Research Article

$W^{1,p}$ Regularity of Weak Solutions to Maxwell's Equations

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In this paper, we study the steady-state Maxwell's equations. The weak solution defined in weak formulation is considered, and the global existence is obtained in general bounded open domain. The interior $W^{1,p}$ ($2 \leq p < \infty$) estimates of the weak solution are obtained, where the coefficient matrix is assumed to be BMO with small seminorm. The main analytical tools are the Vitali covering lemma, the maximal function technique, and the compactness method. We also consider the time-harmonic Maxwell's equations and obtain the interior $W^{1,p}$ estimates.

1. Introduction

It is well known that the classical Maxwell's equations can be written in differential form as follows:

$$\begin{aligned}\varepsilon E_t + \sigma E &= \nabla \times H, \\ \mu H_t + \nabla \times E &= F,\end{aligned}\quad (1)$$

where ε is the permittivity of the electric field, μ the permeability of the magnetic field, and σ the conductivity of the material.

When the material is conductive, the current displacement εE_t can be ignored since it is very small compared with the eddy current σE . Then, we have the following evolution system:

$$\begin{aligned}\mu H_t + \nabla \times (\sigma^{-1} \nabla \times H) &= F, \\ \nabla \cdot H &= 0.\end{aligned}\quad (2)$$

If H is assumed to be time independent, we can obtain

$$\begin{aligned}\nabla \times (A(x) \nabla \times H) &= F, \\ \nabla \cdot H &= 0,\end{aligned}\quad (3)$$

where $A(x) = \sigma^{-1}$.

System (3) is an important mathematical model for the study of the penetration of magnetic field in materials. Yin [1] pointed out that this system is degenerated by the

classical definition (see [2]). Thus, it has a different structure with general elliptic equations, and the regularity should be restudied. The existence of a unique weak solution can be found in [3, 4]. By using Campanato theory, this system has been studied in [1, 3, 5]. They showed that the weak solution is Hölder continuous with the assumption that A is a positive bounded scalar function. In [1, 5], they got the local Hölder continuity. Afterwards, Kang and Kim [3] obtained the global Hölder continuity on the Lipschitz domain. For the higher regularity, the interior $C^{1,\alpha}$ estimate has been given in [5]. The $W^{1,p}$ estimate can be found in [6], in which A^{-1} is assumed to be in the VMO space and the domain is assumed to be C^1 .

In this paper, we establish the existence theorem of weak solution of (3) in general bounded domain and study the $W^{1,p}$ ($2 \leq p < \infty$) regularity with the assumptions that $A(x)$ is defined on $\mathbb{R}^3$ and has the small BMO seminorm (see Definition 2).

Another goal of this paper is to establish the $W^{1,p}$ ($2 \leq p < \infty$) regularity of the following system:

$$\nabla \times (A(x) \nabla \times \hat{E}) + \hat{E} = \nabla \times \hat{F}, \quad (4)$$

which can describe time-harmonic electromagnetic field. We prove that if the matrix A is uniformly positive definite and has the small BMO seminorm, then the weak solution of system (4) belongs to $W^{1,p}$. We weaken the assumption in [7] that A is Lipschitz continuous and also generalize the

assumption in [8] that A is a bounded scalar function and the real part of A has a positive lower bound.

The remaining sections are organized in the following way. In Section 2, we introduce the relevant concepts and lemmas. In Section 3, we state our main theorems and give some remarks concerning them. In Section 4, the proofs of our main results are given.

2. Preliminaries

We introduce some notations and lemmas here.

- (1) $B_r = \{y \in \mathbb{R}^3: |x| < r\}$ is an open ball centered at origin with radius r and $\Omega_r = \Omega \cap B_r$.
- (2) For two vector fields $a \in \mathbb{R}^3$ and $b \in \mathbb{R}^3$, $a \cdot b$ and $a \times b$ define the scalar product and the cross product, respectively
- (3) ∇g is the gradient of g ; $\nabla \cdot u$ is the divergence of u ; and $\nabla \times u$ is the curl of u .
- (4) For a locally integrable function f ,

$$\bar{f}_{B_r(x)} = \frac{1}{|B_r|} \int_{B_r(x)} f(y) dy, \quad (5)$$

is the average of f over $B_r(x)$.

$$\mathcal{M}(f)(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| dy, \quad (6)$$

is called the Hardy–Littlewood maximal function of f . We also use $\mathcal{M}_\Omega(f)$, if f is not defined outside of Ω .

- (5) $|A| = \sqrt{\sum_{i,j=1}^3 a_{ij}^2}$, $\|A\|_\infty = \sup_y |A(y)|$.
- (6) $H_0(\operatorname{div}, \Omega) = \{u \in H_0^1(\Omega; \mathbb{R}^3): \nabla \cdot u = 0 \text{ in } \Omega\}$.
- (7) $H(\operatorname{curl}, \Omega) = \{u \in L^2(\Omega; \mathbb{R}^3): \nabla \times u \in L^2(\Omega; \mathbb{R}^3)\}$.

Definition 1. We say that the matrix $A(x) = (a_{ij}(x))_{3 \times 3}$ is uniformly positive definite if there exists $\lambda > 0$:

$$\lambda^{-1} |\xi|^2 \leq A\xi \cdot \xi \leq \lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^3. \quad (7)$$

Definition 2. The matrix $A(x)$ is called (δ, R) -vanishing if

$$\sup_{0 < r \leq R} \sup_{x \in \mathbb{R}^3} \left(\frac{1}{|B_r|} \int_{B_r(x)} |A(y) - \bar{A}_{B_r}|^2 dy \right)^{1/2} \leq \delta, \quad (8)$$

where $\bar{A}_{B_r}$ is the average of A over $B_r(x)$.

Lemma 1 (see [9]). *Let f be a measurable function in Ω , $\theta > 0$ and $N > 1$ be constants. Then, for any $0 < p < \infty$,*

$$f \in L^p(\Omega) \text{ iff } S = \sum_{k=1}^{\infty} N^{kp} \left| \{x \in \Omega: |f| > \theta N^k\} \right| < \infty, \quad (9)$$

$$C^{-1} S \leq \int_{\Omega} |f|^p dx \leq C(|\Omega| + S),$$

where $C > 0$ is a constant depending only on θ , N , and p .

Lemma 2 (see [10]).

- (a) *If $f \in L^1(\mathbb{R}^n)$, then for every $\alpha > 0$,*

$$|\{x \in \mathbb{R}^n: (\mathcal{M}f)(x) > \alpha\}| \leq \frac{C}{\alpha} \int |f| dx, \quad (10)$$

where C is a constant which depends only on the dimension n .

- (b) *If $f \in L^p(\mathbb{R}^n)$, with $1 < p \leq \infty$, then $\mathcal{M}f \in L^p(\mathbb{R}^n)$ and*

$$\|\mathcal{M}f\|_{L^p} \leq C \|f\|_{L^p}, \quad (11)$$

where C depends only on p and the dimension n .

Lemma 3 (see [11]). *Assume that C and D are measurable sets of $\mathbb{R}^n$, $C \subset D \subset B_1$, and that there exists an $\varepsilon > 0$ such that $|C| < \varepsilon |B_1|$, and for all $x \in B_1$ and for all $r \in (0, 1]$ with $|C \cap B_r(x)| \geq \varepsilon |B_r(x)|$ implying that $B_r(x) \cap B_1 \subset D$. Then*

$$|C| \leq 10^n \varepsilon |D|. \quad (12)$$

Lemma 4 (see [7]). *Let $\Omega \subset \mathbb{R}^3$ be a bounded C^1 domain with connected boundary. Let $f \in L^p(\Omega; \mathbb{R}^3)$ with $\nabla \cdot f = 0$ for some $1 < p < \infty$. Then, there exists $h \in W^{1,p}(\Omega; \mathbb{R}^3)$ such that $\nabla \times h = f$. Moreover, $\nabla \cdot h = 0$ in Ω and $\|h\|_{W^{1,p}(\Omega)} \leq C \|f\|_{L^p(\Omega)}$, where C depends only on p and Ω .*

3. Main Theorems

In the following, we assume that $B_6 \subset B_R \subset \Omega$ and δ is a small positive constant. Our first theorem is the well-posedness in $H^1(\Omega; \mathbb{R}^3)$. Considering the weak solution defined in weak formulation (see Definition 3), we have the following theorem.

Theorem 1. *Let Ω be a bounded open domain and $A(x)$ be uniformly positive definite. Then, for $f \in L^2(\Omega; \mathbb{R}^3)$, the Dirichlet problem*

$$\begin{cases} \nabla \times (A(x) \nabla \times u) = \nabla \times f, \nabla \cdot u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (13)$$

has a unique weak solution $u \in H_0^1(\Omega; \mathbb{R}^3)$ with

$$\|\nabla u\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}, \quad (14)$$

where C depends on λ .

Remark 1. We point out that this theorem holds in general bounded open domain, which may not be Lipschitz, like the Reifenberg flat domain (see [11, 12]). When $\partial\Omega \in C^{0,1}$, this theorem has been proved in many papers (see [1, 3, 4, 13]) because for the Lipschitz domain, the following identity (19) can be easily obtained by performing integration by parts. Here, we will give a proof of (19) in general bounded open domain by density.

Theorem 2. Let $2 \leq p < \infty$ and $B_6 \subset B_R \subset \Omega$. There is a small $\delta = \delta(\lambda, p, R) > 0$ such that for all A with A uniformly positive definite and (δ, R) -vanishing, and for all f with $f \in L^p(B_6; \mathbb{R}^3)$, if u is a weak solution of (13) in B_6 , then u belongs to $W^{1,p}(B_1; \mathbb{R}^3)$ with the estimate

$$\|\nabla u\|_{L^p(B_1)} \leq C \left(\|u\|_{L^p(B_6)} + \|f\|_{L^p(B_6)} \right), \quad (15)$$

where the constant C is independent of u and f .

Remark 2. We remark that our assumption that A is (δ, R) -vanishing weakens the assumption in [6] that A^{-1} belongs to VMO. Since the linear system (13) is degenerate (see [1]), the regularity theory of elliptic systems cannot be applied directly. We established some useful lemmas to handle the difficulty. Our basic tools are the Vitali covering lemma, the Hardy–Littlewood maximal function, and the compactness method, which have been used in [11] to deal with elliptic equations.

Theorem 3. Let $2 \leq p < \infty$ and $B_6 \subset B_R \subset \Omega$. There is a small $\delta = \delta(\lambda, p, R) > 0$ such that for all A with A uniformly positive definite and (δ, R) -vanishing, and for all f with $f \in L^p(B_6; \mathbb{R}^3)$, if $u \in H(\text{curl}, \Omega)$ is a weak solution of

$$\nabla \times (A(x) \nabla \times u) + u = \nabla \times f, \quad \text{in } \Omega, \quad (16)$$

then

$$\|\nabla u\|_{L^p(B_1)} \leq C \left(\|u\|_{L^p(B_6)} + \|f\|_{L^p(B_6)} \right), \quad (17)$$

where the constant C is independent of u and f .

Remark 3. Here, $H(\text{curl}, \Omega) = \{u \in L^2(\Omega; \mathbb{R}^3) : \nabla \times u \in L^2(\Omega; \mathbb{R}^3)\}$. The existence of weak solution of (16) in $H(\text{curl}, \Omega)$ can be found in [8]. We point out that our assumption that A is (δ, R) -vanishing weakens the assumption in [7] that A is Lipschitz continuous.

Remark 4. If $p > 3$, we will have the following interior Hölder estimate:

$$[u]_{C^{0,\alpha}(\bar{B}_1)} \leq C \left(\|u\|_{L^p(B_6)} + \|f\|_{L^p(B_6)} \right), \quad (18)$$

where $\alpha = 1 - 3/p$. We should remark that the Hölder estimate has been established in [8], but they did not give the concrete value of the Hölder exponent.

4. Proofs of Main Theorems

4.1. Proof of Theorem 1. Let us first prove the following important equality.

Lemma 5. Let $\Omega \subset \mathbb{R}^3$ be bounded and $u \in H_0^1(\Omega; \mathbb{R}^3)$. We then have the following identity:

$$\|\nabla u\|_{L^2(\Omega)}^2 = \|\nabla \times u\|_{L^2(\Omega)}^2 + \|\nabla \cdot u\|_{L^2(\Omega)}^2. \quad (19)$$

Proof. Let $\phi \in C_0^\infty(\Omega; \mathbb{R}^3)$, then we have

$$-\Delta \phi = \nabla \times (\nabla \times \phi) - \nabla (\nabla \cdot \phi), \quad \text{in } \Omega. \quad (20)$$

We multiply this identity by u and integrate

$$-\int_{\Omega} \Delta \phi \cdot u dx = \int_{\Omega} \nabla \times (\nabla \times \phi) \cdot u dx - \int_{\Omega} \nabla (\nabla \cdot \phi) \cdot u dx. \quad (21)$$

Since $u \in H_0^1(\Omega; \mathbb{R}^3)$, by using the definition of weak derivative, we have

$$-\int_{\Omega} \Delta \phi \cdot u dx = \int_{\Omega} (\nabla \phi) : (\nabla u) dx. \quad (22)$$

Similarly,

$$\begin{aligned} \int_{\Omega} \nabla \times (\nabla \times \phi) \cdot u dx &= \int_{\Omega} (\nabla \times \phi) \cdot (\nabla \times u) dx, \\ -\int_{\Omega} \nabla (\nabla \cdot \phi) \cdot u dx &= \int_{\Omega} (\nabla \cdot \phi) (\nabla \cdot u) dx. \end{aligned} \quad (23)$$

Hence, we obtain

$$\begin{aligned} \int_{\Omega} (\nabla \phi) : (\nabla u) dx &= \int_{\Omega} (\nabla \times \phi) \cdot (\nabla \times u) dx \\ &\quad + \int_{\Omega} (\nabla \cdot \phi) (\nabla \cdot u) dx. \end{aligned} \quad (24)$$

Then, if we take $\phi \rightarrow u$ in $H_0^1(\Omega; \mathbb{R}^3)$, (19) can be proved by the density.

Remark 5. In the proof of (19), we do not use integration by parts. Thus, we do not need the domain to be Lipschitz.

Remark 6. The identity (19) implies that the norm of $H_0^1(\Omega; \mathbb{R}^3)$ is equivalent to the right hand of it raised to the power 1/2. This also means we can consider the weak solution of (13) in $H_0(\text{div}, \Omega)$ with the norm $\|\nabla \times u\|_{L^2(\Omega)}$ instead of $H_0^1(\Omega; \mathbb{R}^3)$, where $H_0(\text{div}, \Omega) = \{u \in H_0^1(\Omega; \mathbb{R}^3) : \nabla \cdot u = 0 \text{ in } \Omega\}$.

Thus, we can define the weak solution of (13) as follows.

Definition 3. A vector field $u \in H_0(\text{div}, \Omega)$ is said to be a weak solution of (13), if the following identity holds:

$$\int_{\Omega} (A(x) \nabla \times u) \cdot (\nabla \times v) dx = \int_{\Omega} f \cdot (\nabla \times v) dx, \quad (25)$$

for any $v \in H_0(\text{div}, \Omega)$.

Now, we give the proof of Theorem 1.

Proof of Theorem 1. In order to prove the existence and uniqueness of weak solution, we define the bilinear form as follows:

$$B(u, v) = \int_{\Omega} (A(x) \nabla \times u) \cdot (\nabla \times v) dx, \quad (26)$$

for any $u, v \in H_0(\text{div}, \Omega)$. Since $A(x)$ is uniformly positive definite with λ , we have

$$B(u, u) \geq \lambda^{-1} \|\nabla \times u\|_{L^2(\Omega)}^2. \quad (27)$$

This means that the bilinear form $B(u, v)$ is coercive on $H_0(\operatorname{div}, \Omega)$. Moreover,

$$|B(u, v)| \leq \lambda \|\nabla \times u\|_{L^2(\Omega)} \|\nabla \times v\|_{L^2(\Omega)}. \quad (28)$$

Now fix $f \in L^2(\Omega; \mathbb{R}^3)$ and set

$$\langle f, v \rangle := \int_{\Omega} f \cdot (\nabla \times v) dx. \quad (29)$$

This is a bounded linear functional on $H_0(\operatorname{div}, \Omega)$.

Thus, we can apply Lax–Milgram theorem (see [14]) to find a unique function $u \in H_0(\operatorname{div}, \Omega)$ satisfying

$$B(u, v) = \langle f, v \rangle, \quad (30)$$

for all $v \in H_0(\operatorname{div}, \Omega)$; u is consequently the weak solution of (13).

Moreover, we can choose u as a test function to get

$$\|\nabla u\|_{L^2(\Omega)} = \|\nabla \times u\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}, \quad (31)$$

where C depends on λ and the theorem is proved.

4.2. Proof of Theorem 2. For simplicity, we take $R > 6$ and assume $B_6 \subset B_R \subset \Omega$. We will locally approximate solution (13) by a function satisfying a suitable homogeneous problem. We need some lemmas here. The first one is the following energy estimate.

Lemma 6. Assume that u is a weak solution of (13) in B_1 . Then

$$\int_{B_1} \phi^2 |\nabla u|^2 dx \leq C \left(\int_{B_1} \phi^2 |f|^2 dx + \int_{B_1} |\nabla \phi|^2 |u|^2 dx \right), \quad (32)$$

for any $\phi \in C_0^\infty(B_1)$, where C depends on λ .

Proof. First note that $\phi^2 u \in H_0^1(B_1; \mathbb{R}^3)$, so we have

$$\int_{B_1} (A(x) \nabla \times u) \cdot (\nabla \times (\phi^2 u)) dx = \int_{B_1} f \cdot (\nabla \times (\phi^2 u)) dx. \quad (33)$$

Using the identity $\nabla \times (\phi^2 u) = \phi^2 \nabla \times u + (2\phi \nabla \phi) \times u$, we obtain

$$\int_{B_1} \phi^2 |\nabla \times u|^2 dx \leq C \left(\int_{B_1} |\phi|^2 |f|^2 dx + \int_{B_1} |\nabla \phi|^2 |u|^2 dx \right), \quad (34)$$

where we used the Hölder inequality, and C depends on λ .

Moreover, we know that $\phi u \in H_0^1(B_1; \mathbb{R}^3)$. By Lemma 5, we have

$$\int_{B_1} |\nabla(\phi u)|^2 dx = \int_{B_1} |\nabla \times (\phi u)|^2 dx + \int_{B_1} |\nabla \cdot (\phi u)|^2 dx. \quad (35)$$

Therefore,

$$\begin{aligned} \int_{B_1} \phi^2 |\nabla u|^2 dx &\leq C \left(\int_{B_1} |\nabla(\phi u)|^2 dx + \int_{B_1} |\nabla \phi|^2 |u|^2 dx \right) \\ &= C \left(\int_{B_1} |\nabla \times (\phi u)|^2 dx + \int_{B_1} |\nabla \cdot (\phi u)|^2 dx \right. \\ &\quad \left. + \int_{B_1} |\nabla \phi|^2 |u|^2 dx \right) \\ &\leq C \left(\int_{B_1} \phi^2 |f|^2 dx + \int_{B_1} |\nabla \phi|^2 |u|^2 dx \right), \end{aligned} \quad (36)$$

where we have used the fact that $\nabla \cdot u = 0$ in B_1 .

Lemma 7. For any $\varepsilon > 0$, there is a small $\delta = \delta(\varepsilon) > 0$ such that for any weak solution u of (13) in B_4 with

$$\frac{1}{|B_4|} \int_{B_4} |\nabla u|^2 dx \leq 1, \quad (37)$$

$$\frac{1}{|B_4|} \int_{B_4} (|A - \bar{A}_{B_4}|^2 + |f|^2) dx \leq \delta^2,$$

there exists a weak solution v of

$$\begin{cases} \nabla \times (\bar{A}_{B_4} \nabla \times v) = 0, & \text{in } B_4, \\ \nabla \cdot v = 0, & \text{in } B_4, \end{cases} \quad (38)$$

such that

$$\frac{1}{|B_2|} \int_{B_2} |\nabla(u - v)|^2 dx \leq \varepsilon^2. \quad (39)$$

Proof. Firstly, we claim that, for any $\eta > 0$, there is a small $\delta = \delta(\eta) > 0$ and a weak solution v of (38), such that

$$\frac{1}{|B_4|} \int_{B_4} |u - v|^2 dx \leq \eta^2. \quad (40)$$

Suppose it is false. Then, we can find $\eta_0 > 0$ and sequences $\{A_k\}_{k=1}^\infty$, $\{u_k\}_{k=1}^\infty$ and $\{f_k\}_{k=1}^\infty$, such that u_k is a weak solution of

$$\begin{cases} \nabla \times (A_k \nabla \times u_k) = \nabla \times f_k, & \text{in } B_4, \\ \nabla \cdot u_k = 0, & \text{in } B_4, \end{cases} \quad (41)$$

with

$$\begin{aligned} \frac{1}{|B_4|} \int_{B_4} |\nabla u_k|^2 dx &\leq 1, \\ \frac{1}{|B_4|} \int_{B_4} (|A_k - \bar{A}_{B_4}|^2 + |f_k|^2) dx &\leq \frac{1}{k^2}. \end{aligned} \quad (42)$$

But for any weak solution v_k of

$$\begin{cases} \nabla \times (\bar{A}_{B_4} \nabla \times v_k) = 0, & \text{in } B_4, \\ \nabla \cdot v_k = 0, & \text{in } B_4, \end{cases} \quad (43)$$

we have

$$\frac{1}{|B_4|} \int_{B_4} |u_k - v_k|^2 dx > \eta_0^2. \quad (44)$$

By (42) and Poincaré inequality, $\{u_k - \bar{u}_{k_{B_4}}\}_{k=1}^\infty$ is bounded in $H^1(B_4; \mathbb{R}^3)$. So there exist $u_0 \in H^1(B_4; \mathbb{R}^3)$ and a subsequence, still denoted as $\{u_k - \bar{u}_k\}$, such that $u_k - \bar{u}_k \rightarrow u_0$ strongly in $L^2(B_4; \mathbb{R}^3)$ and $\nabla \times u_k \rightarrow \nabla \times u_0$ weakly in $L^2(B_4; \mathbb{R}^3)$. Since $\{\bar{A}_{k_{B_4}}\}_{k=1}^\infty$ is a bounded sequence of constant matrices, there exist a constant matrix A_0 and a subsequence, still denoted as $\{\bar{A}_k\}$, such that $\bar{A}_k \rightarrow A_0$. Combining (42), we know that $\{A_k\}_{k=1}^\infty$ has a subsequence, denoted also as A_k , such that $A_k \rightarrow A_0$ strongly in $L^2(B_4)$. Thus, u_0 satisfies the following system:

$$\begin{cases} \nabla \times (A_0 \nabla \times u_0) = 0, & \text{in } B_4, \\ \nabla \cdot u_0 = 0, & \text{in } B_4. \end{cases} \quad (45)$$

Take $v_k = \bar{u}_k + u_0 - h_k$, where $h_k \in H_0^1(B_4; \mathbb{R}^3)$ satisfies the following system:

$$\begin{cases} \nabla \times (\bar{A}_k \nabla \times h_k) = \nabla \times ((\bar{A}_k - A_0) \nabla \times u_0), & \text{in } B_4, \\ \nabla \cdot h_k = 0, & \text{in } B_4, \\ h_k = 0, & \text{on } \partial B_4. \end{cases} \quad (46)$$

Using Theorem 1, we have

$$\|h_k\|_{L^2(B_4)} \leq C \|\nabla h_k\|_{L^2(B_4)} \leq C \|(\bar{A}_k - A_0) \nabla \times u_0\|_{L^2(B_4)} \leq C |\bar{A}_k - A_0|. \quad (47)$$

Moreover, v_k satisfies system (43), and

$$\begin{aligned} \|u_k - v_k\|_{L^2(B_4)} &\leq \|u_k - \bar{u}_k - u_0\|_{L^2(B_4)} + \|h_k\|_{L^2(B_4)} \\ &\leq \|u_k - \bar{u}_k - u_0\|_{L^2(B_4)} + C |\bar{A}_k - A_0|. \end{aligned} \quad (48)$$

This means

$$\|u_k - v_k\|_{L^2(B_4)} \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (49)$$

But this is a contradiction to (44), and thus, (40) holds and the claim is proved.

Now, we give the proof of (39). It is easy to see that $u - v$ satisfies the following system:

$$\begin{cases} \nabla \times (A(x) \nabla \times (u - v)) = \nabla \times (f - (A(x) - \bar{A}_{B_4}) \nabla \times v), & \text{in } B_4, \\ \nabla \cdot (u - v) = 0, & \text{in } B_4. \end{cases} \quad (50)$$

By Lemma 6, we have

$$\begin{aligned} \frac{1}{|B_2|} \int_{B_2} |\nabla(u - v)|^2 dx &\leq C \left(\frac{1}{|B_3|} \int_{B_3} |f - (A - \bar{A}_{B_4}) \nabla \times v|^2 dx + \frac{1}{|B_4|} \int_{B_4} |u - v|^2 dx \right) \\ &\leq C \left(\frac{1}{|B_4|} \int_{B_4} |f|^2 dx + \frac{1}{|B_4|} \int_{B_4} |A - \bar{A}_{B_4}|^2 dx + \frac{1}{|B_4|} \int_{B_4} |u - v|^2 dx \right). \end{aligned} \quad (51)$$

Here, we used the interior $W^{1,\infty}$ regularity of v (see Theorem 2.2 of [5]). Combining (37) and (40), we conclude

$$\frac{1}{|B_2|} \int_{B_2} |\nabla(u - v)|^2 dx \leq C(\delta^2 + \eta^2) = \varepsilon^2, \quad (52)$$

by taking η and δ satisfying the last identity. This finishes the proof of this lemma.

Lemma 8. *Let u be a weak solution of (13) in Ω^{B_6} . There exists a constant M such that for any $\varepsilon > 0$, there exists a small $\delta = \delta(\varepsilon) > 0$, if*

$$B_1 \cap \{x \in \Omega: \mathcal{M}(|\nabla u|^2) \leq 1\} \cap \{x \in \Omega: \mathcal{M}(|f|^2) \leq \delta^2\} \neq \emptyset, \quad (53)$$

then

$$|\{x \in \Omega: \mathcal{M}(|\nabla u|^2) > M^2\} \cap B_1| < \varepsilon |B_1|. \quad (54)$$

Proof. By assumption (53), there is a point $x_0 \in B_1$ such that for all $r > 0$,

$$\frac{1}{|B_r|} \int_{\Omega_r(x_0)} |\nabla u|^2 dx \leq 1, \quad (55)$$

$$\frac{1}{|B_r|} \int_{\Omega_r(x_0)} |f|^2 dx \leq \delta^2.$$

Since $B_4 \subset B_5(x_0)$, we conclude that

$$\frac{1}{|B_4|} \int_{B_4} |\nabla u|^2 dx \leq \left(\frac{5}{4}\right)^3, \quad (56)$$

$$\frac{1}{|B_4|} \int_{B_4} |f|^2 dx \leq \left(\frac{5}{4}\right)^3 \delta^2.$$

Applying Lemma 7 to $(4/5)^{3/2}u$ and $(4/5)^{3/2}f$, we have

$$\frac{1}{|B_2|} \int_{B_2} |\nabla(u - v)|^2 dx \leq \eta^2. \quad (57)$$

Note that v satisfies system (38). We can find a constant N such that

$$\|\nabla v\|_{L^\infty(B_3)} \leq N. \quad (58)$$

Take $M^2 := \max\{3^3, 4N^2\}$. Now suppose that

$$x_1 \in B_1 \cap \{x \in \Omega: \mathcal{M}_{B_2}(|\nabla(u-v)|^2) \leq N^2\}. \quad (59)$$

When $r \leq 1$, then $B_r(x_1) \subset B_2$. Hence, we have

$$\frac{1}{|B_r|} \int_{B_r(x_1)} |\nabla u|^2 dx \leq \frac{2}{|B_r|} \int_{B_r(x_1)} (|\nabla(u-v)|^2 + |\nabla v|^2) dx \leq 4N^2. \quad (60)$$

When $r > 1$, then $B_r(x_1) \subset B_{3r}(x_0)$. Hence, by (55), we have

$$\frac{1}{|B_r|} \int_{\Omega_r(x_1)} |\nabla u|^2 dx \leq \frac{3^3}{|B_{3r}|} \int_{\Omega_{3r}(x_0)} |\nabla u|^2 dx \leq 3^3. \quad (61)$$

The above two inequalities show that

$$x_1 \in B_1 \cap \{x \in \Omega: \mathcal{M}(|\nabla u|^2) \leq M^2\}. \quad (62)$$

Combining (59) and (62), we have

$$\begin{aligned} B_1 \cap \{x \in \Omega: \mathcal{M}(|\nabla u|^2) > M^2\} &\subset B_1 \\ \cap \{x \in \Omega: \mathcal{M}_{B_2}(|\nabla(u-v)|^2) > N^2\}. \end{aligned} \quad (63)$$

Consequently,

$$\begin{aligned} |\{x \in \Omega: \mathcal{M}(|\nabla u|^2) > M^2\} \cap B_1| &\leq |\{x \in \Omega: \mathcal{M}_{B_2}(|\nabla(u-v)|^2) > N^2\} \cap B_1| \\ &\leq \frac{C}{N^2} \int_{B_2} |\nabla(u-v)|^2 dx \leq \frac{C}{N^2} \eta^2 < \varepsilon |B_1|, \end{aligned} \quad (64)$$

by taking η satisfying the last inequality above. This finishes the proof.

Lemma 9. *Let u be a weak solution of (13) in Ω^{B_6} . There exists a constant M such that for any $\varepsilon > 0$ and $r \in (0, 1]$, there exists a small $\delta = \delta(\varepsilon) > 0$, if*

$$|\{x \in \Omega: \mathcal{M}(|\nabla u|^2) > M^2\} \cap B_r| \geq \varepsilon |B_r|, \quad (65)$$

then

$$B_r \subset \{x \in \Omega: \mathcal{M}(|\nabla u|^2) > 1\} \cup \{x \in \Omega: \mathcal{M}(|f|^2) > \delta^2\}. \quad (66)$$

Proof. We argue by contradiction. If conclusion (66) is false, then

$$B_r \cap \{x \in \Omega: \mathcal{M}(|\nabla u|^2) \leq 1\} \cap \{x \in \Omega: \mathcal{M}(|f|^2) \leq \delta^2\} \neq \emptyset. \quad (67)$$

Let us consider the functions

$$\tilde{u}(x) = \frac{u(rx)}{r},$$

$$\tilde{A}(x) = A(rx), \quad (68)$$

$$\tilde{f}(x) = f(rx),$$

with $x \in B_1$. Then, it is easy to check that $\tilde{u}, \tilde{A}, \tilde{f}$ satisfy the conditions of Lemma 8, and

$$|\{x \in \Omega: \mathcal{M}(|\nabla \tilde{u}|^2) > M^2\} \cap B_1| < \varepsilon |B_1|. \quad (69)$$

Scaling back in the above estimate yields

$$|\{x \in \Omega: \mathcal{M}(|\nabla u|^2) > M^2\} \cap B_r| < \varepsilon |B_r|, \quad (70)$$

which is contradiction to (65).

Now take M, ε , and the corresponding $\delta > 0$ given by Lemma 8.

Lemma 10. *Assume that A is uniformly positive definite and (δ, R) -vanishing. Suppose that u is a weak solution of (13) in $\Omega^{B_{R^{B_6}}}$ and*

$$|\{x \in B_1: \mathcal{M}(|\nabla u|^2) > M^2\}| < \varepsilon |B_1|. \quad (71)$$

Let k be a positive integer and set $\varepsilon_1 = 10^3 \varepsilon$. Then, we have

$$\begin{aligned} |\{x \in B_1: \mathcal{M}(|\nabla u|^2) > M^{2k}\}| &\leq \sum_{i=1}^k \varepsilon_1^i |\{x \in B_1: \mathcal{M}(|f|^2) > \delta^2 M^{2(k-i)}\}| \\ &\quad + \varepsilon_1^k |\{x \in B_1: \mathcal{M}(|\nabla u|^2) > 1\}|. \end{aligned} \quad (72)$$

Proof. We intend to prove by induction on k . For the case $k = 1$, let

$$C = \{x \in B_1: \mathcal{M}(|\nabla u|^2) > M^2\},$$

$$D = \{x \in B_1: \mathcal{M}(|f|^2) > \delta^2\} \cup \{x \in B_1: \mathcal{M}(|\nabla u|^2) > 1\}. \quad (73)$$

Then, in view of (71), Lemmas 9 and 3, we see that $|C| \leq \varepsilon_1 |D|$, and our conclusion is valid for $k = 1$.

Assume that the conclusion is valid for some positive integer $k \geq 2$. Let $u_1 = u/M$ and corresponding $f_1 = f/M$. Then u_1 is the weak solution of

$$\begin{cases} \nabla \times (A \nabla \times u_1) = \nabla \times f_1, & \text{in } B_6, \\ \nabla \cdot u_1 = 0, & \text{in } B_6, \end{cases} \quad (74)$$

and the following inequality holds:

$$|\{x \in B_1: \mathcal{M}(|\nabla u_1|^2) > M^2\}| < \varepsilon |B_1|. \quad (75)$$

Thus, we have

$$\begin{aligned}
& \left| \{x \in B_1 : \mathcal{M}(|\nabla u|^2) > M^{2(k+1)}\} \right| = \left| \{x \in B_1 : \mathcal{M}(|\nabla u_1|^2) > M^{2k}\} \right| \\
& \leq \sum_{i=1}^k \varepsilon_1^i \left| \{x \in B_1 : \mathcal{M}(|f_1|^2) > \delta^2 M^{2(k-i)}\} \right| + \varepsilon_1^k \left| \{x \in B_1 : \mathcal{M}(|\nabla u_1|^2) > 1\} \right| \\
& \leq \sum_{i=1}^{k+1} \varepsilon_1^i \left| \{x \in B_1 : \mathcal{M}(|f|^2) > \delta^2 M^{2(k+1-i)}\} \right| + \varepsilon_1^{k+1} \left| \{x \in B_1 : \mathcal{M}(|\nabla u|^2) > 1\} \right|.
\end{aligned} \tag{76}$$

This estimate in turn completes the induction on k .

Finally, we give the proof of Theorem 2.

Proof of Theorem 2. We will consider the case $p > 2$ only. The case $p = 2$ is Theorem 1. Since $f \in L^p(B_6; \mathbb{R}^3)$, then $\mathcal{M}(|f|^2) \in L^{p/2}(B_6)$ by Lemma 2. In view of Lemma 1, there is a constant C depending only on δ, p, M such that

$$\sum_{k=1}^{\infty} M^{pk} \left| \{x \in B_1 : \mathcal{M}(|f|^2) > \delta^2 M^{2k}\} \right| \leq C \left\| \mathcal{M}(|f|^2) \right\|_{L^{p/2}(B_6)}^{p/2}. \tag{77}$$

Let us suppose that $\|f\|_{L^p(B_6)}$ and $\|\nabla u\|_{L^2(B_6)}$ are small enough such that

$$\begin{aligned}
& \sum_{k=1}^{\infty} M^{pk} \left| \{x \in B_1 : \mathcal{M}(|f|^2) > \delta^2 M^{2k}\} \right| \leq C \left\| \mathcal{M}(|f|^2) \right\|_{L^{p/2}(B_6)}^{p/2} \leq 1, \\
& \left| \{x \in B_6 : \mathcal{M}(|\nabla u|^2) > M^2\} \cap B_1 \right| < \varepsilon |B_1|.
\end{aligned} \tag{78}$$

These assumptions are reasonable, since we can multiply system (13) by a small constant depending on $\|f\|_{L^p(B_6)}$ and $\|\nabla u\|_{L^2(B_6)}$.

Let us compute

$$\begin{aligned}
& \sum_{k=0}^{\infty} M^{pk} \left| \{x \in B_1 : \mathcal{M}(|\nabla u|^2) > M^{2k}\} \right| \\
& \leq \sum_{k=1}^{\infty} M^{pk} \left(\sum_{i=1}^k \varepsilon_1^i \left| \{x \in B_1 : \mathcal{M}(|f|^2) > \delta^2 M^{2(k-i)}\} \right| \right. \\
& \quad \left. + \varepsilon_1^k \left| \{x \in B_1 : \mathcal{M}(|\nabla u|^2) > 1\} \right| \right) \\
& = \sum_{i=1}^{\infty} (M^p \varepsilon_1)^i \left(\sum_{k=i}^{\infty} M^{p(k-i)} \left| \{x \in B_1 : \mathcal{M}(|f|^2) \right. \right. \\
& \quad \left. \left. > \delta^2 M^{2(k-i)}\} \right| \right) + \sum_{k=1}^{\infty} (M^p \varepsilon_1)^k \left| \{x \in B_1 : \mathcal{M}(|\nabla u|^2) > 1\} \right| \\
& \leq C \sum_{k=1}^{\infty} (M^p \varepsilon_1)^k < \infty,
\end{aligned} \tag{79}$$

where we used Lemma 10 and selected ε_1 such that $M^p \varepsilon_1 < 1$. Then by Lemma 1, we get $\mathcal{M}(|\nabla u|^2) \in L^{p/2}(B_1)$, and this gives $u \in W^{1,p}(B_1; \mathbb{R}^3)$. The proof is now completed.

4.3. Proof of Theorem 3. Now we give the proof of Theorem 3

Proof of Theorem 3. Since $\nabla \cdot u = 0$ in B_6 , then, by Lemma 4, there exists $w \in W^{1,p}(B_6; \mathbb{R}^3)$ satisfying $\nabla \times w = u$ and $\|w\|_{W^{1,p}(B_6)} \leq C \|u\|_{L^p(B_6)}$. Thus, we can rewrite system (16) as

$$\begin{aligned}
& \nabla \times (A(x) \nabla \times u) = \nabla \times (f - w), \\
& \nabla \cdot u = 0, \quad \text{in } \Omega.
\end{aligned} \tag{80}$$

Thus, we can use Theorem 2 and get

$$\begin{aligned}
& \|\nabla u\|_{L^p(B_1)} \leq C \left(\|u\|_{L^p(B_6)} + \|f - w\|_{L^p(B_6)} \right) \\
& \leq C \left(\|u\|_{L^p(B_6)} + \|f\|_{L^p(B_6)} + \|w\|_{L^p(B_6)} \right) \\
& \leq C \left(\|u\|_{L^p(B_6)} + \|f\|_{L^p(B_6)} \right).
\end{aligned} \tag{81}$$

This completes the proof.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Authors' Contributions

All authors contributed equally to this work.

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