Hindawi
Journal of Mathematics
Volume 2020, Article ID 1023175, 7 pages
https://doi.org/10.1155/2020/1023175

Research Article

The K-Size Edge Metric Dimension of Graphs

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Received 9 July 2020; Revised 27 November 2020; Accepted 30 November 2020; Published 31 December 2020

Academic Editor: Elena Guardo

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In this paper, a new concept k-size edge resolving set for a connected graph G in the context of resolvability of graphs is defined. Some properties and realizable results on k-size edge resolvability of graphs are studied. The existence of this new parameter in different graphs is investigated, and the k-size edge metric dimension of path, cycle, and complete bipartite graph is computed. It is shown that these families have unbounded k-size edge metric dimension. Furthermore, the k-size edge metric dimension of the graphs $P_m \square P_n$, $P_m \square C_n$ for $m, n \geq 3$ and the generalized Petersen graph is determined. It is shown that these families of graphs have constant k-size edge metric dimension.

1. Introduction

Kelenc et al. [1] recently defined the concept of edge resolvability in graphs and initiated the study of its mathematical properties. The edge metric dimension of graph G is the minimum cardinality of edge resolving set, say X, and is denoted as $\beta_e(G)$. An edge metric generator for G of cardinality $\beta_e(G)$ is an edge metric basis for G [1]. This concept of an edge metric generator may have a weakness with respect to possible uniqueness of the edge identifying a pair of different vertices of the graph. Consider, for example, in a network, a vertex x is identified by a unique edge e in a metric basis X, but if at some point the communication between the vertex x and edge e is blocked, then the vertex x cannot be accessed by the edge metric basis X. To avoid this situation, one can think of defining a metric edge basis in which every vertex can be identified by at least two edges. Inspired by the motivation of idea of k-size resolving sets in graphs by Naeem et al. [2], we present a new concept in the context of edge resolvabililty, called the k-size edge resolving set in graphs.

For an undirected, simple, and connected graph G, the vertex set is $V(G)$ and edge set is $E(G)$. The distance parameter in graphs has been used to distinguish (resolve or determine) the vertices or edges of G. The distance between the vertex α and the edge $\beta = \alpha_1\alpha_2$ in a graph G is given by $d_G(\beta, \alpha) = \min\{d_G(\alpha_1, \alpha), d_G(\alpha_2, \alpha)\}$. Any two edges β_1 and β_2 are resolved by a vertex α of a graph G, whenever $d_G(\alpha, \beta_1) \neq d_G(\alpha, \beta_2)$. A set of vertices X is an edge metric generator for a graph G, whenever every two edges of G are resolved by some vertex of X. The edge metric dimension of graph G is the minimum cardinality of set X and is denoted as $\beta_e(G)$. An edge metric generator for G of cardinality $\beta_e(G)$ is an edge metric basis for G [1].

Definition 1. A set of vertices W is said to be a k-size edge resolving set of a graph G of order $n \geq 2$ if W is an edge resolving set and the size of subgraph $\langle W \rangle$ induced by W is equal to k. The k-size edge metric dimension of G, denoted by $\beta_{kse}(G)$, is the minimum cardinality of a k-size edge resolving set of G. Moreover, the k-size edge resolving set of cardinality $\beta_{kse}(G)$ is represented as $kser$-set, where $k \geq 1$ and belongs to a set of natural number.

Now, we discuss the existence of this new parameter in some simple, nontrivial connected graphs.

Let H be a connected graph having the vertex set $V(H) = \{l_1, l_2, \ldots, l_p, s_1, s_2, \ldots, s_q\} \cup \{t_1, t_2\}$ and edge set $E(H) = \{l_1l_2, l_it_1, p_jt_1, p_jt_2 \colon 1 \leq i \leq p, 1 \leq j \leq q\}$, as shown in

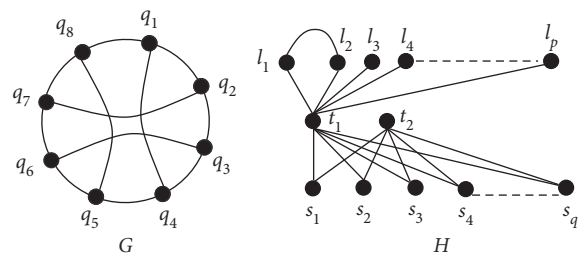

Figure 1: The graphs G and H.

Figure 1. A set $W = \{l_1, l_2, \ldots, l_p, s_2, s_3, \ldots, s_q\}$ is a 1ser-set for H. It can be seen that, for $3 \le p \le q$, the k-size edge resolving set for graph H exist only for $k = 1$.

We observe that the set $S = \{q_1, q_2, q_3\}$ is a minimum $2ser$-set for G of Figure 1. Moreover, for the graph G, $1ser$-set does not exist.

We have the following remark from these two examples.

Remark 1

(i) The existence of $\beta_{kse}(G)$ does not imply the existence of $\beta_{(k+t)se}(G)$ for $t \ge 1$ and vice versa in any nontrivial connected graph G

(ii) $2 \le \beta_{kse}(G) \le n$, for any simple graph G, where $1 \le k \le \binom{n}{2}$

In this paper, we compute the k-size edge metric dimension in several well known families of graphs, Cartesian product graphs $(P_m \square P_n, P_m \square C_n)$, and generalized Petersen graphs $GP(q, 1)$. Moreover, we present some realizable result on k-size edge metric dimension in graphs for $k = 1, 2$.

2. Applications

Resolvability in graphs has diverse applications related to the navigation of robots in networks [3], pattern identification, and image processing. It has also many applications in pharmaceutical chemistry and drugs [4–6]. Few interesting connections between metric generators in graphs and the mastermind game or coin weighing problem have been presented in [7]. The other important results about the metric and edge metric dimension can be found in [8–12].

3. Existence of K-Size Edge Resolving Sets in Well-Known Classes of Graphs

Now, we firstly initiate the study of existence of this new parameter in some basic families of graphs and compute their k-size edge metric dimension.

Lemma 1. *For a path graph $G = P_n (1 \le k \le n-1)$, $\beta_{kse}(G) = k + 1$ if $n \ge 2$.*

Proof. Consider a path graph P_n with vertex set $V(P_n) = \{l_1, l_2, \ldots, l_n\}$ and edge set $E(P_n) = \{l_i l_{i+1}\colon 1 \le i \le n-1\}$. Let $W = \{l_1, l_2, l_3, \ldots, l_{k+1}\}$ be a subset of vertex set of P_n. The code of each edge $l_s l_{s+1} (1 \le s \le k)$ with respect to W is distinct because each edge has the 0 entry at its sth and $(s+1)^{\text{th}}$ place. The code of each edge $l_s l_{s+1}$ is $c_W(l_s l_{s+1}) = (l-1, l-2, l-3, \ldots, l-(k+1))$, for $k+2 \le s \le n-1$, and $c_W(l_s l_{s+1}) = (l-1, l-2, l-3, \ldots, \quad 2, 1, 0)$, for $s = k+1$. Thus, W is an edge resolving set for P_n. Since the size of P_n is $n-1$, it implies that the subgraph $\langle W \rangle$ induced by W has k edges. Therefore, W is k-size edge resolving set for P_n. Hence, $\beta_{kse}(P_n) = k + 1$ for $1 \le k \le n-1$. □

Lemma 2. *For a simple and connected graph G of order $n \ge 6$, $\beta_{kse}(G) = k + 1$ if $G = C_n (1 \le k \le n-2)$.*

Proof. Let C_n: $t_1, t_2, t_3, \ldots, t_n, t_1$ be a cycle graph of order $n \ge 6$, and let $W = \{t_1, t_2, t_3, \ldots, t_{k+1}\}$. We define $l = [n/2]$. The code of each edge $t_p t_{p+1}$ for $1 \le p \le k (1 \le k \le l)$ and for $1 \le p \le l$ $(l+1 \le k \le n-2)$ with respect to W has the 0 entry at its pth and $(p+1)^{\text{th}}$ place. The code of each edge $t_p t_{p+1}$ is $c_W(t_p t_{p+1}) = (p-1, p-2, p-3, \ldots, p-(k+1))$, for $k+1 \le p \le l (1 \le k \le l-1,)$. The qth and $(q+1)^{\text{th}}$ entries of edges $t_p t_{p+1}$ are equal to $l-1$ when n is even $(1 \le q \le k, p = l+q, 1 \le k \le l, 1 \le q \le l, l+1 \le k \le n-2)$ and equal to $l-2$ and $l-1$, respectively, when n is odd $(1 \le q \le k, p = l+q, q, 1 \le k \le l, 1 \le q \le l-1, l+1 \le k \le n-2)$. The code of remaining edges $t_p t p+1$ is $(n-p, n+1-p, n+2-p, \ldots, n+k-p)$, for $l+k+1 \le p \le n, 1 \le k \le l-1$, when n is even and, for $1 \le k \le l-2$, when n is odd. We note that the codes of all the edges are distinct. Therefore, W is a k-size edge resolving set for C_n. Hence, $\beta_{kse}(C_n) = k + 1$ for $1 \le k \le n-2$.

The k-size edge resolving sets of a complete bipartite graph K_{s_1,s_2} exist only for the values of k given in the following result. □

Lemma 3. *For the complete bipartite graph $K_{s_1,s_2} (s_1 = s_2)$ and $s_1, s_2 \ge 4$,*

$$\beta_{kse}\left(K_{s_1,s_2}\right) = \begin{cases} 2s_1 - 2, & k = (s_1 - 1)(s_2 - 1), \\ 2s_1 - 1, & k = s_1 s_2 - s_1 \text{ or } k = s_1(s_2 - 1), \\ 2s_1, & k = s_1 s_2, \end{cases} \tag{1}$$

while, for $s_1 \ge s_2 \ge 1$, we have

$$\beta_{kse}\left(K_{s_1,s_2}\right) = \begin{cases} s_1 + k_1 - 1, & k = k_1(s_1 - 1), 1 \le k_1 \le s_1 - 1, \\ s_1 + k_2, & k = k_2 s_1, 1 \le k_2 \le s_1 - 1. \end{cases} \tag{2}$$

Observation. It cannot be necessary that a k-size edge resolving set has at least k vertices in it.

To justify our above observation, we consider a graph $G = K_4$. A set of vertices $W = \{f_1, f_2, f_3, f_4\}$ is the vertex set of $G = K_4$. One can observe that the set $V(K_4)$ is a 6-size edge resolving set for K_4. Therefore, $\beta_{6se}(K_4) = 4$.

4. K-Size Edge Metric Dimension of Cartesian Product of Graphs

Let $G = P_m \square P_n$ be the Cartesian product of two path graphs P_m and P_n, for $m, n \geq 4$. Let $E_1(G) = \{r_{gh} r_{g(h+1)}: 1 \leq g \leq m, 1 \leq h \leq n\}$ be the set of horizontal edges and $E_2(G) = \{s_{gh} s_{(g+1)h}: 1 \leq g \leq m-1, 1 \leq h \leq n\}$ be the set of vertical edges of $P_m \square P_n$. The graph of $P_5 \square P_6$ is shown in Figure 2. To find distances, we embed G into xy plane in such a way that each vertex is in an ordered pair form. Let the vertices $(0,0)$, $(m-1,0)$, $(0,n-1)$, $(m-1,n-1)$ be the corner vertices of G. In the next two lemmas, we shall discuss size 1, size 2, and size 3 edge metric dimension of $G = P_m \square P_n$ and $G = P_m \square C_n$, for $m, n \geq 5$.

Lemma 4. *Let G be the cartesian product graph $P_m \square P_n$; then, we have*

$$\beta_{kse}(P_m \square P_n) = \begin{cases} 3, & k = 1, \\ 4, & k = 2, \\ 5, & k = 3. \end{cases} \tag{3}$$

Proof. Here, we will prove this result for $k = 3$ (only). For this we consider $W = \{a_1, a_2, a_3, a_4, a_5\}$, where $a_1 = (0,0)$, $a_2 = (1,0)$, $a\ 3 = (m-1,0)$, $a_4 = (0,1)$, and $a_5 = (0,2)$, and prove that W is a $3ser$-set for $P_m \square P_n$. Note that $d((r\ 1, s_1), (r_2, s_2)) = |r_1 - r_2| + |s_1 - s_2|$ is the distance between any two vertices of $P_m \square P_n$. Let $\alpha = (r_1, s_1)(r_2, s_2)$ be an edge. The distances of the edge α from the vertices of W are calculated as follows.

$d(a_1, \alpha) = r_1 + s_1, d(a_3, \alpha) = m - 1 - r_2 + s_1$, $d(a_2, \alpha) = r_1 + s_1 - 1[0,1]$ when $\alpha \in \{r_{gh} r_{g(h+1)}, s_{gh} s_{(g+1)h}: 1 \leq g \leq m, 2 \leq h \leq n\}$, $d(a_2, \alpha) = r_1 + s_1$ when $\alpha \in \{r_{gh} r_{g(h+1)}: 1 \leq g \leq m, h = 1\}$, and $d(a_2, \alpha) = r_1 + s_1 + 1$ when $\alpha \in \{s_{gh} s_{(g+1)h}: 1 \leq g \leq m-1, h = 1\}$; $d(a_4, \alpha) = r_1 + s_1 + 1$, whenever $\alpha \in \{r_{gh} r_{g(h+1)}: g = 1, 1 \leq h \leq n\}$, $d(a_4, \alpha) = r_1 + s_1 - 1$, whenever $\alpha \in \{r_{gh} r_{g(h+1)}, s_{gh} s_{(g+1)h}: 2 \leq g \leq m, 1 \leq h \leq n\}$, $d(a_4, \alpha) = r_1 + s_1$, whenever $\alpha \in \{s_{gh} s_{(g+1)h}: g = 1, 1 \leq h \leq n\}$, $d(a_5, \alpha) = r_1 + s_1 + 2e$, whenever $\alpha \in \{r_{gh} r_{g(h+1)}: g = 1, 1 \leq h \leq n\}$, $d(a_5, \alpha) = r_1 + s_1$, whenever $\alpha \in \{r_{gh} r_{g(h+1)}: g = 2, 1 \leq h \leq n\}$, $d(a_5, \alpha) = r_1 + s_1 + 1$, whenever $\alpha \in \{s_{gh} s_{(g+1)h}: g = 1, 1 \leq h \leq n\}$, $d(a_5, \alpha) = r_1 + s_1 - 1$, whenever $\alpha \in \{s_{gh} s_{(g+1)h}: g = 2, 1 \leq h \leq n\}$, and $d(a_5, \alpha) = r_1 + s_1 - 2$, whenever $\alpha \in \{r_{gh} r_{g(h+1)}, s_{gh} s_{(g+1)h}: 3 \leq g \leq m, 1 \leq h \leq n\}$. Suppose contrary two edges $\alpha_1 = (r_1, s_1)(r_2, s_2)$ and $\alpha_2 = (t_1, u_1)(t_2, u_2)$ are at the same distance from the vertices of W. Thus, we have the following equalities:

$$\begin{aligned} r_1 + s_1 &= t_1 + u_1, \\ m - 1 - r_2 + s_1 &= m - 1 - t_2 + u_1. \end{aligned} \tag{4}$$

The above equalities imply that $s_1 - u_1 = r_2 - t_2$. Thus, it follows that $r_1 + r_2 = t_1 + t_2$. In both the cases $r_1 = r_2$ or $r_1 = r_2 - 1$, and we get $r_1 = t_1$ and $r_2 = t_2$. The equality $r_1 = t_1$ together with $r_1 + s_1 = t_1 + u_1$ implies that $s_1 = u_1$. Both the vertices s_2 and u_2 can either equal to s_1 or equal $s_1 + 1$. One of the edges α_1 or α_2 does not represent an edge if they have distinct values. So, finally we have $\alpha_1 = \alpha_2$, which is a contradiction. Therefore, W is an edge resolving set for $P_m \square P_n$. Moreover, the subgraph induced by W has 3 edges. Hence, we conclude that $\beta_{3se}(P_m \square P_n) = 5$. Similarly, we can prove the result for the values of $k = 1, 2$. Hence, we conclude the result.

We present the following result on the k-size edge metric dimension of the Cartesian product graph $P_m \square C_n$ without proof. □

Lemma 5. *Let G be the cartesian product graph $P_m \square C_n$; then, we have*

$$\beta_{kse}(P_m \square C_n) = \begin{cases} 3, & k = 1, 2, \\ 4, & k = 3. \end{cases} \tag{5}$$

5. K-Size Edge Metric Dimension of Generalized Petersen Graphs

The generalized Petersen graph $GP(q, s)$ is a 3-regular graph containing $2q$ vertices and $3q$ edges. The vertex set of $GP(q, s)$ is $V(GP(q, s)) = \{c_t, d_t: 1 \leq t \leq q\}$, and the edge set is $E(GP(q, s)) = \{c_t c_{t+1}, c_t d_t, d_t d_{t+s}: 1 \leq t \leq q, 1 \leq s < [q/2]\}$. The edges $c_t d_t$ for $1 \leq t \leq q$ are said to be spokes of $GP(q, s)$. The outer cycle of $GP(q, s)$ is said to be the principal cycle $GP(q, s)$.

We will compute the k-size edge metric dimension of $GP(q, s)$ for $k = 1, 2$ and $s = 1$ in the following two sections. Firstly, we will find upper bound for the 1-size edge metric dimension of $GP(q, 1)$.

Lemma 6. *For all $q \geq 7$, we have $\beta_{1se}(GP(q, 1)) \leq 3$.*

Proof. Let $W = \{c_1, c_3, d_1\}$ be a set of vertices of $GP(q, 1)$. □

Case 1. When q is odd.

Here, we define $t_1 = (q + 1/2)$. There is only one edge in the subgraph $\langle W \rangle$ induced by W and the codes of all the edges of $GP(q, 1)$ are given in Tables 1–3.

Case 2. When q is even.

We define $t_2 = (q/2)$. The induced subgraph $\langle W \rangle$ by W has only one edge, and the codes of all the edges of $GP(n, 1)$ are given in Tables 4–6.We observe that codes of all the edges in both cases are distinct. So, W is 1-size edge resolving set for $GP(q, 1)$. Hence, we have $\beta_{1se}(GP(q, 1)) \leq 3$. Now, we will compute upper bound for the size 2 edge metric dimension of generalized Petersen graphs $GP(q, 1)$.

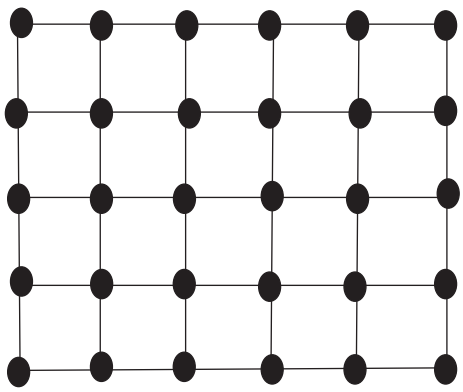

Figure 2: $G = P_5 \square P_6$.

Table 1: Codes of outer edges of $GP(q,1)$ when q is odd.

$d(.,.)$	c_1	c_3	d_1
c_1c_2	0	1	1
c_tc_{t+1}: $2 \le t \le 3$	$t-1$	0	t
c_tt_{t+1}: $4 \le t \le t_1$	$t-1$	$t-3$	t
c_tc_{t+1}: $t = 1+t_1$	$q-t$	$q-t$	$1+q-t$
c_tc_{t+1}: $2+t_1 \le t \le q$	$q-t$	$2+q-t$	$1+q-t$

Table 2: Codes of spokes of $GP(q,1)$ when q is odd.

$d(.,.)$	c_1	c_3	d_1
c_1d_1	0	2	0
c_2d_2	1	1	1
c_3d_3	2	0	2
c_td_t: $4 \le t \le t_1$	$t-1$	$t-3$	$t-1$
c_td_t: $t = 1+t_1$	$1+q-t$	$q-t$	$1+q-t$
c_td_t: $t = 2+t_1$	$1+q-t$	$2+q-t$	$1+q-t$
c_td_t: $3+t_1 \le t \le q$	$1+q-t$	$3+q-t$	$1+q-t$

Table 3: Codes of inner edges of $GP(q,1)$ when q is odd.

$d(.,.)$	c_1	c_3	d_1
d_1d_2	1	2	0
d_2d_3	2	1	1
d_td_{t+1}: $3 \le t \le t_1$	t	$t-2$	$t-1$
d_td_{t+1}: $t = 1+t_1$	$1+q-t$	$1+q-t$	$q-t$
d_td_{t+1}: $2+t_1 \le t \le q$	$1+q-t$	$3+q-t$	$q-t$

Table 4: Codes of outer edges of $GP(q,1)$ when q is even.

$d(.,.)$	c_1	c_3	d_1
c_1c_2	0	1	1
c_tc_{t+1}: $2 \le t \le 3$	$t-1$	0	i
c_tc_{t+1}: $4 \le t \le t_2$	$t-1$	$t-3$	t
c_tc_{t+1}: $t = 1+t_2$	$q-t$	$q-t-1$	$1+q-t$
c_tc_{t+1}: $t = 2+t_2$	$q-t$	$1+q-t$	$1+q-t$
c_tc_{t+1}: $3+t_2 \le t \le q$	$q-t$	$2+q-t$	$1+q-t$

Lemma 7. *For all $q \ge 5$, $\beta_{2se}(GP(q,1)) \le 3$.*

Proof. Let $W = \{c_1, c_2, d_1\}$ be an edge resolving set for $GP(q,1)$. Define $k = [q/2]$. The codes of all the edges of $GP(q,1)$ with respect to W are given Table 7–9.

Table 5: Codes of spokes of $GP(q,1)$ when q is even.

$d(.,.)$	c_1	c_3	d_1
c_1d_1	0	2	0
c_2d_2	1	1	1
c_3d_3	2	0	2
c_td_t: $4 \le t \le t_2+1$	$t-1$	$t-3$	$t-1$
c_td_t: $t = 2+t_2$	t_2-1	t_2-1	t_2-1
c_td_t: $3+t_2 \le t \le q$	$1+q-t$	$3+q-t$	$1+q-t$

Table 6: Codes of inner edges of $GP(q,1)$ when q is even.

$d(.,.)$	c_1	c_3	d_1
$d1d_2$	1	2	0
d_2d_3	2	1	1
d_td_{t+1}: $3 \le t \le t_2$	t	$t-2$	$t-1$
d_td_{t+1}: $t = 1+t_2$	$1+q-t$	$q-t$	$q-t$
d_td_{t+1}: $t = 2+t_2$	$1+q-t$	$2+q-t$	$q-t$
d_td_{t+1}: $3+t_2 \le t \le q$	$q+1-t$	$3+q-t$	$q-t$

Table 7: Codes of outer edges of $GP(q,1)$.

$d(.,.)$	c_1	c_2	d_1
c_1c_2	0	1	1
c_tc_{t+1}: $2 \le t \le k$	$t-1$	$t-2$	t
c_tc_{t+1}: $k+2 \le t \le q$	$q-t$	$1+q-t$	$1+q-t$

Table 8: Codes of spokes of $GP(q,1)$.

$d(.,.)$	c_1	c_2	d_1
c_1d_1	0	1	0
c_2d_2: $2 \le t \le k+1$	$t-1$	$t-2$	$t-1$
c_td_t: $k+2 \le t \le q$	$q-t+1$	$q-t+2$	$q-t+1$

Table 9: Codes of inner edges of $GP(q,1)$.

$d(.,.)$	c_1	c_2	d_1
d_1d_2	1	1	0
d_td_{t+1}: $2 \le t \le k$	t	$t-1$	$t-1$
d_td_{t+1}: $k+2 \le t \le q$	$q-t+1$	$q-t+2$	$q-t$

For $t = 1+k$, the code of outer edge c_tc_{t+1} will be $c_W(c_tc_{t+1}) = (q-t, q-t, q-t+1)$ when q is even and $c_W(c_tc_{t+1}) = (q-t, q-t+1, q-t+1)$ when q is odd.

It seems that codes of all the edges are distinct. So, W is size 2 edge resolving set for $GP(q,1)$. Hence, we have $\beta_{2se}(GP(q,1)) \le 3$.

In the next lemma, we will give the lower bound for the k-size edge metric dimension of $GP(q,1)$ for $k = 1, 2$. □

Lemma 8. *For all $q \ge 7$, we have $\beta_{kse}(GP(q,1)) \ge 3$.*

Proof. First, we will show that there is no edge resolving set of $GP(q,1)$ consisting of two vertices. Contrarily, we suppose that $X = \{A, B\}$ be a set having two vertices of $V(GP(q,1))$. Then, we have the following three possibilities. □

Case 1. When both the vertices A and B are from the principal cycle.

Let us fix a vertex, say $A = c_1$, then B is any other vertex c_t $(2 \le t \le q)$. For $2 \le t \le k-1$, we have $c_W(c_{q-1}c_q) = c_W(d_1 d_q) = (1, t)$. For $t = k$, we have $c_W(c_1c_q) = c_W(c_1d_1) = (0, k-1)$. For $t = k+1$, we have $c_W(c_2c_3) = c_W(c_{q-1}c_q) = (1, k-2)$ when q is even, while $c_W(c_2d_2) = c_W(d_1 d_q) = (1, k-1)$ when q is odd. For $t = k+2$, we have $c_W(c_2c_3) = c_W(d_1d_q) = (1, k-1)$ when q is even, while $c_W(c_2c_3) = c_W(d_1d_2) = (1, q-t+2)$ when t is odd. For $k+3 \le t \le q$, we have $c_W(c_2c_3) = c_W(d_1d_2) = (1, q-t+2)$.

Case 2. When both A and B are the inner vertices.

Let us fix a vertex, say $A = d_1$, then B is any other vertex v_t $(2 \le t \le q)$.

For $2 \le t \le k$, we have $c_W(d_1d_q) = c_W(c_1d_1) = (0, t-1)$. For $t = 1+k$, we have $c_W(d_3d_4) = c_W(d_{q-2}d_{q-1}) = (2, k-3)$ when q is even, while $c_W(d_3d_4) = c_W(c_{q-1}d_{q-1}) = (2, k-3)$ when q is odd. For $k+2 \le t \le q$, we have $c_W(c_1d_1) = c_W(d_1d_2) = (0, q-t+1)$.

Case 3. When A is any vertex from the principal cycle and B is any inner vertex.

Let us fix a vertex, say $A = c_1$; then, B is any inner vertex d_t $(1 \le t \le q)$. For $t = 1$, we have $c_W(c_1c_q) = c_W(c_1c_2) = (0, 1)$. For $2 \le t \le k$, we have $c_w(c_1c_2) = c_W(c_1d_1) = (0, t-1)$. For $t = 1+k$, we have $c_w(c_1c_q) = c_W(c_1d_1) = (0, q-t+1)$ when q is odd; however, $c_w(c_1c_2) = c_W(c_1d_1) = (0, t-1)$ when q is even. For $k+2 \le t \le q$, we have $c_W(c_1c_q) = c_W(c_1d_1) = (0, q-t+1)$.

From the above three cases, we conclude that

$$\beta_e(GP(q,1)) \ge 3. \tag{6}$$

So, there does not exist a size 1 and size 2 edge resolving set of cardinality 2 in $GP(q,1)$. Therefore, it yields that $\beta_{kse}(GP(q,1)) \ge 3$, for the value of $k = 1, 2$.

Lemmas 6–8, we conclude the following main result.

Theorem 1. *For all $q \ge 7$, we have $\beta_{kse}(GP(q,1)) = 3$ when $k = 1, 2$.*

6. Bounds and Some Realizable Results on $\beta_{kse}(G)$

From the earlier discussion, one fundamental question arises. Is the $(k+1)$ size edge metric dimension strictly greater than the k-size metric dimension? To answer this question, we consider following two examples.

Consider the graphs G_1 and G_2 which are depicted in Figure 3. It can be observed that the set $W = \{t_5, t_6, t_7\}$ is an edge resolving set for G_1 and the cardinality of set W is minimum. Moreover, $W_1 = \{t_1, t_5, t_6, t_7\}$ is a $1ser$-set, $W_2 = \{t_5, t_2, t_6, t_7\}$ is a $2ser$-set, and $W_3 = \{t_1, t_2, t_5, t_7\}$ is a $3ser$-set for G_1. Thus, $\beta_e(G_1) = 3$ and $\beta_{1se}(G_1) = \beta_{2se}(G_1) = \beta_{3se}(G_1) = 4$.

While, for the graph $G_2 \cong GP(5,1)$, the set $S = \{p_1, p_2, q_1\}$ is an edge resolving set of the minimum cardinality. Here, the outer vertices are $p_1, \ldots, p_5$ and the inner vertices are $q_1, \ldots, q_5$. It can be easily seen that the sets S_1, S_2, and S_3 are $1ser$-set, $2ser$-set, and $3ser$-set, respectively, where $S_1 = \{p_1, p_2, p_4\}$, $S_2 = \{q_1, q_2, p_1\}$, and $S_3 = \{p_1, p_2, p_3, p_4\}$. Thus, $\beta_e(GP(5,1)) = \beta_{1se}(GP(5,1)) = \beta_{2se}(GP(5,1)) = 3$ and $\beta_{3se}(GP(5,1)) = 4$.

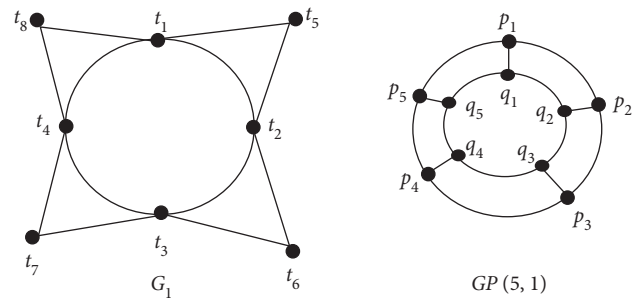

Figure 3: The graphs G_1 and G_2.

From these two examples, it can be observed that if $\beta_{kse}(G)$ exists for $1 \le k \le t$ in a nontrivial connected graph G of order n, then

$$\beta_e(G) \le \beta_{1se}(G) \le \beta_{2se}(G) \le \beta_{3se}(G) \ldots\ldots\ldots \le \beta_{tse}(G). \tag{7}$$

However, the following example shows that the above inequality is not true, in general.

Example 1. Let G be a graph which is constructed from two graphs C_5 and K_4. The vertex set of G is $\{l_1, l_2, l_3, l_4, l_5, p_1, p_2\}$ and the edge set is $\{l_1l_2, l_2l_3, l_3l_4, l_4l_5, l_5l_1, l_1p_1, p_1p_2, p_2l_5, p_1l_5, p_2l_1\}$, as shown in Figure 4. The set $A = \{l_1, l_5, p_1, p_2\}$ is a $6ser$-set of G. However, there is no such set B which resolves all the edges of graph G with cardinality 4 and $\langle B \rangle$ has size 5. So, we take the set $W = \{l_1, l_2, l_3, l_5, p_1\}$ is a $5ser$-set of the minimum cardinality for G. Hence, $\beta_{5se}(G) > \beta_{6se}(G)$.

Next, we characterize some realizable results for $1ser$-set and $2ser$-set in graphs.

Theorem 2. *For a nontrivial, simple, and connected graph G of order t, we have $\beta_{1se}(G) = t$ if and only if $G = P_2$.*

Proof. Lemma 1 implies that if $G = P_2$, then $\beta_{1se}(G) = 2 = t$. Conversely, assume that G be a connected graph of order $t \ge 2$ and $\beta_{1se}(G) = t$. Since the induced subgraph $\langle W \rangle$ has only one edge, therefore it is obvious that $|W| = |V(G)| = 2$. Thus, G is a path graph of order 2.

The following result on the complete graph K_s was presented in [1]. □

Lemma 9 (see [1]). *For any integer $s \ge 2$, $\beta_e(K_s) = s - 1$.*

Theorem 3. *Let G be a complete graph of order $s \ge 3$, then $\beta_{1se}(G)$ exists iff $G = K_3$. Moreover, $\beta_{1se}(G) = 2$.*

Proof. One can observe that if G is a complete graph K_3, then the result holds. Conversely, let G be a complete graph of order $s \ge 4$. Now, by Lemma 7, the induced subgraph G

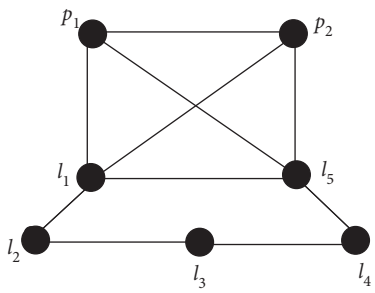

Figure 4: The graph in which $\beta_{5se}(G) > \beta_{6se}(G)$.

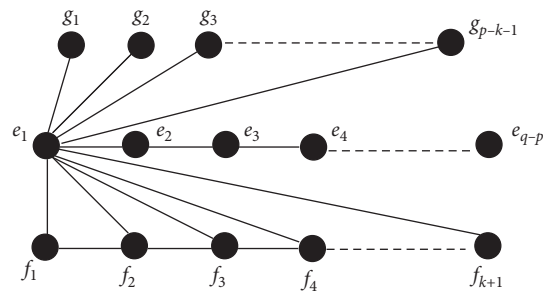

Figure 5: The graph H.

has more than one edge. Therefore, $\beta_{1se}(G)$ does not exist. Hence, the proof is complete. □

Theorem 4. *Let G be a nontrivial connected graph of order $s \geq 3$, then $\beta_{1se}(G) = s - 1$ if and only if $G = P_3$ or $G = K_3 \simeq C_3$.*

Proof. Let $G = P_3$ or $G = K_3 \cong C_3$. From Lemmas 1 and 2, and Theorem 5, we have $\beta_{1se}(G) = 2 = s - 1$. Contrarily, assume that G be a connected graph of order $s \geq 3$ and $\beta_{1se}(G) = s - 1$. For $s = 3$, it is simple to prove that $G = P_3$ or $G = K_3 \simeq C_3$. Now, we will prove the result for $s \geq 4$. For this, we suppose W be a $1ser$-set of G of cardinality $s - 1$. It is easy to see that $|V(G)| - |W| = 1$; therefore, the induced subgraph has surely more than one edge in it (G is a connected graph). It yields that $|W| = |V(G)| = 3$. Thus, $G = P_3$ or $G = K_3 \simeq C_3$. □

Remark 2. The two size edge resolving sets exist in complete bipartite graph $G = K_{s,t}$ if and only if $G \in \{K_{2,1}, K_{2,2}, K_{2,3}, K_{3,1}\}$. Moreover, $\beta_{2se}(K_{2,1}) = 3 = s + t$, $\beta_{2se}(K_{3,1}) = \beta_{2se}(K_{2,2}) = 3 = s + t - 1$, and $\beta_{2se}(K_{2,3}) = 3 = s + t - 2$.

Theorem 5. *For a simple and connected graph G of order $s \geq 3$, $\beta_{2se}(G) = s$ if and only if $G = P_3 \cong K_{2,1}$.*

Proof. By Lemma 1 and Remark 2, if $G = P_3 \cong K_{2,1}$, then $\beta_{2se}(G) = 3 = s$. Conversely, suppose that $\beta_{2se}(G) = s$ for a connected graph G of order $s \geq 3$. Let W be a $2ser$-set of order s. Since G is a connected graph and induced subgraph has two edges which implies that $|W| = |V(G)| = 3$, thus $G = P_3 \simeq K_{2,1}$.

Now, we study a sufficient condition for a pair (q, p) of positive integers to be realizable as the order and the k-size edge metric dimension of a connected graph, respectively. □

Theorem 6. *For a pair (q, p) of positive integers with $q \geq p \geq 2$, there exists a connected graph G of order q and $\beta_{kse}(G) = p$, where $k + 1 \leq p \leq q$.*

Proof. For $k \geq 1$, we consider the following two cases according to the choice of p.

(i) For $p = k + 1$, let $G = P_q$ be a path graph of order $q \geq 2$. Lemma 1 implies that $\beta_{kse}(P_q) = p$, where $1 \leq k \leq q - 1$. For $p = q$, let $G = C_q$ be a cycle of order $q \geq 3$, then it seems that $\beta_{kse}(C_q) = p$, where $k = p$.

(ii) For $k + 2 \leq p \leq q - 1$, let H be a connected graph of order $q \geq 6$ obtained from paths P_{q-p}: $e_1, e_2, e_3 \ldots, e_{q-p}$ ($q - p \geq 2$, where $p \geq 4$), P_{k+1}: $f_1, f_2, f_3, \ldots, f_{k+1}$ ($1 \leq k \leq p - 3$) and $p - k - 1$ vertices $g_1, g_2, \ldots, g_{p-k-1}$ with $g_i \sim e_1$ and $c_j \sim e_1$ for $1 \leq i \leq k + 1$ and $1 \leq j \leq p - k - 1$, as shown in Figure 5. Firstly, we prove that $\beta_{kse}(H) \leq p$. For this, let $S = \{f_1, f_2, \ldots, f_{k+1}, g_1, g_2 \ldots, g_{p-k-1}\} \subseteq V(H)$. Since, the subgraph induced by $\langle S \rangle$ has k edges and $c_S(e_i e_{i+1}) = (i, i, i, \ldots, i, i)$ for each $1 \leq i \leq q - p - 1$, it implies that S is a $kser$-set for G, and hence $\beta_{kse}(H) \leq l$. Now, to prove $\beta_{kse}(G) \geq l$, we suppose contrarily that $\beta_{kse}(G) \leq l - 1$ and W be a $kser$-set for H with $|W| \leq p - 1$. If W contains at least $p - k - 2$ vertices from C, say g_i ($1 \leq i \leq p - k - 2$) and the size of $\langle W \rangle$ is equal to k, it follows that $W = V(P_{k+1}) \cup \{g_1, g_2, \ldots, g_{p-k-2}\}$. However, still we have $c_W(e_1 e_2) = c_W(e_1 g_{p-k-1}) = (1, 1, 1, \ldots, 1, 1)$. Thus, $\beta_{kse}(H) \geq p$ which yields that $\beta_{ks}(H) = p$. □

7. Conclusions

In this work, we have introduced a new variant, namely, the k-size edge metric dimension of graphs, and initiated its study by finding the k-size edge metric dimension of several well-known classes of graphs. We have characterized the graphs having k-size edge metric dimension n and $n - 1$. Moreover, we have computed the k-size edge metric dimension of the Cartesian product graphs $P_m \square P_n$ and $P_m \square C_n$ for the values of $k = 1, 2, 3$. In addition, we have proved that the k-size edge metric dimension of generalized Petersen graphs $GP(q, 1)$ is 3 for the values of $k = 1, 2$. Some realizable results on k-size edge resolvability are also presented in this paper [8–12].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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