

## Research Article

# Numerical Approximation of Riccati Fractional Differential Equation in the Sense of Caputo-Type Fractional Derivative

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Received 15 June 2020; Revised 6 August 2020; Accepted 11 August 2020; Published 7 September 2020

Academic Editor: Ljubisa Kocinac

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The Riccati differential equation is a well-known nonlinear differential equation and has different applications in engineering and science domains, such as robust stabilization, stochastic realization theory, network synthesis, and optimal control, and in financial mathematics. In this study, we aim to approximate the solution of a fractional Riccati equation of order  $0 < \beta < 1$  with Atangana–Baleanu derivative (ABC). Our numerical scheme is based on Laplace transform (LT) and quadrature rule. We apply LT to the given fractional differential equation, which reduces it to an algebraic equation. The reduced equation is solved for the unknown in LT space. The solution of the original problem is retrieved by representing it as a Bromwich integral in the complex plane along a smooth curve. The Bromwich integral is approximated using the trapezoidal rule. Some numerical experiments are performed to validate our numerical scheme.

## 1. Introduction

In applied mathematics, the fractional differential equation (FDE) is an equation which contains derivatives of arbitrary order. Fractional derivatives first appeared in 1695 [1]. Recently, the research community took more interest in fractional calculus because of its applications in engineering and other sciences [2–4]. Many physical systems show fractional order behavior that may vary with time or space. Fractional derivatives have many kinds. The three important and most commonly used fractional derivatives are Grünwald–Letnikov derivative, Riemann–Liouville fractional derivative, and Caputo derivative [5]. However, these classical fractional derivatives have a singular kernel, and hence, they may face difficulties in describing the nonlocality of real-world dynamics. In order to handle the nonlocal systems in a better way, recently, new fractional derivatives with nonsingular kernels are defined such as the Caputo–Fabrizio (CF) derivative and Atangana–Baleanu (ABC) derivative [6, 7]. Fractional derivatives with nonsingular kernels become more valuable due to the fact that numerous phenomena cannot properly be modeled by fractional derivatives with singular kernels [8].

In this work, we aim to approximate a Riccati differential equation (RDE) with ABC derivative. RDEs have many applications such as random processes, optimal control, and diffusion process [9]. The RDE of fractional order has been studied by many authors; for example, in [10], the authors developed the Adomain decomposition method for the solution of RDE of fractional order. In [11], some analytic techniques are presented for the solution of RDE. The authors [12] obtained the solution of RDE using the differential transform method. In [13], the authors have developed a Laplace-transform Adomain decomposition method for the solution of RDE. Other works on the analytic solution of RDE can be found in [14, 15] and the references therein.

Most of the time, the exact/analytical solution of FDEs cannot be found, so numerical approximations must be utilized [16–19]. Numerous numerical methods have been developed for numerical approximation of FDEs such as the Chebyshev collocation method [20], variation iteration method [16], reproducing kernel Hilbert space method [18, 19], and homotopy perturbation method ([21] and references therein). The numerical solutions of fractional-order RDE have been studied by a large number of

researchers. For example, the authors [20] studied fractional-order RDE with ABC derivatives, and they established the existence and uniqueness results using Banach fixed-point theorem. A reproducing kernel Hilbert space method [19] for approximating RDEs and Bernoulli differential equations of fractional order with ABC derivative has been presented. In [18], the authors developed an iterative reproducing kernel Hilbert space method for numerical approximation of fractional RDE. The authors [22] studied the numerical solution of fractional-order RDE using the modified homotopy perturbation method. Numerical solution of fractional-order RDEs using Bernstein polynomials is considered [23]. A fractional Chebyshev finite difference method [24] for numerical investigation of RDE of fractional order is proposed. Jafari and Tajadodi [25] proposed a variational iteration method for solving the RDE of fractional order. The authors in [26] proposed a hybrid method based on the quasilinearization technique and reproducing kernel method for RDE of fractional order. In [27], the solution of fractional-order RDE is approximated using fractional polynomial approximations. The authors in [28] developed a method based on finite difference and Padé-variational iteration method for solving the RDE of non-integer order. A numerical method based on the path following methods and the Tau Legendre is presented [29] for the solution of fractional-order RDE. The authors of [30] proposed a modified variational iteration method based on Adomian polynomials for the solution of RDE. Khashan et al. [31] have utilized the Haar wavelets for the approximation of fractional-order RDE. The authors of [32] utilized the finite difference scheme for the approximation of RDE. Other works on the numerical approximation of the solution of fractional-order RDE can be found in [33–42] and references therein. In this work, we approximate the solution of fractional-order RDE with ABC derivative of the following form:

$$\begin{aligned} D_{\mathbf{x}}^{\beta} \chi(\mathbf{x}) &= F(\mathbf{x}, \chi(\mathbf{x})), \quad \mathbf{x} \in [0, 1], \beta \in [0, 1], \\ \chi(0) &= \alpha, \end{aligned} \quad (1)$$

here,  $D_{\mathbf{x}}^{\beta}$  denotes Atangana–Baleanu fractional derivative of order  $\beta$ .

## 2. Preliminaries

In this section, we present some basic results and definitions. For details about fractional calculus, we refer [43–49], and for details about modeling, we refer [50, 51] and references therein.

*Definition 1.* The Mittag–Leffler (ML) function with one parameter is defined as [52, 53]

$$E_{\beta}(\mathbf{x}) = \sum_{m=0}^{\infty} \frac{\mathbf{x}^m}{\Gamma(m\beta + 1)}, \quad \beta > 0, \quad -\infty < \mathbf{x} < \infty. \quad (2)$$

*Definition 2.* The two-parameter Mittag–Leffler (ML) function is defined as [53, 54]

$$E_{\beta, \mu}(\mathbf{x}) = \sum_{m=0}^{\infty} \frac{\mathbf{x}^m}{\Gamma(m\beta + \mu)}, \quad \beta > 0, \quad -\infty < \mathbf{x} < \infty. \quad (3)$$

*Definition 3.* The ABC fractional derivative of order  $0 < \beta < 1$  of  $\chi^{**} \in \mathcal{S}^1(a, b)$  with base point  $a$  at  $\mathbf{x} \in (a, b)$  is defined as [7]

$$D_{\mathbf{x}}^{\beta} \chi(\mathbf{x}) = \frac{M(\beta)}{1-\beta} \int_a^{\mathbf{x}} \chi'(s) E_{\beta} \left[ \frac{-\beta}{1-\beta} (\mathbf{x}-s)^{\beta} \right] ds, \quad (4)$$

where  $\mathcal{S}^1$  is a first-order Sobolev space equipped with  $L^2$ -norm over the region  $\Omega \subset \mathbb{R}$ , which is defined as

$$\mathcal{S}^1(\Omega) = \{ \chi \in L^2(\Omega); \chi' \in L^2(\Omega) \}, \quad (5)$$

and the term  $M(\beta)$  is given as

$$M(\beta) = 1 - \beta + \frac{\beta}{\Gamma(\beta)}. \quad (6)$$

*Definition 4.* The (ABC) fractional integral of order  $0 < \beta < 1$  of  $\chi \in \mathcal{S}^1(a, b)$  with base point  $a$  at  $\mathbf{x} \in (a, b)$  is defined as [7]

$$\mathcal{I}_{\mathbf{x}}^{\beta} \chi(\mathbf{x}) = \frac{1-\beta}{M(\beta)} \chi(\mathbf{x}) + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_a^{\mathbf{x}} \chi(s) (\mathbf{x}-s)^{\beta-1} ds. \quad (7)$$

*Definition 5.* The Laplace transform of a piecewise continuous function  $\chi(\mathbf{x})$  is defined as

$$\mathcal{L}\{\chi(\mathbf{x})\} = \hat{\chi}(s) = \int_0^{\infty} e^{-s\mathbf{x}} \chi(\mathbf{x}) d\mathbf{x}. \quad (8)$$

*Definition 6.* The LT of ML function with one parameter is defined as [4]

$$\mathcal{L}\{E_{\beta}(-\rho\mathbf{x}^{\beta})\} = \frac{s^{\beta-1}}{s^{\beta} + \rho}. \quad (9)$$

*Definition 7.* The LT of two-parameter ML function is defined as [4]

$$\mathcal{L}\{\mathbf{x}^{\mu-1} E_{\beta, \mu}(-\rho\mathbf{x}^{\beta})\} = \frac{s^{\beta-\mu}}{s^{\beta} + \rho}. \quad (10)$$

*Definition 8.* If  $0 < \beta \leq 1$ , then the LT of the ABC derivative is defined by [7]

$$\mathcal{L}\{D_{\mathbf{x}}^{\beta} \chi(\mathbf{x})\} = \frac{s^{\beta} \hat{\chi}(s) - s^{\beta-1} \chi(0)}{s^{\beta} (1-\beta) + \beta}. \quad (11)$$

### 3. Laplace Transform Method for Fractional Riccati Equation with ABC Derivative

In this section, we give a detailed description of our proposed numerical scheme for approximating the inverse Laplace transform. First, we will apply the Laplace transform to the given fractional problem, which will transform it to an algebraic equation. After solving the reduced equation, the solution of the original problem can be obtained by representing it as a contour integral in the left half of the complex plane. The trapezoidal rule is then utilized to approximate the contour integral. Applying the Laplace transform to equation (1), we obtain

$$\frac{s^\beta \widehat{\chi}(s) - s^{\beta-1} \chi(0)}{s^\beta (1 - \beta) + \beta} = \widehat{F}, \tag{12}$$

which can be written in simplified form as

$$\widehat{\chi}(s) = s^{-\beta} \widehat{G}(s), \tag{13}$$

where

$$\widehat{G}(s) = s^{\beta-1} \alpha + \widehat{F}(s^\beta (1 - \beta) + \beta). \tag{14}$$

In our method, first we represent the solution  $\chi(\mathbf{x})$  of original problem (1) as a contour integral:

$$\chi(\mathbf{x}) = \frac{1}{2\pi i} \int_{\Gamma} e^{s\mathbf{x}} \widehat{\chi}(s) ds, \tag{15}$$

where, for  $\text{Re } s \geq \omega$ ,  $\omega$  is appropriately large and  $\Gamma$  is an initially appropriately chosen line  $\Gamma_0$  perpendicular to the real axis in the complex plane, with  $\text{Im } s \rightarrow \pm\infty$ . Integral (15) is just the inverse transform of  $\widehat{\chi}(s)$ , with the condition that it must be analytic to the right of  $\Gamma_0$ . To make sure the contour of integration remains in the domain of analyticity of  $\widehat{\chi}(\mathbf{x})$ , we select  $\Gamma$  as a deformed contour in the set  $\Sigma_\phi^\vartheta = \{s \neq 0: |\arg s| < \phi\} \cup \{0\}$ , which behaves as a pair of asymptotes in the left half plane, with  $\text{Re } s \rightarrow -\infty$  when  $\text{Im } s \rightarrow \pm\infty$ , which force  $e^{s\mathbf{x}}$  to decay towards both ends of  $\Gamma$ . In our work, we choose  $\Gamma$  as

$$s(\lambda) = \vartheta + \delta(1 - \sin(\eta - i\lambda)), \quad \lambda \in \mathbb{R}, \quad (\Gamma), \tag{16}$$

where

$$\begin{aligned} \delta &> 0, \\ 0 &< \eta < \phi - \frac{\pi}{2}, \\ \vartheta &> 0. \end{aligned} \tag{17}$$

By writing  $s = x + iy$ , we notice that (16) is the left branch of the following hyperbola:

$$\left(\frac{x - \vartheta - \delta}{\delta \sin \eta}\right)^2 - \left(\frac{y}{\delta \cos \eta}\right)^2 = 1. \tag{18}$$

The asymptotes for (18) are  $y = \pm (x - \vartheta - \delta) \cot \eta$  and  $x$ -intercept at  $s = \vartheta + \delta(1 - \sin \eta)$ . Condition (17) confirms

that  $\Gamma$  lies in the sector  $\Sigma_\phi^\vartheta = \vartheta + \Sigma_\phi \subset \Sigma_\phi$  and grows into the left half plane. From (16) and (15), we obtain

$$\chi(\mathbf{x}) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{s(\lambda)\mathbf{x}} \widehat{\chi}(s(\lambda)) s'(\lambda) d\lambda. \tag{19}$$

The trapezoidal rule is used for the approximation of equation (19) with step  $k$  as follows:

$$\chi_k(\mathbf{x}) = \frac{k}{2\pi i} \sum_{j=-N}^N e^{s_j \mathbf{x}} \widehat{\chi}(s_j) s'_j, \tag{20}$$

where  $\lambda_j = jk$ ,  $s_j = s(\lambda_j)$ , and  $s'_j = s'(\lambda_j)$ .

### 4. Error Analysis

In the process of obtaining the solution of problem equations (1)–(13), the fractional integrodifferential equation is first transformed to an algebraic equation using Laplace transform, and this causes no error. The transformed equation is then solved for the unknown in the Laplace space. Finally, the solution is obtained using inverse Laplace transform via integral representation (19). The integral is then approximated using quadrature rule. In the process of approximating integral (19), convergence is achieved at different rates depending on the path  $\Gamma$ . In approximating integral (19), the convergence order relies on the step  $k$  of the quadrature rule and the time domain  $[t_0, T]$ . The proof for the order of quadrature error is given in the next theorem.

**Theorem 1** ([54] Theorem 2.1). *Let  $\chi(\mathbf{x})$  be the solution of (1) with  $\widehat{\chi}(s)$  being analytic in  $\Sigma_\phi^\vartheta$ . Let  $\Gamma \subset \Omega_r \subset \Sigma_\phi^\vartheta$ , and define  $b > 0$  by  $\cosh(b) = 1/\theta \tau_1 \sin(\eta)$ , where  $\tau_1 = t_0/T$ ,  $0 < \tau_0 < T$ , and  $0 < \theta < 1.0$  and let  $\delta = \theta \text{ liner } N/bT$ . Then, for equation (20), with  $k = (b/N) \leq (\text{liner}/\log 2)$ , we have  $|\chi(\mathbf{x}) - \chi_k(\mathbf{x})| \leq CQ e^{\vartheta \tau_1} l(\rho_r N) e^{-\mu N} (\|\chi_0\| + \|\widehat{F}\|_{\Sigma_\phi^\vartheta})$ , for  $\mu = \text{liner}(1 - \theta)/b$ ,  $\rho_r = \theta \text{ liner } \tau_1 \sin(\eta - r_1)/b$ ,  $\bar{r} = 2\pi r_1$ ,  $r_1 > 0$ ,  $\tau_0 \leq t \leq T$ ,  $C = C_{\eta, r_1, \beta_1}$ , and  $l(x) = \max(1, \log(1/x))$ . Hence, the error estimate for the proposed scheme is*

$$\begin{aligned} \text{error}_{\text{est}} &= |\chi_k(\mathbf{x}) - \chi(\mathbf{x})| \\ &= O(e^{-\mu N}) l(\rho_r N). \end{aligned} \tag{21}$$

### 5. Results and Discussion

Most of the time analytical methods cannot be applied to handle a real-world problem. So we need numerical methods to approximate the solutions of the problems. In this section, we consider fractional-order Riccati equations to validate our method.

*Problem 1.* Here, we consider the fractional Riccati equation as follows:

$$D_x^\beta \chi(\mathbf{x}) = 2 - \chi^2(\mathbf{x}), \quad \mathbf{x}, \beta \in [0, 1], \tag{22}$$

with  $\chi(0) = \sqrt{2}$ .

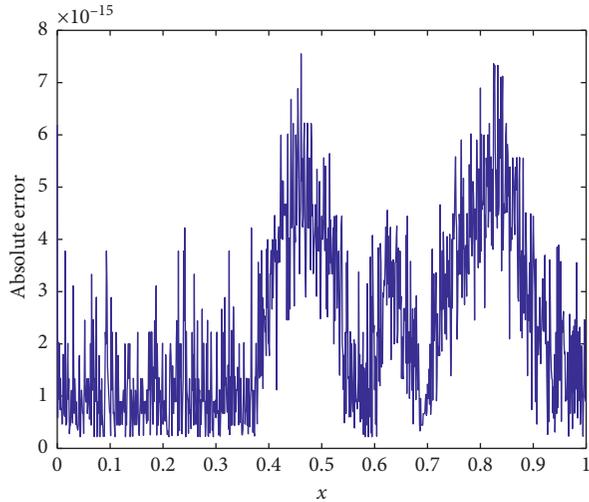
The problem has exact solution  $\chi(\mathbf{x}) = \sqrt{2}$ . In this experiment, the optimal parameters utilized are  $\theta = 0.10$ ,  $\eta = 0.1647$ ,  $\tau = (t_0/T)$ ,  $r = 0.2397$ ,  $\bar{r} = 2\pi r$ ,  $\vartheta = 2.0$ , and  $[t_0, T]$

TABLE 1: The results are obtained for different fractional orders and  $N = 250$ .

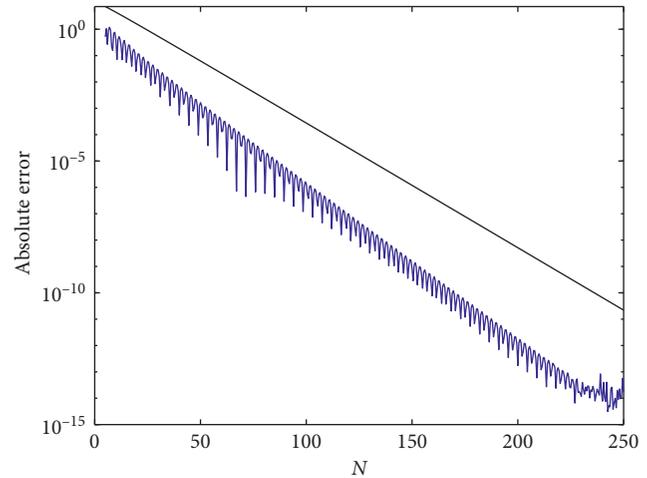
$x$	$\beta = 1$	$\beta = 0.9$	$\beta = 0.8$	$\beta = 0.5$	$\beta = 0.25$
0.1	1.414213562373095	1.414213562373095	1.414213562373095	1.414213562373095	1.414213562373095
0.2	1.414213562373094	1.414213562373094	1.414213562373094	1.414213562373094	1.414213562373094
0.3	1.414213562373096	1.414213562373095	1.414213562373096	1.414213562373095	1.414213562373096
0.4	1.414213562373100	1.414213562373100	1.414213562373100	1.414213562373100	1.414213562373100
0.5	1.414213562373101	1.414213562373101	1.414213562373101	1.414213562373101	1.414213562373101
0.6	1.414213562373092	1.414213562373093	1.414213562373092	1.414213562373093	1.414213562373093
0.7	1.414213562373096	1.414213562373096	1.414213562373097	1.414213562373096	1.414213562373097
0.8	1.414213562373100	1.414213562373100	1.414213562373101	1.414213562373100	1.414213562373101
0.9	1.414213562373099	1.414213562373100	1.414213562373099	1.414213562373100	1.414213562373100
1	1.414213562373098	1.414213562373098	1.414213562373097	1.414213562373097	1.414213562373097

TABLE 2: The absolute errors obtained for different fractional orders  $\beta$  and  $N$  at  $x = 1$ .

$N$	$\beta = 1$	$\beta = 0.9$	$\beta = 0.8$	$\beta = 0.5$	$\beta = 0.25$
30	$1.47 \times 10^{-2}$				
60	$1.76 \times 10^{-4}$				
90	$1.87 \times 10^{-6}$				
120	$2.56 \times 10^{-8}$				
150	$2.99 \times 10^{-10}$				
180	$4.34 \times 10^{-12}$				
210	$7.36 \times 10^{-14}$	$7.58 \times 10^{-14}$	$7.62 \times 10^{-14}$	$7.71 \times 10^{-14}$	$7.68 \times 10^{-14}$
250	$2.76 \times 10^{-15}$	$2.90 \times 10^{-15}$	$2.20 \times 10^{-15}$	$2.22 \times 10^{-15}$	$2.32 \times 10^{-15}$
[20]	$1.01 \times 10^{-16}$				



(a)



(b)

FIGURE 1: (a) The absolute error for fractional order  $\beta = 1$  and  $N = 250$  corresponding to Problem 1. (b) The comparison between absolute error and  $error_{est}$  at  $x = 1$ ,  $N = 250$ ,  $\theta = 0.10$ ,  $\eta = 0.1541$ ,  $r = 0.1387$ ,  $\bar{r} = 2\pi r$ ,  $\vartheta = 2.0$ , and  $\beta = 1$  for Problem 1. A good agreement between them is observed.

$= [0.5, 5]$ . The quadrature nodes are generated using the MATLAB commands  $\lambda = -N:k:N$ . The values of approximate solution for different fractional orders  $\beta$  and  $x \in [0, 1]$  are depicted in Table 1. The absolute errors for various quadrature nodes  $N$  and fractional order  $\beta$  are depicted in Table 2. The method produced almost exact

values for different values of fractional order  $\beta$ . The plots of absolute error for fractional orders  $\beta = 1$  are displayed in Figure 1(a), where Figure 1(b) shows the comparison between the absolute error and error estimate for the fractional order  $\beta = 1$ . Figures 2(a) and 2(b) show the error functions for fractional orders  $\beta = 1$  and  $\beta = 0.5$ ,

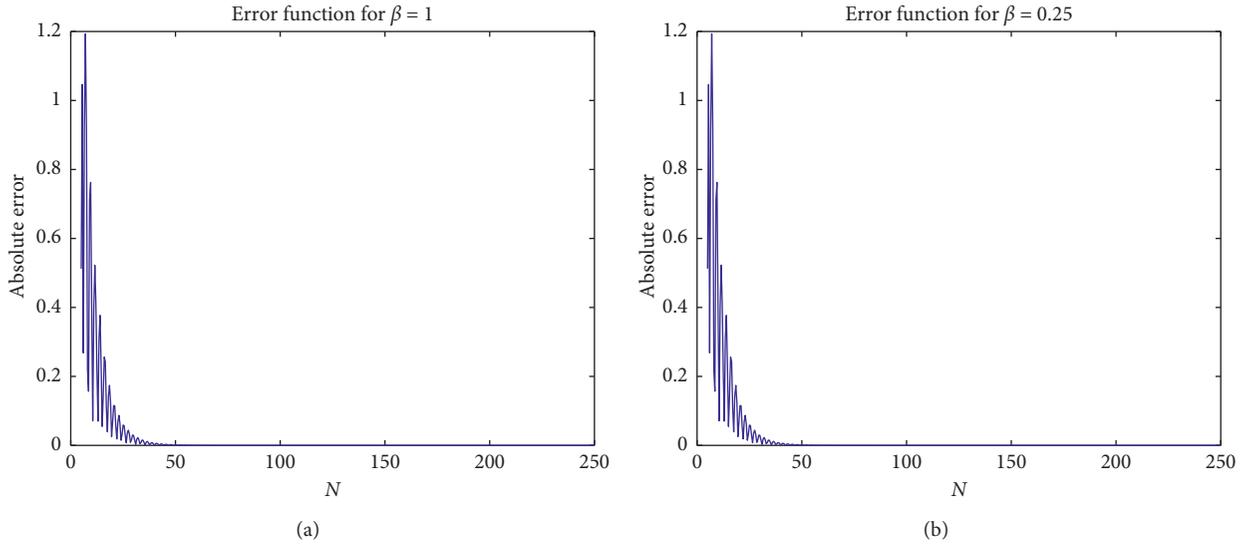


FIGURE 2: (a) The error function for fractional order  $\beta = 1$ . (b) The error function for  $\beta = 0.25$  corresponding to Problem 1.

TABLE 3: The results are obtained for different fractional orders and  $N = 130$ .

$t$	$\beta = 1$	$\beta = 0.9$	$\beta = 0.8$	$\beta = 0.7$
0.1	0.0999999999999997	0.231168929042977	0.378890254407207	0.540603871314406
0.2	0.1999999999999935	0.339457009420931	0.492611583158604	0.654981683985162
0.3	0.2999999999999814	0.442211492048623	0.594981117995457	0.752630554959894
0.4	0.400000000000399	0.541529712645641	0.690611916885552	0.840788989032917
0.5	0.500000000000783	0.638368560911693	0.781525188669415	0.922501829604263
0.6	0.600000000000226	0.733276357205215	0.868847629153083	0.999420047678484
0.7	0.7000000000001987	0.826607814998613	0.953292789025402	1.072567200353862
0.8	0.79999999997327	0.918610710161770	1.035350520844109	1.142633846360632
0.9	0.89999999996349	1.009467601183178	1.115376469957896	1.210114016809452
1	0.99999999994301	1.099318390756997	1.193639720785398	1.275376747522151

respectively. It can be seen that the method can solve fractional Riccati equation with ABC derivative efficiently.

**Problem 2.** Here, we consider the fractional Riccati equation as follows:

$$D_x^\beta \chi(\mathbf{x}) = \mathbf{x}^3 \chi^2(\mathbf{x}) - 2\mathbf{x}^4 \chi(\mathbf{x}) + \mathbf{x}^5 + 1, \quad \mathbf{x}, \beta \in [0, 1], \tag{23}$$

with  $\chi(0) = 0$ .

The problem has exact solution  $\chi(\mathbf{x}) = \mathbf{x}$ . In this experiment, the optimal parameters utilized are  $\theta = 0.10$ ,  $\eta = 0.1647$ ,  $\tau = (t_0/T)$ ,  $r = 0.2397$ ,  $\bar{r} = 2\pi r$ ,  $\vartheta = 3.0$ , and  $[t_0, T] = [0.5, 5]$ . The quadrature nodes are generated using the MATLAB commands  $\lambda = -N:k:N$ . The values of approximate solution for different fractional orders  $\beta$  and  $\mathbf{x} \in [0, 1]$  are depicted in Table 3. The method produced exact values for  $\beta = 1$ . The plots of numerical numerical solutions for different fractional orders are displayed in Figure 3, where Figure 4(a) shows the absolute error for the fractional order  $\beta = 1$  and Figure 4(b) shows the error function for  $\beta = 1$ .

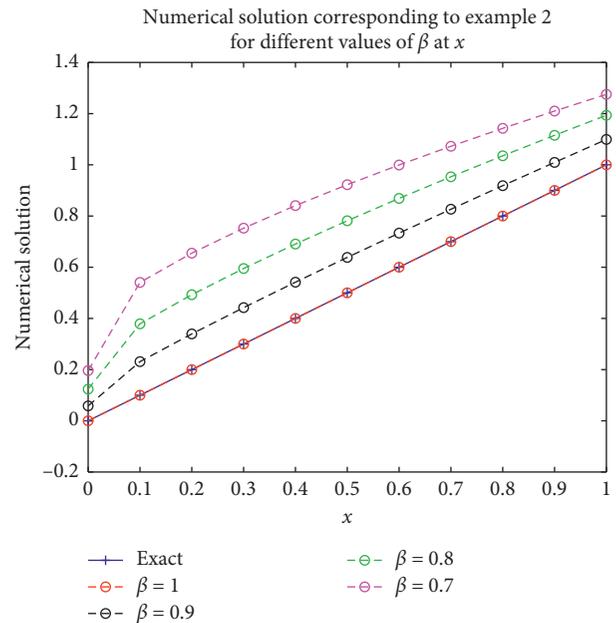


FIGURE 3: The numerical solutions for different fractional orders  $\beta$  are depicted corresponding to Problem 2.

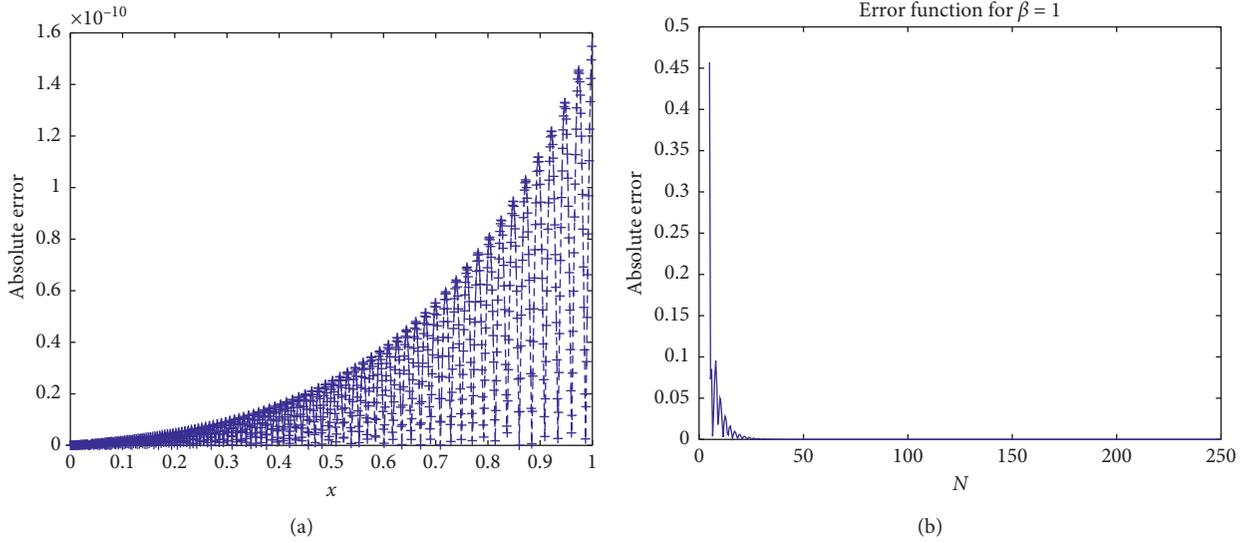


FIGURE 4: (a) The absolute error at  $\mathbf{x} = 1$  and  $\beta = 1$  for Problem 2. (b) The error function for  $\beta = 1$  and various quadrature nodes corresponding to Problem 2.

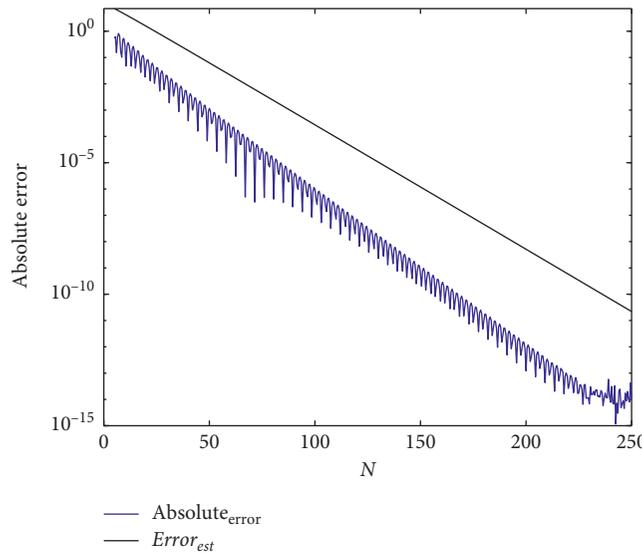


FIGURE 5: The comparison between absolute error and  $\text{error}_{\text{est}}$  at  $\mathbf{x} = 1$  and  $\beta = 1$  for Problem 3. A good agreement between them is observed.

**Problem 3.** Here, we consider the fractional Riccati equation as follows:

$$D_{\mathbf{x}}^{\beta} \chi(\mathbf{x}) = A_0(\mathbf{x}) + \chi(\mathbf{x}) - \chi^2(\mathbf{x}), \quad \mathbf{x}, \beta \in [0, 1], \quad (24)$$

with

$$\chi(0) = 1, \quad (25)$$

where,

$$A_0(\mathbf{x}) = \frac{\beta + 1}{\Gamma(\beta + 1)} \mathbf{x} \left( E_{\beta, 2} \left( \frac{-\beta}{1 - \beta} \mathbf{x}^{\beta} \right) - 1 \right) - (\mathbf{x}^{\beta+1} + 1) + (\mathbf{x}^{\beta+1} + 1)^2. \quad (26)$$

The problem has exact solution  $\chi(\mathbf{x}) = \mathbf{x}^{(\beta+1)} + 1$ . In this experiment, the same set of optimal parameters is utilized. The values of approximate solution for different fractional orders  $\beta$  and  $\mathbf{x} \in [0, 1]$  are depicted in Table 4. The absolute errors for various values of  $N$  and fractional order  $\beta$  are shown in Table 5. Figure 5 shows the comparison between the absolute error and  $\text{error}_{\text{est}}$ , and a good agreement between them is observed. The plots of numerical and exact solutions for different fractional orders are displayed in Figures 6–9. It can be seen that the proposed method has produced good results and the exact and numerical solutions are in good agreement. This shows that this method can

TABLE 4: The results are obtained for different fractional orders and  $N = 130$ .

$x$	$\beta = 1$	$\beta = 0.9$	$\beta = 0.8$	$\beta = 0.7$
0.1	1.010000000081090	1.013000508297990	1.016475393193733	1.020156483509860
0.2	1.039999999904037	1.048519612196495	1.057370650922991	1.065488607772575
0.3	1.090000000107037	1.104831237979493	1.119029345287758	1.130473075366759
0.4	1.160000000118295	1.181081596140041	1.199776210363150	1.212773114819303
0.5	1.250000000104207	1.276696291556891	1.298525747082721	1.310930820826873
0.6	1.359999999746213	1.391244013660800	1.414484253729630	1.423908271904983
0.7	1.48999999970782	1.524380587892006	1.547031402579680	1.550910987123471
0.8	1.639999999836653	1.675820412830372	1.695661169283945	1.691301889020929
0.9	1.810000000220293	1.845319910066929	1.859948222231272	1.844553239275344
1	1.99999999918102	2.032667074163037	2.039527028680209	2.010217216849950

TABLE 5: The results are obtained for different fractional orders and quadrature nodes  $N$ .

$N$	$\beta = 1$	$\beta = 0.9$	$\beta = 0.5$
30	$2.03 \times 10^{-2}$	$5.28 \times 10^{-2}$	$1.65 \times 10^{-1}$
60	$9.17 \times 10^{-5}$	$3.27 \times 10^{-2}$	$1.85 \times 10^{-1}$
90	$2.65 \times 10^{-6}$	$3.27 \times 10^{-2}$	$1.86 \times 10^{-1}$
120	$7.07 \times 10^{-8}$	$3.27 \times 10^{-2}$	$1.86 \times 10^{-1}$
150	$5.84 \times 10^{-10}$	$3.27 \times 10^{-2}$	$1.86 \times 10^{-1}$
180	$9.34 \times 10^{-12}$	$3.27 \times 10^{-2}$	$1.86 \times 10^{-1}$
210	$3.40 \times 10^{-13}$	$3.27 \times 10^{-2}$	$1.86 \times 10^{-1}$
250	$1.10 \times 10^{-14}$	$3.27 \times 10^{-2}$	$1.86 \times 10^{-1}$
252	$3.36 \times 10^{-15}$	$3.27 \times 10^{-2}$	$1.86 \times 10^{-1}$
[20]	$1.01 \times 10^{-15}$		

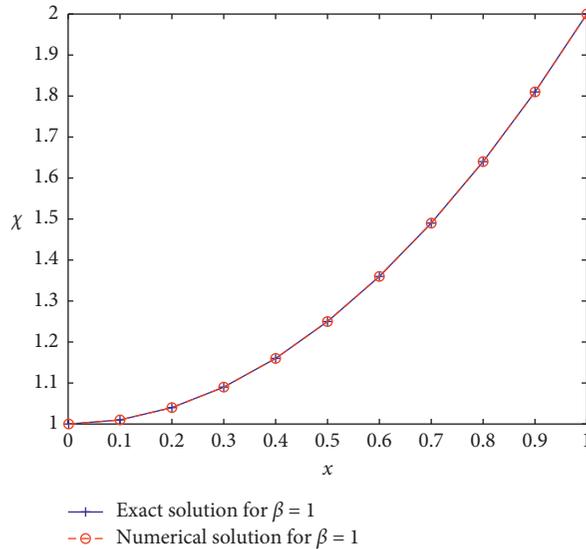


FIGURE 6: The numerical and exact solutions for fractional order  $\beta = 1$ .

solve fractional Riccati equation with ABC derivative efficiently.

$$D_x^\beta \chi(x) = -\chi(x) - \chi^2(x) + (E_\beta(-x^\beta))^2, \quad x, \beta \in [0, 1], \tag{27}$$

*Problem 4.* Here, we consider the fractional Riccati equation as follows:

with  $\chi(0) = 1$ .

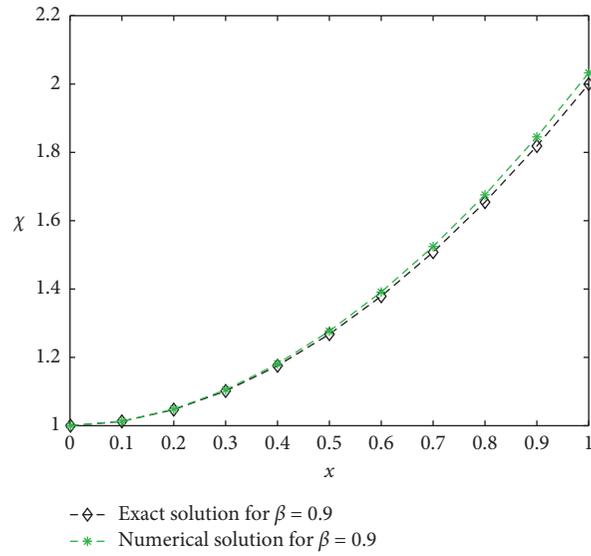


FIGURE 7: The numerical and exact solutions for fractional order  $\beta = 0.9$ .

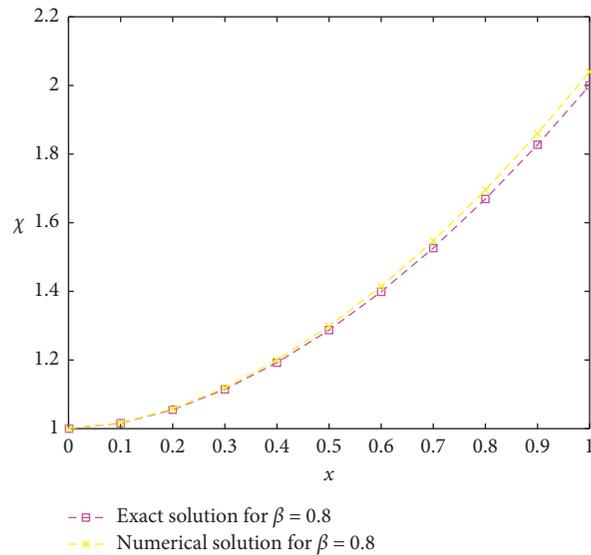


FIGURE 8: The numerical and exact solutions for fractional order  $\beta = 0.8$ .

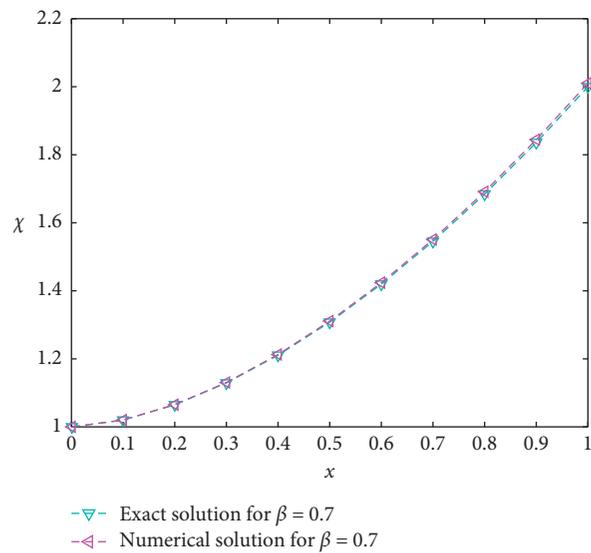


FIGURE 9: The numerical and exact solutions for fractional order  $\beta = 0.7$ .

TABLE 6: The results are obtained for different fractional orders and  $N = 250$ .

$x$	$\beta = 1$	$\beta = 0.95$	$\beta = 0.9$	$\beta = 0.85$
0.1	0.904837418035961	0.878343285218955	0.851704707572839	0.824831134877866
0.2	0.818730753077981	0.795360306677717	0.773306387788940	0.752558353585973
0.3	0.740818220681718	0.722946506074075	0.707343502809177	0.693926556750935
0.4	0.670320046035642	0.659000108719911	0.650476193469615	0.644550914567384
0.5	0.606530659712630	0.602175814084098	0.600817516047575	0.602146031627945
0.6	0.548811636094031	0.551473582617326	0.557090577010358	0.565261898262492
0.7	0.496585303791411	0.506098747938276	0.518352782107219	0.532883565755115
0.8	0.449328964117216	0.465396594634668	0.483872422015733	0.504257605945836
0.9	0.406569659740603	0.428815700079375	0.453062839223986	0.478802532603470
1	0.367879441171453	0.395884829945778	0.425443065181985	0.456057014564435

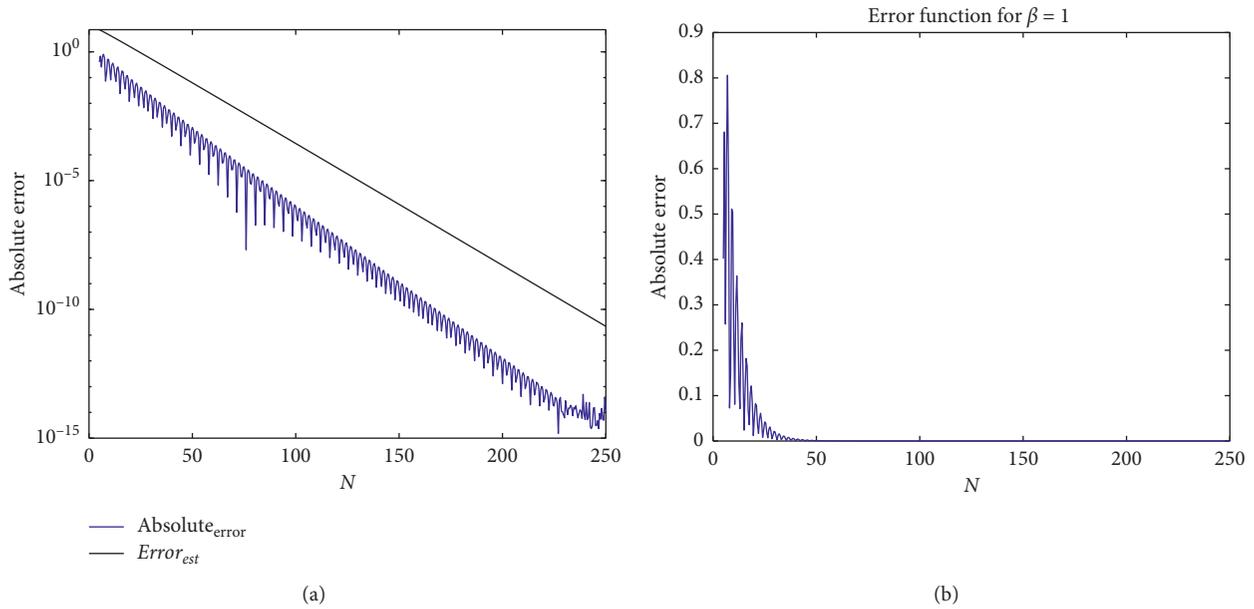


FIGURE 10: (a) Comparison between the absolute error and  $error_{est}$  at  $x = 1$  and  $\beta = 1$  for Problem 4; a good agreement between them is observed. (b) The error function for  $\beta = 1$  and various quadrature nodes corresponding to Problem 4.

TABLE 7: The results are obtained for different fractional orders and  $N = 240$ .

$x$	$\beta = 1$	$\beta = 0.95$	$\beta = 0.5$
0.1	$7.08 \times 10^{-16}$	$4.70 \times 10^{-2}$	$8.30 \times 10^{-2}$
0.2	$7.88 \times 10^{-16}$	$5.16 \times 10^{-2}$	$1.23 \times 10^{-1}$
0.3	$9.43 \times 10^{-16}$	$5.48 \times 10^{-2}$	$1.58 \times 10^{-1}$
0.4	$1.66 \times 10^{-15}$	$5.70 \times 10^{-2}$	$1.88 \times 10^{-1}$
0.5	$2.55 \times 10^{-15}$	$5.81 \times 10^{-2}$	$2.15 \times 10^{-1}$
0.6	$2.55 \times 10^{-15}$	$5.83 \times 10^{-2}$	$2.38 \times 10^{-1}$
0.7	$3.10 \times 10^{-15}$	$5.75 \times 10^{-2}$	$2.56 \times 10^{-1}$
0.8	$4.21 \times 10^{-15}$	$5.56 \times 10^{-2}$	$2.70 \times 10^{-1}$
0.9	$9.10 \times 10^{-15}$	$5.29 \times 10^{-2}$	$2.79 \times 10^{-1}$
1	$8.43 \times 10^{-15}$	$4.91 \times 10^{-2}$	$2.84 \times 10^{-1}$

The problem has exact solution  $\chi(x) = E_{\beta}(-x^{\beta})$ . In this experiment, the same set of optimal parameters is utilized. The values of approximate solution for different fractional orders  $\beta$  and  $x \in [0, 1]$  are depicted in Table 6. Figure 10(a) displays the comparison between the

absolute error and error estimate. Figure 10(b) shows the error function for  $\beta = 1$ .

*Problem 5.* Here, we consider the fractional Riccati equation as follows:

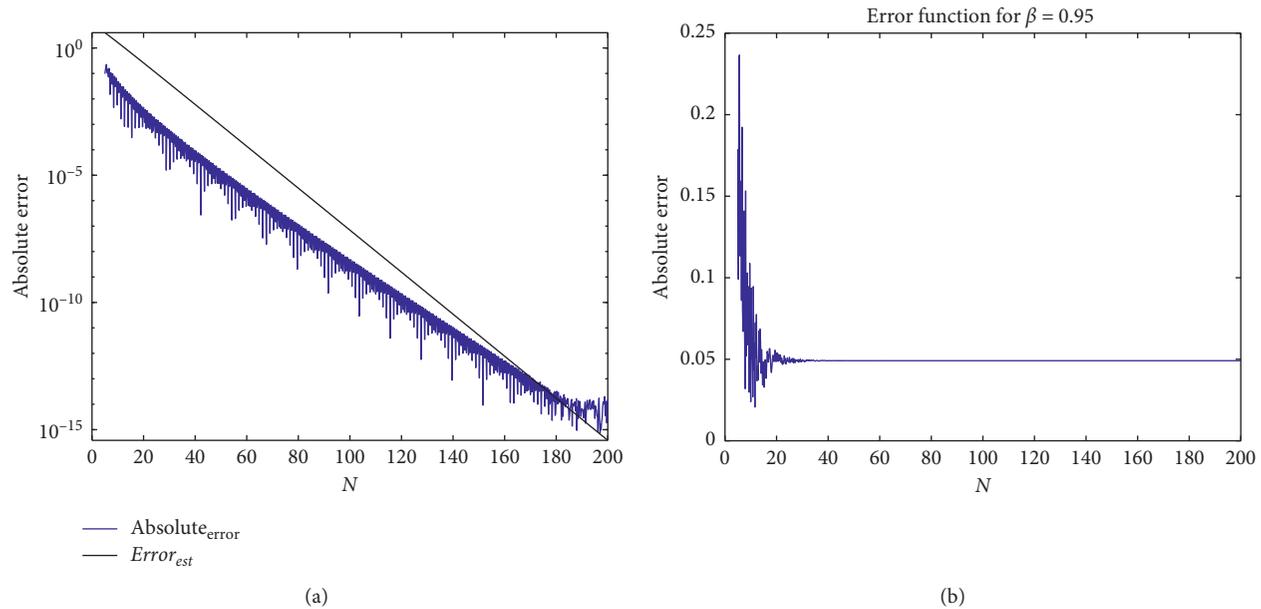


FIGURE 11: (a) Comparison between the absolute error and  $\text{error}_{\text{est}}$  at  $\mathbf{x} = 1$  and  $\beta = 1$  for Problem 5; a good agreement between them is observed. (b) The error function for  $\beta = 0.95$  and various quadrature nodes corresponding to Problem 5.

$$D_{\mathbf{x}}^{\beta} \chi(\mathbf{x}) = \frac{\beta^2 \mathbf{x}^{1-\beta}}{\Gamma(3-\beta)} - \frac{2\beta \mathbf{x}^{1-\beta}}{\Gamma(3-\beta)} - \frac{2\mathbf{x}^{2-\beta}}{\Gamma(3-\beta)} + \mathbf{x}^4 + 2\beta \mathbf{x}^3 + \beta^2 \mathbf{x}^2 - (\chi(\mathbf{x}))^2, \quad \mathbf{x}, \beta \in [0, 1], \quad (28)$$

with  $\chi(0) = 0$ .

The problem has exact solution  $\chi(\mathbf{x}) = -\mathbf{x}^2 - \beta \mathbf{x}$ . In this experiment, the same set of optimal parameters is utilized. The values of approximate solution for different fractional orders  $\beta$  and  $\mathbf{x} \in [0, 1]$  are displayed in Table 7. The comparison between absolute error and error estimate is shown in Figure 11(a), and Figure 11(b) shows the error function for fractional order  $\beta = 0.95$ .

## 6. Conclusion

In this work, we developed a numerical scheme based on LT and inverse LT for approximation of the solution of fractional Riccati equations with ABC derivative. The inverse LT is approximated using the quadrature rule. The proposed method approximated the fractional Riccati equation with ABC derivative accurately and efficiently. From the results, it can be seen that this method is an excellent alternative for approximation of such type of equations.

## Data Availability

Data are included within this paper.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors' Contributions

Xin Liu improved literature review and gave numerical applications of results and present examples. Kamran wrote the paper, and Yukun Yao gave error analysis and plotted error functions.

## Acknowledgments

This research was partitively supported by the funds of HEC Pakistan.

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