq t \leq q)</td>
<td>(1+q-t)</td>
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Table 6: Codes of inner edges of \(GP(q,1)\) when \(q\) is even.

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<tr>
<td>(d_1d_3)</td>
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<td>1</td>
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<tr>
<td>(d_1d_{t_1}; \quad 3 \leq t \leq t_2)</td>
<td>(t)</td>
<td>(t-2)</td>
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<td>(d_1d_{t_1}; \quad t=1+t_2)</td>
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<td>(q-t)</td>
</tr>
<tr>
<td>(d_1d_{t_1}; \quad 3+t_2 \leq t \leq q)</td>
<td>(q+1-t)</td>
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Table 7: Codes of outer edges of \(GP(q,1)\).

<table>
<thead>
<tr>
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<th>(c_3)</th>
<th>(d_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c_1c_2)</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(c_1c_3); (2 \leq t \leq k)</td>
<td>(t-1)</td>
<td>(t-2)</td>
<td>(t)</td>
</tr>
<tr>
<td>(c_1c_{t_1}; \quad k \leq t \leq q)</td>
<td>(q-t)</td>
<td>(1+q-t)</td>
<td>(1+q-t)</td>
</tr>
</tbody>
</table>

Table 8: Codes of spokes of \(GP(q,1)\).

<table>
<thead>
<tr>
<th>(d(___))</th>
<th>(c_1)</th>
<th>(c_3)</th>
<th>(d_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d_1d_2)</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>(d_1d_3)</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(d_1d_{t_1}; \quad 3 \leq t \leq k)</td>
<td>(t)</td>
<td>(t-2)</td>
<td>(t-1)</td>
</tr>
<tr>
<td>(d_1d_{t_1}; \quad t=1+t_2)</td>
<td>(1+q-t)</td>
<td>(q-t)</td>
<td>(q-t)</td>
</tr>
<tr>
<td>(d_1d_{t_1}; \quad k \geq t \leq q)</td>
<td>(q-t+1)</td>
<td>(q-t+2)</td>
<td>(q-t+1)</td>
</tr>
</tbody>
</table>

Table 9: Codes of inner edges of \(GP(q,1)\).

For \(t = 1 + k\), the code of outer edge \(c_tc_{t+k}\) will be \(c_W(c_tc_{t+k}) = (q-t, q-t, q-t+1)\) when \(q\) is even and \(c_W(c_tc_{t+k}) = (q-t, q-t+1, q-t+1)\) when \(q\) is odd.

It seems that codes of all the edges are distinct. So, \(W\) is size 2 edge resolving set for \(GP(q,1)\). Hence, we have \(\beta_{2se}(GP(q,1)) \leq 3\).

In the next lemma, we will give the lower bound for the \(k\)-size edge metric dimension of \(GP(q,1)\) for \(k = 1, 2\).

Lemma 8. For all \(q \geq 7\), we have \(\beta_{kse}(GP(q,1)) \geq 3\).

Proof. First, we will show that there is no edge resolving set of \(GP(q,1)\) consisting of two vertices. Contrarily, we suppose that \(X = \{A, B\}\) be a set having two vertices of \(V(GP(q,1))\). Then, we have the following three possibilities.
Case 1. When both the vertices $A$ and $B$ are from the principal cycle.

Let us fix a vertex, say $A = c_1$, then $B$ is any other vertex $c_i$ ($2 \leq i \leq q$). For $2 \leq i \leq k - 1$, we have $c_{W}(c_{i-1}c_i) = c_{W}(d_i)$ ($1, t$). For $t = k$, we have $c_{W}(c_1c_k) = c_{W}(c_1d_k) = (0, k - 1)$. For $t = k + 1$, we have $c_{W}(c_1c_k) = c_{W}(c_1d_k)$ when $q$ is even, while $c_{W}(c_1c_k) = c_{W}(d_i)$ when $q$ is odd. For $t = k + 2$, we have $c_{W}(c_1c_k) = c_{W}(d_i)$ when $q$ is odd, while $c_{W}(c_1c_k) = c_{W}(d_i)$ when $q$ is even. For $k + 3 \leq t \leq q$, we have $c_{W}(c_1c_k) = c_{W}(d_i) = (1, q - t + 2)$.

Case 2. When both $A$ and $B$ are the inner vertices.

Let us fix a vertex, say $A = d_1$, then $B$ is any other vertex $v_j$ ($2 \leq j \leq q$).

For $2 \leq j \leq k$, we have $c_{W}(d_1d_j) = c_{W}(c_1d_j) = (0, t - 1)$. For $t = k + 1$, we have $c_{W}(d_1d_j) = c_{W}(d_1d_{j-1}) = (2, k - 3)$ when $q$ is even, while $c_{W}(d_1d_j) = c_{W}(d_1d_{j-1}) = (k - 1, t - 2)$ when $q$ is odd. For $k + 2 \leq j \leq q$, we have $c_{W}(d_1d_j) = (0, q - t + 1)$.

Case 3. When $A$ is any vertex from the principal cycle and $B$ is any inner vertex.

Let us fix a vertex, say $A = c_1$; then, $B$ is any inner vertex $d_i$ ($1 \leq i \leq q$).

For $t = 1$, we have $c_{W}(c_1c_2) = (0, 1)$. For $2 \leq t \leq k$, we have $c_{W}(c_1c_2) = c_{W}(d_1)$ ($1, t - 1$). For $t = k + 1$, we have $c_{W}(c_1c_2) = c_{W}(d_1) = (0, q - t + 1)$ when $q$ is odd; however, $c_{W}(c_1c_2) = c_{W}(d_1) = (0, t - 1)$ when $q$ is even. For $k + 2 \leq q$, we have $c_{W}(c_1c_2) = c_{W}(d_1) = (0, q - t + 1)$.

From the above three cases, we conclude that $\beta_{se}(GP(q, 1)) \geq 3$.

(6)

So, there does not exist a size 1 and size 2 edge resolving set of cardinality 2 in $GP(q, 1)$. Therefore, it yields that $\beta_{se}(GP(q, 1)) \geq 3$, for the value of $k = 1, 2$.

Lemmas 6–8, we conclude the following main result.

Theorem 1. For all $q \geq 7$, we have $\beta_{se}(GP(q, 1)) = 3$ when $k = 1, 2$.

6. Bounds and Some Realizable Results on $\beta_{se}(G)$

From the earlier discussion, one fundamental question arises: Is the $(k + 1)$ size edge metric dimension strictly greater than the $k$-size metric dimension? To answer this question, we consider following two examples.

Consider the graphs $G_1$ and $G_2$ which are depicted in Figure 3. It can be observed that the set $W = \{e_1, e_5, e_6, e_7\}$ is an edge resolving set for $G_1$ and the cardinality of set $W$ is minimum. Moreover, $W_1 = \{e_1, e_5, e_6, e_7\}$ is a 1ser-set, $W_2 = \{e_1, e_5, e_6, e_7\}$ is a 2ser-set, and $W_3 = \{e_1, e_5, e_6, e_7\}$ is a 3ser-set for $G_1$. Thus, $\beta_{se}(G_1) = 3$ and $\beta_{se}(G_1) = 4$.

While, for the graph $G_2 \equiv GP(5, 1)$, the set $S = \{p_1, p_2, p_3\}$ is an edge resolving set of the minimum cardinality. Here, the outer vertices are $p_1, \ldots, p_5$ and the inner vertices are $q_1, \ldots, q_5$. It can be easily seen that the sets $S_1, S_2$, and $S_3$ are 1ser-set, 2ser-set, and 3ser-set, respectively, where $S_1 = \{p_1, p_2, p_3\}$, $S_2 = \{q_1, q_2, p_3\}$, and $S_3 = \{p_1, p_2, p_3, p_4, p_5\}$.

Thus, $\beta_{se}(GP(5, 1)) = 3$ and $\beta_{se}(GP(5, 1)) = 4$.

From these two examples, it can be observed that if $\beta_{se}(G)$ exists for $1 \leq k \leq t$ in a nontrivial connected graph $G$ of order $n$, then

$$\beta_{se}(G) \leq \beta_{se}(G) \leq \beta_{se}(G) \leq \beta_{se}(G) \leq \beta_{se}(G).$$

(7)

However, the following example shows that the above inequality is not true, in general.

Example 1. Let $G$ be a graph which is constructed from two graphs $C_5$ and $K_4$. The vertex set of $G$ is $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and the edge set is $\{1, 2, 3, 4, 5, 6, 7, 8\}$. The set $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ is a 6ser-set of $G$. However, there is no such set $B$ which resolves all the edges of graph $G$ with cardinality 4 and $B$ has size 5. So, we take the set $W = \{1, 2, 3, 4, 5, 6, 7, 8\}$ is a 5ser-set of the minimum cardinality for $G$. Hence, $\beta_{se}(G) > \beta_{se}(G)$.

Next, we characterize some realizable results for 1ser-set and 2ser-set in graphs.

Theorem 2. For a nontrivial, simple, and connected graph $G$ of order $t$, we have $\beta_{se}(G) = t$ if and only if $G = P_r$.

Proof. Lemma 1 implies that if $G = P_2$, then $\beta_{se}(G) = 2 = t$. Conversely, assume that $G$ be a connected graph of order $t \geq 2$ and $\beta_{se}(G) = t$. Since the induced subgraph $W$ has only one edge, therefore it is obvious that $|W| = |V(G)| = 2$. Thus, $G$ is a path graph of order 2.

The following result on the complete graph $K_s$ was presented in [1].

Lemma 9 (see [1]). For any integer $s \geq 2$, $\beta_{se}(K_s) = s - 1$.

Theorem 3. Let $G$ be a complete graph of order $s \geq 3$, then $\beta_{se}(G)$ exists if $G = K_s$. Moreover, $\beta_{se}(G) = 2$.

Proof. One can observe that if $G$ is a complete graph $K_s$, then the result holds. Conversely, let $G$ be a complete graph of order $s \geq 4$. Now, by Lemma 7, the induced subgraph $G$
Figure 4: The graph in which $\beta_{s\text{se}}(G) > \beta_{t\text{se}}(G)$.

has more than one edge. Therefore, $\beta_{t\text{se}}(G)$ does not exist. Hence, the proof is complete.

Theorem 4. Let $G$ be a nontrivial connected graph of order $s \geq 3$, then $\beta_{t\text{se}}(G) = s - 1$ if and only if $G = P_3$ or $G = K_3 = C_3$.

Proof. Let $G = P_3$ or $G = K_3 = C_3$. From Lemmas 1 and 2, and Theorem 5, we have $\beta_{l\text{se}}(G) = 2 = s - 1$. Contrarily, assume that $G$ is a connected graph of order $s \geq 3$ and $\beta_{t\text{se}}(G) = s - 1$. For $s = 3$, it is simple to prove that $G = P_3$ or $G = K_3 = C_3$. Now, we will prove the result for $s \geq 4$. For this, we suppose $W$ be a $1\text{ser}$-set of $G$ of cardinality $s - 1$. It is easy to see that $|V(G)| - |W| = 1$; therefore, the induced subgraph has surely more than one edge in it ($G$ is a connected graph). It yields that $|W| = |V(G)| = 3$. Thus, $G = P_3$ or $G = K_3 = C_3$.

Remark 2. The two size edge resolving sets exist in complete bipartite graph $G = K_{s,t}$ if and only if $G \in \{K_{2,1}, K_{2,2}, K_{2,3}, K_{3,1}\}$. Moreover, $\beta_{2\text{se}}(K_{2,1}) = 3 = s + t$, $\beta_{2\text{se}}(K_{2,2}) = 2 = s + t - 1$, and $\beta_{2\text{se}}(K_{2,3}) = 3 = s + t - 2$.

Theorem 5. For a simple and connected graph $G$ of order $s \geq 3$, $\beta_{t\text{se}}(G) = s$ if and only if $G = P_3 \cong K_{1,2}$.

Proof. By Lemma 1 and Remark 2, if $G = P_3 \cong K_{1,2}$, then $\beta_{2\text{se}}(G) = 3 = s$. Conversely, suppose that $\beta_{2\text{se}}(G) = s$ for a connected graph $G$ of order $s \geq 3$. Let $W$ be a $2\text{ser}$-set of order $s$. Since $G$ is a connected graph and induced subgraph has two edges which implies that $|W| = |V(G)| = 3$, thus $G = P_3 \cong K_{1,2}$.

Now, we study a sufficient condition for a pair $p, q$ of positive integers to be realizable as the order and the $k$-size edge metric dimension of a connected graph, respectively.

Theorem 6. For a pair $(q, p)$ of positive integers with $q \geq p \geq 2$, there exists a connected graph $G$ of order $q$ and $\beta_{k\text{se}}(G) = p$, where $k + 1 \leq p \leq q$.

Proof. For $k \geq 1$, we consider the following two cases according to the choice of $p$.

(i) For $p = k + 1$, let $G = P_q$ be a path graph of order $q \geq 2$. Lemma 1 implies that $\beta_{k\text{se}}(P_q) = p$, where $1 \leq k \leq q - 1$. For $p = q$, let $G = C_q$ be a cycle of order $q \geq 3$, then it seems that $\beta_{k\text{se}}(C_q) = p$, where $k = p$.

(ii) For $k + 2 \leq p \leq q - 1$, let $H$ be a connected graph of order $q \geq 6$ obtained from paths $P_{q-p}$, $e_2, e_3, \ldots, e_q$ ($q - p \geq 2$, where $p \geq 4$). $P_{q-1}$: $f_1$, $f_2$, $f_3$, $f_{k+1}$ ($1 \leq k \leq p - 3$) and $p - k - 1$ vertices $g_1, g_2, \ldots, g_{p-k-1}$ with $g_1 \sim e_1$ and $c_i \sim e_i$ for $1 \leq i \leq k + 1$ and $1 \leq j \leq p - k - 1$, as shown in Figure 5. Firstly, we prove that $\beta_{k\text{se}}(H) \leq p$. For this, let $S = \{f_1, f_2, \ldots, f_{k+1}, g_1, g_2, \ldots, g_{p-k-1}\}$ be a $k\text{ser}$-set of $H$. Since, the subgraph induced by $\langle S \rangle$ has $k$ edges and $\text{c}_S(e_i, e_{i+1}) = (i, i, \ldots, i, i)$ for each $1 \leq i \leq q - p - 1$, it implies that $S$ is a $k\text{ser}$-set for $H$, and hence $\beta_{k\text{se}}(H) \leq l$. Now, to prove $\beta_{k\text{se}}(G) \geq l$, we suppose contrarily that $\beta_{k\text{se}}(G) \leq l - 1$ and $W$ be a $k\text{ser}$-set for $H$ with $|W| \leq p - 1$. If $W$ contains at least $p - k - 2$ vertices from $G$, say $g_i$ ($1 \leq i \leq p - k - 2$) and the size of $\langle W \rangle$ is equal to $k$, it follows that $W = V(P_{k+1}) \cup \{g_1, g_2, \ldots, g_{p-k-2}\}$. However, still we have $\text{c}_W(e_i, e_{i+1}) = e_i (e_1, e_{p-k-1}) = (1, 1, 1, \ldots, 1, 1)$. Thus, $\beta_{k\text{se}}(H) \geq p$ which yields that $\beta_{k\text{se}}(H) = p$.

7. Conclusions

In this work, we have introduced a new variant, namely, the $k$-size edge metric dimension of graphs, and initiated its study by finding the $k$-size edge metric dimension of several well-known classes of graphs. We have characterized the graphs having $k$-size edge metric dimension of several well-known classes of graphs. We have introduced the $k$-size edge metric dimension of the Cartesian product graphs $P_m \square P_n$ and $P_m \odot C_n$ for the values of $k = 1, 2, 3$. In addition, we have proved that the $k$-size edge metric dimension of generalized Petersen graphs $GP(q, k)$ is $3$ for the values of $k = 1, 2$. Some realizable results on $k$-size edge resolvability are also presented in this paper [8–12].

Data Availability

No data were used to support this study.
Conflicts of Interest

The authors declare that they have no conflicts of interest.

References