Hindawi
Journal of Mathematics
Volume 2020, Article ID 1235842, 7 pages
https://doi.org/10.1155/2020/1235842

Research Article

Operator (p, η)-Convexity and Some Classical Inequalities

Chuanjun Zhang,[1] Muhammad Shoaib Saleem,[2] Waqas Nazeer,[3] Naqash Shoukat,[2] and Yongsheng Rao[4]

[1]*School of Mathematics and Big Data, Guizhou Normal College, Guiyang 550018, China*
[2]*Department of Mathematics, University of Okara, Okara, Pakistan*
[3]*Department of Mathematics, Government College University, Lahore, Pakistan*
[4]*Institute of Computing Science and Technology, Guangzhou University, 510006 Guangzhou, China*

Correspondence should be addressed to Waqas Nazeer; nazeer.waqas@gmail.com and Yongsheng Rao; rysheng@gzhu.edu.cn

Received 11 April 2020; Accepted 8 June 2020; Published 4 July 2020

Academic Editor: Efthymios G. Tsionas

In this paper, we will introduce the definition of operator (p, η)-convex functions, we will derive some basic properties for operator (p, η)-convex function, and also check the conditions under which operations' function preserves the operator (p, η)-convexity. Furthermore, we develop famous Hermite–Hadamard, Jensen type, Schur type, and Fejér's type inequalities for this generalized function.

1. Introduction and Preliminary

Convexity plays an essential part in optimization theory and nonlinear programming. Although, different results have been derived under convexity, most of the real-world problems are nonconvex in nature. So, it is always appreciable to study nonconvex functions, which are near to convex function approximately [1, 2].

In the twentieth century, many famous mathematicians give recognition of the subject of convex functions such as Jensen, Hermite, Holder, and Stolz [3–10]. Throughout the twentieth century, an exceptional research activity was carried out and important results were obtained in convex analysis, geometric functional analysis, and nonlinear programming [11–14]. Among the most important of all the inequalities related to convex function is doubtlessly the Hermite–Hadamard inequality:

$$u\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b u(l)\mathrm{d}l \leq \frac{u(a)+u(b)}{2}. \tag{1}$$

The above inequality is very useful in many mathematical contexts and also put up as a tool for demonstrating some interesting estimations, and the literature above inequality is famously known as Hermite–Hadamard inequality [15]. If u is concave, then the couple inequalities in (1) hold in reversed direction. For more studies of Hermite–Hadamard-type inequalities, we refer [8, 9, 16]. The weighted version of Hermite–Hadamard inequality is known as Fejér Inequality, and for the famous work on Fejér Inequality, we refer [17–25].

In [6], Dragomir obtained some Hermite–Hadamard inequalities, which hold for convex function of self-adjoint operators in Hilbert spaces and slaked applications for special cases of interest. For interesting works on operator convex functions, we refer [3, 5, 7].

For simplicity, now onward, we will utilize the given notations:

H is Hilbert space

$<.,.>$ is an inner product

$B(H) = \{C/C : H \longrightarrow H \text{ be bounded } \textit{linear} \text{ operator } \textit{linear operator}\}$

$B(H)^+$ is all positive operators in $B(H)$

K is a convex subset of $B(H)^+$

$\rho(D) = \left\{\lambda \in C\colon (D-\lambda E)^{-1} \in L(X)\right\}$

$Sp(D) = C/\rho(D)$

For $C, D \in K$, $[C, D]:\ = \{(1-s)C + sD\colon s \in [0,1]\}$.

Also, let $\eta\colon C \times C \longrightarrow D$ be a bifunction for appropriate $C, D \subseteq R$. Considering self-adjoint $C, D \in B(H)$, we write, for every $l \in H$, $C \leq D$if $< Cl, l > \ \leq \ < Dl, l >$.

If u is a function on $Sp(C)$ which is a real-valued continuous function and S is a bounded self-adjoint operator, for any $s \in Sp(C)$, then $u(s) \geq 0$ implies that $u(C) \geq 0$. Furthermore, if u and v are both real-valued function on $Sp(S)$ such that $u(s) \leq v(s)$ for any $s \in Sp(C)$, then $u(C) \leq v(C)$.

Definition 1 (see [6]). Assume $u\colon I \subseteq R \longrightarrow R$ be a function, and we call it the operator convex function, if

$$u(sC + (1-s)D) \leq su(C) + (1-s)u(D), \tag{2}$$

for all $s \in [0,1]$ and for every C and D, which are bounded self-adjoint operators in $B(H)$, and I contains spectra of C and D. The function u is called operator concave if the above inequality is reversed.

Definition 2 (see [4]). Considering $u\colon I \longrightarrow \mathbb{R}$ a function, it is called η-convex function if the following inequality holds:

$$u(sl + (1-s)m) \leq u(m) + s\eta(u(l), u(m)), \tag{3}$$

where $s \in [0,1]$ and for all $l, m \in I$.

Definition 3 (see [26]). Let $u\colon I \longrightarrow \mathbb{R}$ be a function, and we call it operator η-convex function, if the next inequality is maintained,

$$u(sC + (1-s)D) \leq u(C) + s\eta(u(C), u(D)), \tag{4}$$

for all $s \in [0,1]$ and for every C and D, which are bounded self-adjoint operators in $B(H)$, where I contains spectra of C and D. The above function u is called operator η-concave function, if the above inequality is reversed.

Remark 1. Equation (4) reduces to the operator convex function for $\eta(l, m) = l - m$.

Definition 4 (see [27]). Suppose a function $u\colon I \longrightarrow \mathbb{R}$, and we call it p-convex function, if

$$u\left(\left[s^p + (1-s)m^p\right]^{1/p}\right) \leq su(l) + (1-s)u(m), \tag{5}$$

for all $l, m \in I$, $s \in [0,1]$, and I is a p-convex set.

Definition 5. Let $\eta\colon C \times C \longrightarrow D$ be a bifunction for appropriate $C, D \subseteq \mathbb{R}$ and I be a p-convex set; then, we call $u\colon I \longrightarrow \mathbb{R}\,(p, \eta)$-convex function, if

$$u\left(\left[sl^p + (1-s)m^p\right]^{1/p}\right) \leq u(m) + s\eta(u(l), v(m)), \tag{6}$$

for all $l, m \in I$ and $s \in [0,1]$.

The paper is organized as follows. Section 2 is devoted for some basic properties, and Section 2.1 is devoted to Schur-type inequality for operator (p, η)-convexity. However, Sections 2.2–2.4 are devoted for Hermite–Hadamard-, Jensen-, and Fejér-type inequalities, respectively.

2. Basic Properties

Now, we are ready to set forth the definition of operator (p, η)-convex function.

Definition 6. Considering $u\colon I \longrightarrow \mathbb{R}$ a function, we call it operator (p, η)-convex function, if the following inequality is maintained:

$$u\left(\left[sC^p + (1-s)D^p\right]^{1/p}\right) \leq u(D) + s\eta(u(C), u(D)), \tag{7}$$

for all $s \in [0,1]$ and for every C and D which are bounded self-adjoint operators in $B(H)$, where I contains spectra of C and D.

The above function u in (7) is known as operator (p, η)-concave function, if the above inequality is reversed.

Example 1. Let $u\colon I \longrightarrow R$ be a function, where $u(C) = C^p$ and $\eta(C, D) \geq C - D$ also $C \geq 0$; then, u is operator (p, η)-convex function.

Proof. Take

$$\begin{aligned} u\left[\left(sC^p + (1-s)D^p\right)^{1/p}\right] &= sC^p + (1-s)D^p \\ &= sC^p + D^p - sD^p \\ &= D^p + sC^p - sD^p \\ &= D^p + s\left(C^p - D^p\right) \\ &= D^p + s(u(C) - u(D)) \\ &\leq u(D) + s\eta(u(C), u(D)). \end{aligned} \tag{8}$$

Hence, u is an operator (p, η)-convex function. □

Proposition 1. *Considering $u, v\colon I \longrightarrow R$ as two operators (p, η) convex functions, the following holds:*

(i) If η is additive, then $u + v$ is operator (p, η)-convex function

(ii) If η is nonnegatively homogenous, then, for any $c \geq 0$, $cu\colon I \longrightarrow R$ is an operator (p, η)-convex function

Proof.

(i) Using operator (p, η)-convexity, we have

$$u\left(\left[sC^p + (1-s)D^p\right]^{1/p}\right) \leq (u(C) + s\eta(u(C), u(D))), \tag{9}$$

$$v\left(\left[sC^p + (1-s)D^p\right]^{1/p}\right) \leq v(D) + s\eta(v(C), v(D)), \tag{10}$$

for all C, D and $s \in [0,1]$, where I contains the spectra of C and D.

By summing up the above inequalities (9) and (10),

$$\begin{aligned}(u+v)\left(\left[sC^{p}+(1-s)D^{p}\right]^{1/p}\right) &= [u(D)+s\eta(u(C),u(D))]+[v(D)+s\eta(v(C),v(D))]\\ &\leq v(D)+v(D)+s[\eta(u(C),u(D))+\eta(u(C),u(D))]\\ &= u(D)+v(D)+s[\eta(u(C)+v(D),u(D)+v(D))]\\ &= (u+v)(D)+[\eta((u+v)(C),(u+v)(D))],\end{aligned} \tag{11}$$

implies that $u+v$ is an operator (p,η)-convex.

(ii) Consider

$$\begin{aligned}(cu)\left(\left[sC^{p}+(1-s)D^{p}\right]^{1/p}\right) &= cu\left(\left[sC^{p}+(1-s)D^{p}\right]^{1/p}\right)\\ &\leq c[u(D)+s\eta(u(C),u(D))]\\ &= cu(D)+s\eta(u(C),u(D))\\ &= cu(D)+s\eta(cu(C),cu(D))\\ &= (cu)(D)+s\eta((cu)(U),(cu)(D)),\end{aligned} \tag{12}$$

implies that cu is an operator (p,η)-convex function. □

Theorem 1. *Assume $u_j\colon I \longrightarrow R$, $j\in J$, is the nonempty collection of operator (p,η)-convex functions such that*

(a) There exist $\alpha\in[0,\infty)$ and $\beta\in[-1,\infty)$ such that $\eta(C,D)=\alpha C+\beta D$ for all C, D whose spectra contained in I

(b) For each $C\in I$, $sup_{j\in J}u_j(C)$ exists in R; then, $u\colon I\longrightarrow R$ is defined by $u(C)=sup_{j\in J}u_j(C)$ for each $C\in I$ is operator (p,η)-convex function.

Proof. For any $C,D\in I$ and $s\in[0,1]$, we have

$$\begin{aligned}u\left[sC^{p}+(1-s)D^{p}\right]^{1/p} &= \sup_{j\in J}u_j\left[sC^{p}+(1-s)D^{p}\right]^{1/p}\\ &\leq \sup_{j\in J}\left(u_j(D)+s\eta\left(u_j(C),u_j(D)\right)\right)\\ &= \sup_{j\in J}\left(u_j(D)+s\left(\alpha u_j(C)+\beta u_j(D)\right)\right)\\ &= \sup_{j\in J}(1+\beta s)u_j(D)+\alpha s u_j(C)\\ &\leq (1+\beta s)\sup_{j\in J}u_j(D)+\alpha s\sup_{j\in J}u_j(C)\\ &= (1+\beta s)u(D)+\alpha s u(C)\\ &= u(D)+s(\alpha u(C)+\beta u(D))\\ &= u(C)+s\eta(u(C),u(C)).\end{aligned} \tag{13}$$

□

2.1. Schur-Type Inequality

Theorem 2. *Let $\eta\colon C\times C\longrightarrow B$ be a bifunction for appropriate $C,D\subseteq R$ and let u be a function defined on interval I such that l is operator (p,η)-convex function. Then, for all $C_1,C_2,C_3\in I$ such that $C_1<C_2<C_3$ and $C_3^p-C_1^p$, $C_3^p-C_2^p, C_2^p-C_1^p\in(0,1)$, the following inequality holds:*

$$u(C_3)\left(C_3^p-C_1^p\right)-u(C_2)\left(C_3^p-C_1^p\right)+\left(C_3^p-C_2^p\right)\eta(u(C_1),u(C_3))\geq 0. \tag{14}$$

Proof. Let u be an operator (p,η)-convex function and let $C_1,C_2,C_3\in I$ be given. Then, we have

$$\begin{aligned}&\frac{C_3^p-C_2^p}{C_2^p-C_1^p},\frac{C_2^p-C_1^p}{C_3^p-C_1^p}\in(0,1)\\ &\frac{C_3^p-C_2^p}{C_3^p-C_1^p}+\frac{C_2^p-C_1^p}{C_3^p-C_1^p}=1.\end{aligned} \tag{15}$$

Invoking (4), for $s=(C_3^p-C_2^p/C_3^p-C_1^p)$, $C=C_1$, and $D=C_3$, we have $C_2^p=sC^p+(1-s)D^p$ and

$$u(C_2)\leq u(C_3)+\frac{C_3^p-C_2^p}{C_3^p-C_1^p}\eta(u(C_1),u(C_3)). \tag{16}$$

Assuming $C_3^p-C_1^p>0$ and after the multiplication on the above inequality by $C_3^p-C_1^p$, we will obtain inequality (14). □

2.2. Hermite–Hadamard-Type Inequalities. Next, we employ the Hermite–Hadmard-type inequality for the above said generalization.

Theorem 3. *Assume $u\colon I\longrightarrow R$ be operator (p,η)-convex function for any C and D, whose spectra is contained in I with condition $C<D$; then, the next estimate holds:*

$$\begin{aligned}&u\left(\frac{C^p+D^p}{2}\right)^{1/p}-\frac{p}{2(D^p-C^p)}\int_a^b u^{p-1}\eta\\ &\cdot\left(u\left(C^p+D^p-u^p\right)^{1/p},u(l)\right)\mathrm{d}l\\ &\leq\frac{p}{D^p-C^p}\int_a^b l^{p-1}u(l)\mathrm{d}l\\ &\leq\frac{u(C)+u(D)}{2}+\frac{1}{4}[\eta(u(C),u(D))+\eta(u(D),u(C))].\end{aligned} \tag{17}$$

Proof. Take $S^p = sC^p + (1-s)D^p$ and $T^p = (1-s)C^p + sD^p$, which implies

$$\frac{C^p + D^p}{2} = \frac{S^p + T^p}{2},$$
$$u\left(\frac{C^p + Dp}{2}\right)^{1/p} = u\left(\frac{S^p + T^p}{2}\right)^{1/p}. \tag{18}$$

By definition of operator (p,η)-convex function, we have

$$u\left(\frac{C^p + D^p}{2}\right)^{1/p} = u\left(\frac{1}{2}\left(\left(sC^p + (1-s)D^p\right)^{1/p}\right)^p + \frac{1}{2}\left(\left((1-s)C^p + sC^p\right)^{1/p}\right)^p\right)^{1/p}$$
$$\leq u\left((1-s)C^p + sV^p\right)^{1/p} + \frac{1}{2}\eta\left(u\left(sC^p + (1-s)D^p\right)^{1/p}, u\left((1-s)C^p + sV^p\right)^{1/p}\right). \tag{19}$$

Integrating the above inequality w.r.t "x" on $[0,1]$, we will obtain

$$u\left(\frac{C^p + D^p}{2}\right)^{1/p} \leq \int_0^1 u\left((1-s)C^p + sV^p\right)^{1/p}\mathrm{d}s + \frac{1}{2}\int_0^1 \eta\left(u\left(sC^p + (1-s)D^p\right)^{1/p}, u\left((1-s)C^p + sV^p\right)^{1/p}\right)\mathrm{d}s, \tag{20}$$

which implies

$$u\left(\frac{C^p + D^p}{2}\right)^{1/p} - \frac{p}{2(D^p - C^p)}\int_a^b l^{p-1}\eta \cdot \left(u\left(C^p + D^p - l^p\right)^{1/p}, u(l)\right)\mathrm{d}l \leq \frac{p}{D^p - C^p}\int_a^b l^{p-1}u(l)\mathrm{d}l. \tag{21}$$

Now,

$$\int_a^b l^{p-1}u(l)\mathrm{d}l = \frac{D^p - C^p}{p}\int_0^1 u\left(sC^p + (1-s)D^p\right)^{1/p}\mathrm{d}s \leq \frac{D^p - C^p}{p}\left(u(b) + \int_0^1 s\eta(u(C), u(D))\mathrm{d}s\right), \tag{22}$$

which implies

$$\frac{p}{D^p - C^p}\int_a^b l^{p-1}u(l)\mathrm{d}l \leq u(D) + \int_0^1 s\eta(u(C), u(D))\mathrm{d}s. \tag{23}$$

Similarly,

$$\frac{p}{D^p - C^p}\int_a^b l^{p-1}u(l)\mathrm{d}l \leq u(C) + \int_0^1 s\eta(u(D), u(C))\mathrm{d}s. \tag{24}$$

Summing up (21) and (23) yields

$$\frac{p}{D^p - C^p}\int_a^b l^{p-1}u(l)\mathrm{d}l \leq \frac{u(C) + u(D)}{2} + \frac{1}{4}\left[\eta(u(C), u(D)) + \eta(u(D), u(C))\right]. \tag{25}$$

Combining (21) and (25) and small calculation yields (17). □

Remark 2. (17) is the classical Hermite–Hadamard-type inequality for the operator convex function for $\eta(l,m) = l - m$ and $p = 1$.

2.3. Jensen-Type Inequalities

Lemma 1. *Suppose u: $I \longrightarrow R$ be an operator (p,η)-convex function, for C_1 and C_2, where I contains the spectra of C and D and $\alpha_1 + \alpha_2 = 1$, and we have*

$$u(\alpha_1 C_1 + \alpha_2 C_2) \leq u(C_2) + \alpha\eta(u(C_1), u(C_2)). \tag{26}$$

Also, when $n > 2$, for $C_1, C_2, \ldots, C_n$, whose spectra is contained in I, where $\sum_{i=1}^n \alpha_i = 1$ and $T_i = \sum_{j=1}^i \alpha_j$, we have

$$u\left(\sum_{i=1}^n \alpha_i C_i^p\right)^{1/p} = u\left(T_{n-1}\left(\sum_{i=1}^{n-1}\frac{\alpha_i}{T_{n-1}}C_i^p\right)^{1/p} + \alpha_n C_n\right) \leq u(C_n) + T_{n-1}\eta\left(u\left(\sum_{i=1}^{n-1}\frac{\alpha_i}{T_{n-1}}C_i^p\right)^{1/p}, u(C_n)\right). \tag{27}$$

Now, in the proof of next theorem, we will utilize the above lemma.

Theorem 4 (Jensen-type inequality). *Let $w_1, w_2, \ldots, w_n \in R^+$ with $n \geq 2$ and for $C_1, C_2, \ldots, C_n$, whose spectra is contained in I. Let u: $I \longrightarrow R$ be an operator (p,η)-convex function and η be nondecreasing and nonnegatively sublinear in the first variable; then, we have the following inequality:*

$$u\left(\left[\frac{1}{W_n}\sum_{i=1}^{n} w_i C_i^p\right]^{1/p}\right) \le u(C_n) + \sum_{i=1}^{n}\left(\frac{W_i}{W_n}\right)\eta_u \cdot (C_i, C_{i+1}, \ldots, C_n), \tag{28}$$

where $W_n = \sum_{i=1}^{n} w_i$, *also*

$$\eta_u(C_i, C_{i+1}, \ldots, C_n) = \eta(\eta_l(C_i, C_{i+1}, \ldots, C_{n-1}), u(C_n)), \tag{29}$$

and $\eta_u(C) = u(C)$ *for all* C *whose spectra contained in* I.

Proof. Since η is nondecreasing and nonnegatively sublinear in the first variable, so from the above lemma it yields that

$$\begin{aligned}
u\left(\left[\frac{1}{W_n}\sum_{i=1}^{n} w_i C_i^p\right]^{1/p}\right) &= u\left(\left[\frac{w_n}{W_n}C_n^p + \sum_{i=1}^{n-1}\frac{w_i}{W_n}C_i^p\right]^{1/p}\right)\\
&= u\left(\left[\frac{W_{n-1}}{W_n}\sum_{i=1}^{n-1}\frac{w_i}{W_{n-1}}C_i^p + \frac{w_n}{W_n}C_n^p\right]^{1/p}\right)\\
&\le u(C_n) + \frac{W_{n-1}}{W_n}\eta\left(u\left(\left[\sum_{i=1}^{n-1}\frac{w_i}{W_{n-1}}C_i^p\right]^{1/p}\right), u(C_n)\right)\\
&= u(C_n) + \frac{W_{n-1}}{W_n}\eta\left(u\left[\frac{W_{n-2}}{W_{n-1}}\sum_{i=1}^{n-2}\frac{w_i}{W_{n-2}}C_i^p + \frac{w_{n-1}}{W_{n-1}}C_{n-1}^p\right]^{1/p}, u(C_n)\right)\\
&\le u(C_n) + \frac{W_{n-1}}{W_n}\eta\left(u(u_{n-1}) + \frac{W_{n-2}}{W_{n-1}}\eta\left(u\left[\sum_{i=1}^{n-2}\frac{w_i}{W_{n-2}}C_i^p\right]^{1/p}, u(C_{n-1})\right), u(C_n)\right)\\
&\le u(C_n) + \frac{W_{n-1}}{W_n}\eta(u(C_{n-1}), u(C_n))\\
&\quad + \frac{W_{n-2}}{W_n}\eta\left(\eta\left(u\left[\sum_{i=1}^{n-2}\frac{w_i}{W_{n-2}}C_i^p\right]^{1/p}, u(l_{n-1})\right), u(l_n)\right)\\
&\;\;\vdots\\
&\le u(C_n) + \frac{W_{n-1}}{W_n}\eta(u(C_{n-1}), u(C_n)) + \frac{W_{n-2}}{W_n}\eta(\eta(u(C_{n-2}), u(C_n)), u(C_n))\\
&\quad + \cdots + \frac{W_1}{Wn}\eta(\eta(\cdots\eta(u(C_1), u(C_2)), u(C_3)\cdots), u((C_{n-1}), u(C_n)))\\
&= u(C_n) + \frac{W_{n-1}}{w_n}\eta_u(C_{n-1}, C_n) + \frac{W_{n-2}}{W_n}\eta_u(C_{n-2}, C_{n-1}, C_n)\\
&\quad + \cdots + \frac{W_1}{W_n}\eta_u(C_1, C_2, \ldots, C_{n-1}, C_n)\\
&= u(C)_n + \sum_{i=1}^{n-1}\left(\frac{W_i}{W_n}\right)\eta_u(C_i, C_{i+1}, \ldots, C_n).
\end{aligned} \tag{30}$$

Hence, the proof is completed. □

Remark 3 (28) is the Jensen-type inequality for operator η-convex functions for $p = 1$.

Remark 4 (28) is the Jensen-type inequality for the operator convex function for $p = 1$ and $\eta(l, m) = l - m$.

2.4. Fejér-Type Inequality

Theorem 5. *Let u, v be nonnegative operator (p, η)-convex functions $a, b \in IC < D$ such that $uv \in L_1[a, b]$; then,*

$$\frac{p}{D^p - C^p}\int_a^b u^{p-1}u(l)v(l)\mathrm{d}l \le C(C, D) + \frac{1}{2}D(C, D), \quad (31)$$

where

$$C(C, D) = u(D)Cv(D) + \frac{1}{3}\eta(u(C), u(D))\eta(v(C), v(D)),$$
$$D(C, D) = v(D)\eta(v(C), v(D)) + v(D)\eta(u(C), u(D)). \quad (32)$$

Proof. Since u and v are operator (p, η)-convex functions, we have

$$u\left(\left[sU^p + (1-s)D^p\right]^{1/p}\right) \le u(D) + s\eta(u(C), u(D)),$$
$$v\left(\left[sU^p + (1-s)D^p\right]^{1/p}\right) \le v(D) + s\eta(v(C), v(D)), \quad (33)$$

for all $s \in [C, D]$. Since u and v are nonnegative, so

$$\begin{aligned} &u\left(\left[sU^p + (1-s)D^p\right]^{1/p}\right)v\left(\left[sU^p + (1-s)D^p\right]^{1/p}\right)\\ &\le u(D)v(D) + su(D)\eta(v(C), v(D)) + sv(D)\eta(u(C), u(D))\\ &\quad + s^2\eta(u(C), u(D))\eta(v(C), v(D)). \end{aligned} \quad (34)$$

Integrating (34) over $(0, 1)$, we will obtain the following inequality:

$$\begin{aligned} &\int_0^1 u\left(\left[sU^p + (1-s)D^p\right]^{1/p}\right)v\left(\left[sU^p + (1-s)D^p\right]^{1/p}\right)\mathrm{d}s\\ &\le \int_0^1 u(D)v(D)\mathrm{d}s + \int_0^1 su(D)\eta(v(C), v(D))\mathrm{d}s\\ &\quad + \int_0^1 sv(D)\eta(u(C), u(D))\mathrm{d}s\\ &\quad + \int_0^1 s^2\eta(u(C), u(D))\eta(v(C), v(D))\mathrm{d}s. \end{aligned} \quad (35)$$

Setting $u = [sC^p + (1-s)D^p]^{1/p}$, we obtain

$$\begin{aligned} \frac{p}{D^p - C^p}\int_a^b u^{p-1}u(l)v(l)\mathrm{d}l &\le u(D)v(D) + \frac{1}{2}u(D)\eta(v(C), v(D))\\ &\quad + \frac{1}{2}v(D)\eta(u(C), u(D)) + \frac{1}{3}\eta(u(C), u(D))\eta(v(C), v(D)). \end{aligned} \quad (36)$$

Then,

$$\frac{p}{D^p - C^p}\int_a^b u^{p-1}u(l)g(l)\mathrm{d}l \le C(C, D) + \frac{1}{2}D(C, D). \quad (37)$$

□

Remark 5. If we put $p = 1$ and $\eta(l, m) = l - m$ in (31), then it reduces for operator convex functions.

3. Conclusion

In this report, we introduced the definition of operator (p, η)-convex functions and derived some basic properties for operator (p, η)-convex function. We also gave the conditions under which operations' function preserves the operator (p, η)-convexity. Furthermore, we developed famous Hermite–Hadamard, Jensen-type, Schur-type, and Fejér-type inequalities for this generalized function.

Data Availability

All data used in this study are included within this paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors have equally contributed to the article.

Acknowledgments

This work was supported by the Doctoral Program of Guizhou Normal College in 2020 (no. 2020BS001), Higher Education Content and Curriculum System Reform Project of Guizhou Province in 2019 (no. 2019083), Specialized Fund for Science and Technology Platform and Talent Team Project of Guizhou Province (no. QianKeHePingTaiRenCai [2016] 5609), the Key Disciplines of Guizhou Province–Computer Science and Technology (ZDXK [2018]007), First-Class C Discipline Project of Guizhou Normal College in 2019 (no. 2019YLXKC02), and the Key Supported Disciplines of Guizhou Province–Computer Application Technology (no. QianXueWeiHeZi ZDXK [2016]20).

References

[1] Y. C. Kwun, G. Farid, W. Nazeer, S. Ullah, and S. M. Kang, "Generalized riemann-liouville *k*-fractional integrals associated with ostrowski type inequalities and error bounds of hadamard inequalities," *IEEE Access*, vol. 6, pp. 64946–64953, 2018.

[2] S. M. Kang, G. Farid, W. Nazeer, and B. Tariq, "Hadamard and Fejr-Hadamard inequalities for extended generalized fractional integrals involving special functions," *Journal of Inequalities and Applications*, vol. 2018, no. 1, p. 119, 2018.

[3] B. Li, "Refinements of Hermite Hadamardhs type inequalities for operator convex functions," *International Journal of Contemporary Mathematical*, vol. 8, pp. 9–12, 2013.

[4] M. R. Delavar and S. S. Dragomir, "Trapezoidal type inequalities related to *h*-convex functions with applications. Revista de la Real Academia de Ciencias Exactas, Fsicas y Naturales," *Serie A. Matemticas*, vol. 113, no. 2, pp. 1487–1498, 2019.

[5] S. S. Dragomir, "Hermite-Hadamard's type inequalities for convex functions of selfadjoint operators in Hilbert spaces," *Linear Algebra and Its Applications*, vol. 436, no. 5, pp. 1503–1515, 2012.

[6] A. O. Akdemir, E. Set, M. E. zdemir, and . Yildiz, "On some new inequalities of Hadamard type for *h*-convex functions," *AIP Conference Proceedings*, vol. 1470, no. 1, pp. 35–38, 2012.

[7] M. Grbz and M. E. Zdemir, "On some inequalities for product of different kinds of convex functions," *Turkish Journal of Science*, vol. 5, no. 1, pp. 23–27, 2020.

[8] A. O. Akdemir, M. E. zdemir, and S. Varoanec, "On some inequalities for *h*-concave functions," *Mathematical and Computer Modelling*, vol. 55, no. 3-4, pp. 746–753, 2012.

[9] M. E. Ozdemir, A. O. Akdemir, and E. Set, "On (h-m)-convexity and Hadamard-type inequalities," 2011, https://arxiv.org/abs/1103.6163.

[10] T. H. Dinh and K. T. B. Vo, "Some inequalities for operator *h*-convex functions," *Linear and Multilinear Algebra*, vol. 66, no. 3, pp. 580–592, 2018.

[11] S. Kang, G. Abbas, G. Farid, and W. Nazeer, "A generalized fejér-hadamard inequality for harmonically convex functions via generalized fractional integral operator and related results," *Mathematics*, vol. 6, no. 7, p. 122, 2018.

[12] G. Farid, W. Nazeer, M. Saleem, S. Mehmood, and S. Kang, "Bounds of Riemann-Liouville fractional integrals in general form via convex functions and their applications," *Mathematics*, vol. 6, no. 11, p. 248, 2018.

[13] Y. C. Kwun, M. S. Saleem, M. Ghafoor, W. Nazeer, and S. M. Kang, "Hermite-Hadamard-type inequalities for functions whose derivatives are η-convex via fractional integrals," *Journal of Inequalities and Applications*, vol. 2019, p. 44, 2019.

[14] G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, Cambridge University Press, Cambridge, MA, USA, 1952.

[15] J. E. Peajcariaac and Y. L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Press, Cambridge, MA, USA, 1992.

[16] A. W. Roberts, "Convex functions," in *Handbook of Convex Geometry*, pp. 1081–1104, Elsevier Science, Amsterdam, Netherlands, 1993.

[17] M. Z. Sarikaya, "On new Hermite Hadamard Fejr type integral inequalities," *Studia Universitatis Babeş-Bolyai Mathematica*, vol. 57, no. 3, pp. 377–386, 2012.

[18] M. Bombardelli and S. Varoanec, "Properties of *h*-convex functions related to the Hermite Hadamard Fejr in equalities," *Computers & Mathematics with Applications*, vol. 58, no. 9, pp. 1869–1877, 2009.

[19] I. Iscan, "Hermite-Hadamard-Fejer type inequalities for convex functions via fractional integrals," 2014, https://arxiv.org/abs/1409.5245.

[20] K.-L. Tseng, S.-R. Hwang, and S. Dragomir, "Fejér-type inequalities (I)," *Journal of Inequalities and Applications*, vol. 2010, no. 1, Article ID 531976, 2010.

[21] H. Chen and U. N. Katugampola, "Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for generalized fractional integrals," *Journal of Mathematical Analysis and Applications*, vol. 446, no. 2, pp. 1274–1291, 2017.

[22] M. Eshaghi, S. S. Dragomir, and M. R. Delavar, "An inequality related to etaeta-convex functions (II)," *International Journal of Nonlinear Analysis and Applications*, vol. 6, no. 2, pp. 27–33, 2015.

[23] M. Z. Sarkaya, F. Ertural, and F. Yldrm, "On the Hermite-Hadamard-Fejr type integral inequality for s-convex function," *Konuralp Journal of Mathematics*, vol. 6, no. 1, pp. 35–41, 2014.

[24] K. L. Tseng, S. R. Hwang, and S. S. Dragomir, "On some new inequalities of Hermite-Hadamard-Fejr type involving convex functions," *Demonstratio Mathematica*, vol. 40, no. 1, pp. 51–64, 2007.

[25] B. Micherda and T. Rajba, "On some Hermite-Hadamard-Fejér inequalities for (k, h)-convex functions," *Mathematical Inequalities & Applications*, vol. 15, no. 4, pp. 931–940, 2012.

[26] C. D. Horvath, "Contractibility and generalized convexity," *Journal of Mathematical Analysis and Applications*, vol. 156, no. 2, pp. 341–357, 1991.

[27] K. S. Zhang and J. P. Wan, "P-convex functions and their properties," *Pure and Applied Mathematics*, vol. 23, no. 1, pp. 130–133, 2007.