Research Article

Multiplication Operators on Weighted Nakano (sss)

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In this article, we investigate the sufficient conditions on weighted Nakano sequence space to be premodular Banach (sss). We examine some topological and geometrical structures of the multiplication operators defined on weighted Nakano prequasi-normed (sss).

1. Introduction

Summability is very important in mathematical models and has numerous implementations, such as normal series theory, approximation theory, ideal transformations, and fixed point theory. For more details, see [1–4]. By $\mathbb{C}^{\mathbb{N}}$, c , ℓ_{∞} , ℓ^r , and c_0 , we signify the spaces of each, convergent, bounded, r -absolutely summable, and convergent to zero sequences of complex numbers. $\mathbb{N}$ indicates the set of nonnegative integers. Let $p = (p_n)$ and $(\beta_n)_{n \in \mathbb{N}}$ be sequences of positive reals, and the weighted Nakano sequence space [5] is defined by

$$\ell_{\beta}^{(p_n)} = \{x = (x_k) \in \mathbb{C}^{\mathbb{N}} : \sigma(\lambda x) < \infty, \text{ for some } \lambda > 0\}, \quad (1)$$

where $\sigma(x) = \sum_{k=0}^{\infty} \beta_k |x_k|^{p_k}$. And $(\ell_{\beta}^{(p_n)}, \|\cdot\|)$ is a Banach space, where $\|x\| = \inf\{\eta > 0 : \sigma(x/\eta) \leq 1\}$.

When $(p_n) \in \ell_{\infty}$, we have

$$\ell_{\beta}^{(p_n)} = \{x = (x_k) \in \mathbb{C}^{\mathbb{N}} : \sigma(\lambda x) < \infty, \text{ for any } \lambda > 0\}. \quad (2)$$

- (1) If $\beta_n = 1$, for all $n \in \mathbb{N}$, then $\ell_{\beta}^{(p_n)}$ will reduce to $\ell^{(p_n)}$ (see [6, 7])
- (2) If $\beta_n = (1/p_n)$, for all $n \in \mathbb{N}$, then $\ell_{\beta}^{(p_n)}$ will reduce to $\ell^{(p_n)}$ which is studied by many authors (see [8–10])

By $B(W, Z)$, we will indicate the set of every operators which are linear and bounded between Banach spaces W and Z , and if $W = Z$, we write $B(W)$. The multiplication operators have a wide field of mathematics in functional analysis, for instance, in eigenvalue distributions theorem, geometric structure of Banach spaces, and theory of fixed point. For more details, see [11–16]. The s -numbers [17] have many examples such as the r th approximation number, denoted by $\alpha_r(V)$, is defined by $\alpha_r(V) = \inf\{\|V - B\| : B \in B(W, Z) \text{ and } \text{rank}(B) \leq r\}$, and the r th Kolmogorov number, denoted by $d_r(V)$, is defined by

$d_r(V) = \inf_{\dim W \leq r} \sup \|\omega\|_{\leq 1} \inf_{v \in W} \|V\omega - v\|$. The following notations will be used in the sequel:

$$\begin{aligned} X^s &= \{X^s(W, Z)\}, \quad \text{where } X^s(W, Z) = \left\{V \in B(W, Z) \left(S_j(V) \right)_{j=0}^{\infty} \in X\right\}, \\ X^{\text{app}} &= \{X^{\text{app}}(W, Z)\}, \quad \text{where } X^{\text{app}}(W, Z) = \left\{V \in B(W, Z) \left(\alpha_j(V) \right)_{j=0}^{\infty} \in X\right\}, \\ X^{\text{Kol}} &= \{X^{\text{Kol}}(W, Z)\}, \quad \text{where } X^{\text{Kol}}(W, Z) = \left\{V \in B(W, Z) \left(d_j(V) \right)_{j=0}^{\infty} \in X\right\}. \end{aligned} \quad (3)$$

A few of operator ideals in the class of Hilbert spaces or Banach spaces are defined by distinct scalar sequence spaces, such as the ideal of compact operators B_c formed by $(d_r(V))$ and c_0 . Pietsch [4] studied the quasi-ideals $(\ell^r)^{\text{app}}$ for $r \in (0, \infty)$, the ideals of Hilbert Schmidt operators between Hilbert spaces constructed by ℓ^2 , and the ideals of nuclear operators generated by ℓ^1 . He explained that $\bar{F} = (\ell^r)^{\text{app}}$ for $r \in [1, \infty)$, where $\bar{F}$ is the closed class of all finite rank operators, and the class $(\ell^r)^{\text{app}}$ became simple Banach and small [1]. The strictly inclusion $(\ell^j)^{\text{app}}(W, Z) \subsetneq B(W, Z)$, whenever $j > r > 0$, W and Z are infinite-dimensional Banach spaces investigated through Makarov and Faried [2]. Faried and Bakery [18] gave a generalization of the class of quasi-operator ideal which is the prequasi-operator ideal, and they examined several geometric and topological structure of $(\ell_M^S)^S$ and $(ces(r))^S$. On sequence spaces, Mursaleen and Noman [19, 20] investigated the compact operators on some difference sequence spaces. The multiplication operators on $(ces(r), \|\cdot\|)$ with the Luxemburg norm $\|\cdot\|$ elaborated by Komal et al. [21] and Bakery and Abou Elmatty [5] gave the sufficient (not necessary) conditions on $\ell_{\beta}^{(p_n)}$ such that $S_{\ell_{\beta}^{(p_n)}}$ constructed a simple Banach prequasi operator ideal. The prequasi-operator ideal $S_{\ell_{\beta}^{(p_n)}}^{\text{app}}$ became small and strictly contained for different weights and powers. The aim of this article to explain some results of $(\ell_{\beta}^{(p_n)})_{\tau}$ equipped with the prequasi-norm τ . Firstly, we give the sufficient conditions on weighted Nakano sequence space to form premodular Banach (sss). Secondly, we give the necessity and sufficient conditions on weighted Nakano sequence space equipped with the prequasi-norm such that the multiplication operator defined on it is bounded, approximable, invertible, Fredholm, and closed range operator.

2. Preliminaries and Definitions

Definition 1 (see [4]). An operator $V \in B(W)$ is called approximable if there are $D_r \in F(W)$, for every $r \in \mathbb{N}$ and $\lim_{r \rightarrow \infty} \|V - D_r\| = 0$.

By $Y(W, Z)$ and $B_c(W, Z)$, we will indicate the space of all approximable and compact operators from W to Z , respectively.

Theorem 1 (see [4]). If W is Banach space with $\dim(W) = \infty$, then $F(W) \subsetneq Y(W) \subsetneq B_c(W) \subsetneq B(W)$.

Definition 2 (see [22]). An operator $V \in B(W)$ is called Fredholm if $\dim(R(V))_c < \infty$, $\dim(\ker V) < \infty$, and $R(V)$ is closed, where $(R(V))_c$ denotes the complement of range V .

The sequence $e_j = (0, 0, 1, 0, 0, \dots)$ with 1 in the j th coordinate, for every $j \in \mathbb{N}$, will be used in the sequel.

Definition 3 (see [3]). The space of linear sequence spaces Y is called (sss) if

- (1) If $e_r \in Y$ with $r \in \mathbb{N}$,
- (2) Let $u = (u_r) \in \mathbb{C}^{\mathbb{N}}$, $v = (v_r) \in Y$, and $|u_r| \leq |v_r|$, for every $r \in \mathbb{N}$, then $u \in Y$. This means Y be solid,"
- (3) If $(u_r)_{r=0}^{\infty} \in Y$, then $(u_{r/2})_{r=0}^{\infty} \in Y$, wherever $(r/2)$ means the integral part of $(r/2)$

Definition 4 (see [23]). A subspace of the (sss) Y_{τ} is called a premodular (sss) if there is a function: $Y \rightarrow [0, \infty)$ verifying the conditions:

- (i) $\tau(y) \geq 0$, for each $y \in Y$ and $\tau(y) = 0 \Leftrightarrow y = \theta$, where θ is the zero element of Y
- (ii) There exists $a \geq 1$ such that $\tau(\eta y) \leq a|\eta| \tau(y)$, for all $y \in Y$, and $\eta \in \mathbb{C}$
- (iii) For some $b \geq 1$, $\tau(y + z) \leq b(\tau(y) + \tau(z))$, for every $y, z \in Y$
- (iv) $|y_r| \leq |z_r|$ with $r \in \mathbb{N}$, implies $\tau((y_r)) \leq \tau((z_r))$
- (v) For some $b_0 \geq 1$, $\tau((y_r)) \leq \tau((y_{(r/2)})) \leq b_0 \tau((y_r))$
- (vi) If $y = (y_r)_{r=0}^{\infty} \in Y$, and $d > 0$, then there is $r_0 \in \mathbb{N}$ with $\tau((y_r)_{r=0}^{\infty}) < d$
- (vii) There is $t > 0$ with $\tau(y, 0, 0, 0, \dots) \geq t|y| \tau(1, 0, 0, 0, \dots)$, for any $y \in \mathbb{C}$

Definition 5 (see [23]). The (sss) Y_{τ} is called prequasi-normed (sss) if τ satisfies the Parts (i)–(iii) of Definition 4, and when the space Y is complete under τ , then Y_{τ} is called a prequasi-Banach (sss).

Theorem 2 (see [23]). A prequasi-norm (sss) Y_{τ} , whenever it is premodular (sss).

The inequality [24]: $|a_i + b_i|^{q_i} \leq 2^{H-1}(|a_i|^{q_i} + |b_i|^{q_i})$, where $q_i \geq 0$, for all $i \in \mathbb{N}$ and $H = \max\{1, \sup_i q_i\}$, will be used in the sequel.

3. Main Results

3.1. Prequasi-Norm on $\ell_\beta^{(p_n)}$. We investigate the sufficient conditions on the sequence space $\ell_\beta^{(p_n)}$ equipped with prequasi-norm τ to form prequasi-Banach and closed (sss).

Theorem 3. $(\ell_\beta^{(p_n)})_\tau$ where $\tau(x) = [\sum_{k=0}^\infty \beta_k |x_k|^{p_k}]^{(1/H)}$, for all $x \in \ell_\beta^{(p_n)}$, is a premodular (sss), if the following conditions are satisfied:

- (a1) The sequence $(p_n) \in \mathbb{R}^{+\mathbb{N}} \cap \ell_\infty$ is increasing
- (a2) Either (β_n) is a monotonic decreasing or monotonic increasing such that there is $C \geq 1$ for which $\beta_{2n} + 1 \leq C\beta_n$

Proof. Firstly, we have to prove $\ell_\beta^{(p_n)}$ is a (sss):

- (1-i) Let $x, y \in \ell_\beta^{(p_n)}$. Since (p_n) is bounded, we obtain

$$\begin{aligned} \tau(X + Y) &= \left[\sum_{n=0}^\infty \beta_n |x_n + y_n|^{p_n} \right]^{(1/H)} \leq \left[\sum_{n=0}^\infty \beta_n |x_n|^{p_n} \right]^{(1/H)} \\ &\quad + \left[\sum_{n=0}^\infty \beta_n |y_n|^{p_n} \right]^{(1/H)} = \tau(X) + \tau(Y) < \infty, \end{aligned} \quad (4)$$

then $x + y \in \ell_\beta^{(p_n)}$.

- (1-ii) Let $\lambda \in \mathbb{C}$ and $x \in \ell_\beta^{(p_n)}$. Since (p_n) is bounded, we have

$$\begin{aligned} \tau(\lambda X) &= \left[\sum_{n=0}^\infty \beta_n |\lambda x_n|^{p_n} \right]^{(1/H)} \leq \sup_n |\lambda|^{(p_n/H)} \left[\sum_{n=0}^\infty \beta_n |x_n|^{p_n} \right]^{(1/H)} \\ &\leq a |\lambda| \tau(X) < \infty. \end{aligned} \quad (5)$$

- (i) Then, $\lambda x \in \ell_\beta^{(p_n)}$. Therefore, by using Parts (1-i) and (1-ii) we have that the space $\ell_\beta^{(p_n)}$ is linear. Also, $e_n \in \ell_\beta^{(p_n)}$, for all $n \in \mathbb{N}$, since

$$\tau(e_n) = \left[\sum_{i=0}^\infty \beta_i |e_n(i)|^{p_i} \right]^{(1/H)} = (\beta_n)^{(1/H)}. \quad (6)$$

- (2) Let $|x_n| \leq |y_n|$, for all $n \in \mathbb{N}$ and $y \in \ell_\beta^{(p_n)}$. Since $\beta_n > 0$, for all $n \in \mathbb{N}$, then $\tau(x) = [\sum_{n=0}^\infty \beta_n |x_n|^{p_n}]^{(1/H)} \leq [\sum_{n=0}^\infty \beta_n |y_n|^{p_n}]^{(1/H)} = \tau(y) < \infty$, we get $x \in \ell_\beta^{(p_n)}$.
- (3) Let $(x_n) \in \ell_\beta^{(p_n)}$ and (β_n) be an increasing sequence. There exists $C \geq 1$ such that $\beta_{2n} + 1 \leq C\beta_n$ and (p_n) be an increasing; then, we have

$$\begin{aligned} \tau(x_{[n/2]}) &= \left[\sum_{n=0}^\infty \beta_n |x_{[n/2]}|^{p_n} \right]^{(1/H)} = \left[\sum_{n=0}^\infty \beta_{2n} |x_n|^{p_{2n}} + \sum_{n=0}^\infty \beta_{2n+1} |x_n|^{p_{2n+1}} \right]^{(1/H)} \\ &\leq \left[\sum_{n=0}^\infty \beta_{2n} |x_n|^{p_n} + \sum_{n=0}^\infty \beta_{2n+1} |x_n|^{p_n} \right]^{(1/H)} \leq (2C)^{(1/H)} \left[\sum_{n=0}^\infty \beta_n |x_n|^{p_n} \right]^{(1/H)} = (2C)^{(1/H)} \tau(x_n), \end{aligned} \quad (7)$$

then $(x_{[n/2]}) \in \ell_\beta^{(p_n)}$. Secondly, we show that the functional τ on $\ell_\beta^{(p_n)}$ is a premodular:

- (i) Evidently, $\tau(x) \geq 0$ and $\tau(x) = 0 \Leftrightarrow x = \theta$
- (ii) There is a steady $a = \max \{1, \sup_n |\lambda|^{(p_n/H)-1}\} \geq 1$ such that $\tau(\lambda x) \leq a |\lambda| \tau(x)$, for all $x \in \ell_\beta^{(p_n)}$ and $\lambda \in \mathbb{C}$
- (iii) There exists $K \geq 1$ such that $\tau(x + y) \leq K(\tau(x) + \tau(y))$, for all $x, y \in \ell_\beta^{(p_n)}$
- (iv) Clearly, from the proof part (2) of Theorem 3, the condition is clear since the weighted Nakano sequence space is solid
- (v) It is obtained from (3) that $b_0 = (2C)^{(1/H)} \geq 1$
- (vi) It is clear that $\overline{F} = \ell_\beta^{(p_n)}$
- (vii) There exists a steady $0 < \xi \leq |\lambda|^{(p_0/H)-1}$, for $\lambda \neq 0$ or $\xi > 0$, for $\lambda = 0$ such that $\tau(\lambda, 0, 0, 0, \dots) \geq \xi |\lambda| \tau(1, 0, 0, 0, \dots)$ $\square$

Theorem 4. If conditions (a1) and (a2) are satisfied, then $(\ell_\beta^{(p_n)})_\tau$ where $\tau(x) = [\sum_{k=0}^\infty \beta_k |x_k|^{p_k}]^{(1/H)}$ for all $x \in \ell_\beta^{(p_n)}$, is a prequasi-Banach (sss).

Proof. Let the conditions be verified. By Theorem 3, the space $(\ell_\beta^{(p_n)})_\tau$ is a premodular (sss). Therefore, from Theorem 2, the space $(\ell_\beta^{(p_n)})_\tau$ is a prequasi-normed (sss). To prove that $(\ell_\beta^{(p_n)})_\tau$ is a prequasi-Banach (sss), suppose $x_n = (x_k^n)_{k=0}^\infty$ be a Cauchy sequence in $(\ell_\beta^{(p_n)})_\tau$, then for every $\varepsilon \in (0, 1)$, there exists a number $n_0 \in \mathbb{N}$ such that, for all $n, m \geq n_0$, one has

$$\tau(X^n - X^m) = \left[\sum_{j=0}^\infty \beta_j |x_j^n - x_j^m|^{p_j} \right]^{(1/H)} < \varepsilon. \quad (8)$$

Hence, for $n, m \geq n_0$ and $j \in \mathbb{N}$, we get $|x_j^n - x_j^m| < \varepsilon$.

So, (x_j^m) is a Cauchy sequence in $\mathbb{C}$ for fixed $j \in \mathbb{N}$, and this gives $\lim_{m \rightarrow \infty} x_j^m = x_j^0$ for $j \in \mathbb{N}$. Hence, $\tau(x_n - x_0) < \varepsilon$, for all $n \geq n_0$. Finally, to prove that $x_0 \in \ell_\beta^{(p_n)}$, we have $\tau(x_0) = \tau(x_0 - x_n + x_n) \leq \tau(x_n - x_0) + \tau(x_n) < \infty$.

So, $x_0 \in \ell_\beta^{(p_n)}$. This means that $(\ell_\beta^{(p_n)})_\tau$ is a prequasi-Banach (sss). $\square$

Theorem 5. If conditions (a1) and (a2) are satisfied, then $(\ell_\beta^{(p_n)})_\tau$ where $\tau(x) = [\sum_{k=0}^\infty \beta_k |x_k|^{p_k}]^{(1/H)}$, for all $x \in \ell_\beta^{(p_n)}$, is a prequasi-closed (sss).

Proof. Let the conditions be verified. By Theorem 3, the space $(\ell_\beta^{(p_n)})_\tau$ is a premodular (sss). Therefore, from Theorem 2, the space $(\ell_\beta^{(p_n)})_\tau$ is a prequasi-normed (sss). To prove that $(\ell_\beta^{(p_n)})_\tau$ is a prequasi-closed (sss), suppose $X^n = (x_k^n)_{k=0}^\infty \in (\ell_\beta^{(p_n)})_\tau$ and $\lim_{n \rightarrow \infty} \tau(X^n - X^0) = 0$, then for every $\varepsilon \in (0, 1)$, there exists a number $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, one has

$$\varepsilon > \tau(X^n - X^0) = \left[\sum_{j=0}^\infty \beta_j |x_j^n - x_j^0|^{p_j} \right]^{(1/H)} \geq \left[\beta_j |x_j^n - x_j^0|^{p_j} \right]^{(1/H)}. \quad (9)$$

Hence, for $n \geq n_0$ and $j \in \mathbb{N}$, we get $|x_j^n - x_j^0| < \varepsilon$.

So, (x_j^n) is a convergent sequence in $\mathbb{C}$ for fixed $j \in \mathbb{N}$, and this gives $\lim_{n \rightarrow \infty} x_j^n = x_j^0$ for fixed $j \in \mathbb{N}$. Finally, to prove that $x_0 \in \ell_\beta^{(p_n)}$, we have $\tau(x_0) = \tau(x_0 - x_n + x_n) \leq \tau(x_n - x_0) + \tau(x_n) < \infty$.

So, $x_0 \in \ell_\beta^{(p_n)}$. This means that $(\ell_\beta^{(p_n)})_\tau$ is a prequasi-closed (sss). $\square$

3.2. Bounded Multiplication Operator on $\ell_\beta^{(p_n)}$. Here after, we investigate some topological and geometric structure of the multiplication operator acting on $\ell_\beta^{(p_n)}$. In this section, we examine the sufficient conditions on the sequence space $\ell_\beta^{(p_n)}$ equipped with prequasi-norm τ such that the multiplication operator defined on $(\ell_\beta^{(p_n)})_\tau (\ell_\beta^{(p_n)})_\tau$ is bounded and isometry.

Definition 6. Let $\kappa \in \mathbb{C}^\mathbb{N} \cap \ell\infty$ and W_τ be a prequasi-normed (sss). An operator $V_\kappa: W_\tau \rightarrow W_\tau$ is called multiplication operator if $V_\kappa \omega = \kappa \omega = (K_r \omega_r)_{r=0}^\infty \in W$, for all $\omega \in W$. If $V_\kappa \in B(W)$, we name it a multiplication operator generated by κ .

Theorem 6. Let $\kappa \in \mathbb{C}^\mathbb{N}$ and conditions (a1) and (a2) be satisfied; then, $\kappa \in \ell\infty$, if and only if $V_\kappa \in B(\ell_\beta^{(p_n)})_\tau$ where $\tau(x) = [\sum_{k=0}^\infty \beta_k |x_k|^{p_k}]^{(1/H)}$, for all $x \in \ell_\beta^{(p_n)}$.

Proof. Let the conditions be satisfied. Assume $\kappa \in \ell\infty$. Therefore, there is $\varepsilon > 0$ with $|\kappa_r| \leq \varepsilon$, for every $r \in \mathbb{N}$. For $x \in (\ell_\beta^{(p_n)})_\tau$, since (p_r) is bounded from above with $p_r > 0$, for all $r \in \mathbb{N}$, then

$$\begin{aligned} \tau(V_\kappa X) &= \tau(KX) = \tau((K_r X_r)_{r=0}^\infty) = \left[\sum_{r=0}^\infty \beta_r |\kappa_r x_r|^{p_r} \right]^{(1/H)} \\ &\leq \left[\sum_{r=0}^\infty \beta_r \varepsilon |x_r|^{p_r} \right]^{(1/H)} \leq \sup_r \varepsilon^{(p_r/H)} \left[\sum_{r=0}^\infty \beta_r |x_r|^{p_r} \right]^{(1/H)} \\ &= D\tau(x). \end{aligned} \quad (10)$$

This gives $V_\kappa \in B(\ell_\beta^{(p_n)})_\tau$. Conversely, assume that $V_\kappa \in B(\ell_\beta^{(p_n)})_\tau$. Let us assume $\kappa \notin \ell\infty$; hence, for each $j \in \mathbb{N}$, there is $i_j \in \mathbb{N}$ such that $\kappa_{i_j} > j$. Since $(p_n) \in \mathbb{R}^{+\mathbb{N}} \cap \ell\infty$ is increasing, we have

$$\begin{aligned} \tau(V_\kappa e_{i_j}) &= \tau(K e_{i_j}) = \tau(K_r(e_{i_j})_{r=0}^\infty)_{r=0}^\infty \\ &= \left[\sum_{r=0}^\infty \beta_r |\kappa_r(e_{i_j})_r|^{p_r} \right]^{(1/H)} = [\beta_{i_j} |\kappa_{i_j}|]^{(p_{i_j}/H)} \\ &> [\beta_{i_j} |j|]^{(p_{i_j}/H)} = [\beta_{i_j} |j|]^{(p_{i_j}/H)} \tau(e_{i_j}). \end{aligned} \quad (11)$$

This shows that $V_\kappa \in B(\ell_\beta^{(p_n)})_\tau$. Therefore, $\kappa \in \ell\infty$. $\square$

Theorem 7. Pick up $\kappa \in \mathbb{C}^\mathbb{N}$ and $(\ell_\beta^{(p_n)})_\tau$ be a prequasi-normed (sss), with $\tau(x) = [\sum_{k=0}^\infty \beta_k |x_k|^{p_k}]^{(1/H)}$, for all $x \in \ell_\beta^{(p_n)}$. Then, $|\kappa_r| = 1$, for all $r \in \mathbb{N}$, if and only if, V_κ is an isometry.

Proof. Suppose $|\kappa_r| = 1$, for all $r \in \mathbb{N}$. Hence,

$$\begin{aligned} \tau(V_\kappa X) &= \tau(KX) = \tau(K_r X_r)_{r=0}^\infty = \left[\sum_{r=0}^\infty \beta_r |\kappa_r x_r|^{p_r} \right]^{(1/H)} \\ &= \left[\sum_{r=0}^\infty \beta_r |x_r|^{p_r} \right]^{(1/H)} = \tau(x), \end{aligned} \quad (12)$$

for all $x \in (\ell_\beta^{(p_n)})_\tau$. Therefore, V_κ is an isometry. Conversely, Assume that $|\kappa_i| < 1$ for some $i = i_0$. Since $(p_r) \in \mathbb{R}^{+\mathbb{N}} \cap \ell\infty$ is increasing, we obtain

$$\begin{aligned} \tau(V e_{i_0}) &= \tau(K e_{i_0}) = \tau(K_r(e_{i_0})_{r=0}^\infty)_{r=0}^\infty = \left[\sum_{r=0}^\infty \beta_r |\kappa_r(e_{i_0})_r|^{p_r} \right]^{(1/H)} \\ &< \left[\sum_{r=0}^\infty \beta_r |(e_{i_0})_r|^{p_r} \right]^{(1/H)} = \tau(e_{i_0}). \end{aligned} \quad (13)$$

When $|\kappa_{i_0}| > 1$, we can prove that $\tau(V_\kappa e_{i_0}) > \tau(e_{i_0})$. Therefore, in both cases, we have a contradiction. So, $|\kappa_r| = 1$, for every $r \in \mathbb{N}$. $\square$

3.3. Approximable Multiplication Operator on $\ell_\beta^{(p_n)}$. In this section, we introduce the sufficient conditions on the sequence space $\ell_\beta^{(p_n)}$ equipped with prequasi-norm τ such that the multiplication operator defined on $(\ell_\beta^{(p_n)})_\tau$ is an approximable and compact. By card (A), we indicate the cardinality of the set A.

Theorem 8. If $\kappa \in \mathbb{C}^\mathbb{N}$ and $(\ell_\beta^{(p_n)})_\tau$ is a prequasi-normed (sss), where $\tau(x) = [\sum_{k=0}^\infty \beta_k |x_k|^{p_k}]^{(1/H)}$, for all $x \in \ell_\beta^{(p_n)}$, then, $V_\kappa \in Y((\ell_\beta^{(p_n)})_\tau)$ if and only if $(K_r)_{r=0}^\infty \in c_0$.

Proof. Assume that $V_\kappa \in Y((\ell_\beta^{(p_n)})_\tau)$. Therefore, $V_\kappa \in BB_c(\ell_\beta^{(p_n)})_\tau$, to prove that the sequence $(K_r)_{r=0}^\infty$ belongs to c_0 . Suppose $(K_r)_{r=0}^\infty \notin c_0$. Hence, there is $\delta > 0$ such that the set $A_\delta = \{r \in \mathbb{N} : |\kappa_r| \geq \delta\}$ has card $(A_\delta) = \infty$. Assume $a_i \in A_\delta$, for all $i \in \mathbb{N}$. Hence, $\{e_{a_i} : a_i \in A_\delta\}$ is an infinite bounded set in $(\ell_\beta^{(p_n)})_\tau$. Let

$$\begin{aligned}
\tau(V_K e_{a_i} - V_K e_{a_j}) &= \tau(K e_{a_i} - K e_{a_j}) = \tau(\kappa_r((e_{a_i})_r - (e_{a_j})_r))_{r=0}^\infty \\
&= \left[\sum_{r=0}^\infty \beta_r \left| \kappa_r((e_{a_i})_r - (e_{a_j})_r) \right|^{p_r} \right]^{(1/H)} \\
&\geq \left[\sum_{r=0}^\infty \beta_r \left| \delta((e_{a_i})_r - (e_{a_j})_r) \right|^{p_r} \right]^{(1/H)} \\
&\geq \inf_r \delta^{(p_r/H)} \tau(e_{a_i} - e_{a_j}),
\end{aligned} \tag{14}$$

for all $a_i, a_j \in A\delta$. This shows $\{e_{a_i}; a_i \in B\delta\} \in \ell_\infty$ which cannot have a convergent subsequence under V_κ . This proves that $V_\kappa \notin B_c(\ell_\beta^{(p_n)})_\tau$. Then, $V_\kappa \notin Y((\ell_\beta^{(p_n)})_\tau)$, this gives a contradiction. So, $\lim_{i \rightarrow \infty} \kappa_i = 0$. Conversely, let $\lim_{i \rightarrow \infty} \kappa_i = 0$. Then, for each $\delta > 0$, the set $A_\delta = \{i \in \mathbb{N}; |\kappa_i| \geq \delta\}$ has $\text{card}(A_\delta) < \infty$. Hence, for every $\delta > 0$, the space $((\ell_\beta^{(p_n)})_\tau)_{A_\delta} = \{x = (x_i) \in \mathbb{C}^{A_\delta}\}$ is finite dimensional. Then, $V_\kappa|_{((\ell_\beta^{(p_n)})_\tau)_{A_\delta}}$ is a finite rank operator. For every $i \in \mathbb{N}$, define $\kappa_i \in \mathbb{C}^\mathbb{N}$ by $(\kappa_i)_j = \begin{cases} \kappa_j, & j \in A_{(1/i)}, \\ 0, & \text{otherwise.} \end{cases}$

It is clear that V_{κ_i} has rank $(V_{\kappa_i}) < \infty$ as $\dim((\ell_\beta^{(p_n)})_\tau)_{A_{(1/i)}} < \infty$, for each $i \in \mathbb{N}$. Therefore, since $(p_n) \in \mathbb{R}^{+\mathbb{N}} \cap \ell_\infty$ is increasing, we obtain

$$\begin{aligned}
\tau((V_K - V_{\kappa_i})X) &= \tau((k_j - (\kappa_i)_j)x_j)_{j=0}^\infty = \left[\sum_{j=0}^\infty \beta_j \left| \kappa_j - (\kappa_i)_j \right|^{p_j} \right]^{(1/H)} \left[\sum_{j=0, j \notin A_{(1/i)}}^\infty \beta_j \left| \kappa_j - (\kappa_i)_j \right|^{p_j} \right]^{(1/H)} \\
&\quad + \left[\sum_{j=0, j \in A_{(1/i)}}^\infty \beta_j \left| \kappa_j - (\kappa_i)_j \right|^{p_j} \right]^{(1/H)} \\
&= \left[\sum_{j=0, j \notin A_{(1/i)}}^\infty \beta_j \left| \kappa_j \right|^{p_j} \right]^{(1/H)} \leq \sup_j \left(\frac{1}{j} \right)^{(p_j/H)} \left[\sum_{j=0, j \notin A_{(1/i)}}^\infty \beta_j \left| x_j \right|^{p_j} \right]^{(p_j/H)} \\
&> \sup_j \left(\frac{1}{j} \right)^{(p_j/H)} \tau(x) = \left(\frac{1}{j} \right)^{(p_0/H)} \tau(x).
\end{aligned} \tag{15}$$

This implies that $\|B_c - V_{\kappa_i}\| \leq (1/j)^{(p_0/H)}$ and that V_κ is a limit of finite rank operators. Therefore, V_κ is an approximable operator. $\square$

Theorem 9. Let $\kappa \in \mathbb{C}^\mathbb{N}$ and $(\ell_\beta^{(p_n)})_\tau$ be a prequasi-normed (sss), where $\tau(x) = [\sum_{r=0}^\infty \beta_r |x_r|^{p_r}]^{(1/H)}$, for all $x \in (\ell_\beta^{(p_n)})_\tau$. Then, $V_\kappa \in B_c(\ell_\beta^{(p_n)})_\tau$ if and only if, $(\kappa_i)_{i=0}^\infty \in c_0$.

Proof. It is simple so overlooked. $\square$

Corollary 1. If $\kappa \in \mathbb{C}^\mathbb{N}$, conditions (a1) and (a2) are satisfied, then $B_c((\ell_\beta^{(p_n)})_\tau) \subset B((\ell_\beta^{(p_n)})_\tau)$, where $\tau(x) = [\sum_{r=0}^\infty \beta_r |x_r|^{p_r}]^{(1/H)}$, for all $x \in (\ell_\beta^{(p_n)})_\tau$.

Proof. Since I is a multiplication operator on $(\ell_\beta^{(p_n)})_\tau$ generated by $\kappa = (1, 1, \dots)$, therefore, $I \notin B_c((\ell_\beta^{(p_n)})_\tau)$ and $I \in B((\ell_\beta^{(p_n)})_\tau)$. $\square$

3.4. Fredholm Multiplication Operator on $\ell_\beta^{(p_n)}$. In this section, we give the sufficient conditions on the sequence space $\ell_\beta^{(p_n)}$ equipped with prequasi-norm τ such that the multiplication operator defined on $(\ell_\beta^{(p_n)})_\tau$ has closed range, invertible, and Fredholm.

Theorem 10. If $\kappa \in \mathbb{C}^\mathbb{N}$, $(\ell_\beta^{(p_n)})_\tau$ is prequasi-Banach (sss), where $\tau(x) = [\sum_{r=0}^\infty \beta_r |x_r|^{p_r}]^{(1/H)}$, for all $x \in (\ell_\beta^{(p_n)})_\tau$, then κ is bounded away from zero on $(\ker(\kappa))^\perp$, if and only if, $R(V_\kappa)$ is closed.

Proof. Let the sufficient condition be satisfied. Therefore, there is $\varepsilon > 0$ with $|\kappa_i| \geq \varepsilon$, for all $i \in (\ker(\kappa))^\perp$, to show $R(V_\kappa)$ is closed. Assume d be a limit point of $R(V_\kappa)$. Therefore, there is $V_\kappa x_i$ in $(\ell_\beta^{(p_n)})_\tau$, for all $i \in \mathbb{N}$ such that $\lim_{i \rightarrow \infty} V_\kappa x_i = d$. Obviously, $(V_\kappa x_i)$ is a Cauchy sequence. Since $(p_n) \in \mathbb{R}^{+\mathbb{N}} \cap \ell_\infty$ is increasing, one has

$$\begin{aligned}
\tau(V_k x_i - V_k x_j) &= \left[\sum_{r=0}^{\infty} \beta_r |\kappa_r(x_i)_r - \kappa_r(x_j)_r|^{p_r} \right]^{(1/H)} = \left[\sum_{r=0, r \in (\ker(\kappa))^c}^{\infty} \beta_r |\kappa_r(x_i)_r - \kappa_r(x_j)_r|^{p_r} + \sum_{r=0, r \notin (\ker(\kappa))^c}^{\infty} \beta_r |\kappa_r(x_i)_r - \kappa_r(x_j)_r|^{p_r} \right]^{(1/H)} \\
&\geq \left[\sum_{r=0, r \in (\ker(\kappa))^c}^{\infty} \beta_r |\kappa_r(x_i)_r - \kappa_r(x_j)_r|^{p_r} \right]^{(1/H)} = \left[\sum_{r=0}^{\infty} \beta_r |\kappa_r((y_i)_r - (y_j)_r)|^{p_r} \right]^{\frac{1}{H}} > \inf_{\varepsilon} \varepsilon^{(p_r/H)} \left[\sum_{r=0}^{\infty} \beta_r |(y_i)_r - (y_j)_r|^{p_r} \right]^{(1/H)},
\end{aligned} \tag{16}$$

where

$$(y_i)_r = \begin{cases} (x_i)_r, & r \in (\ker(\kappa))^c, \\ 0, & r \notin (\ker(\kappa))^c. \end{cases} \tag{17}$$

This shows that (y_i) is a Cauchy sequence in $(\ell_{\beta}^{(p_n)})_{\tau}$. Since $(\ell_{\beta}^{(p_n)})_{\tau}$ is complete, there is $x \in (\ell_{\beta}^{(p_n)})_{\tau}$ such that $\lim_i y_i = x$. Since V_{κ} is continuous, then $\lim_i V_{\kappa} y_i = V_{\kappa} x$. However, $\lim_i V_{\kappa} x_i = \lim_i V_{\kappa} y_i = d$. Hence, $V_{\kappa} x = d$. Therefore, $d \in R(V_{\kappa})$. This shows that $R(V_{\kappa})$ is closed. Conversely, let $R(V_{\kappa})$ be closed. Therefore, V_{κ} be bounded away from zero on $((\ell_{\beta}^{(p_n)})_{\tau})_{(\ker(\kappa))^c}$. Hence, there exists $\varepsilon > 0$ such that $\tau(V_{\kappa} x) \geq \varepsilon \tau(x)$, for all $x \in ((\ell_{\beta}^{(p_n)})_{\tau})_{(\ker(\kappa))^c}$.

Assume $B = \{r \in (\ker(\kappa))^c : |\kappa_r| > \varepsilon\}$. If $B \neq \emptyset$, then for $i_0 \in B$, we obtain

$$\begin{aligned}
\tau(V_{\kappa} e_{n_0}) &= \tau(K_r(e_{n_0})_r)_{r=0}^{\infty} = \left[\sum_{r=0}^{\infty} \beta_r |\kappa_r(e_{n_0})_r|^{p_r} \right]^{(1/H)} \\
&< \left[\sum_{r=0}^{\infty} \beta_r |\varepsilon(e_{n_0})_r|^{p_r} \right]^{(1/H)} \\
&\leq \sup \varepsilon^{(p_r/H)} \left[\sum_{r=0}^{\infty} \beta_r |(e_{n_0})_r|^{p_r} \right]^{(1/H)} \\
&= \sup \varepsilon^{(p_r/H)} \tau(e_{n_0}),
\end{aligned} \tag{18}$$

and this gives contradiction. So, $B = \emptyset$ such that $|\kappa_r| \geq \varepsilon$, for all $r \in (\ker(\kappa))^c$. This completes the proof of the theorem. $\square$

Theorem 11. If $\kappa \in \mathbb{C}^{\mathbb{N}}$ and $(\ell_{\beta}^{(p_n)})_{\tau}$ is a prequasi-Banach (sss), with $\tau(\omega) = \sum_{r=0}^{\infty} \beta_r |\omega_r|^{p_r} \left(\frac{1}{H} \right)$, for all $(\omega) \in \ell_{\beta}^{(p_n)}$, there are $b > 0$ and $B > 0$ such that $b < |\kappa_r| < B$, for all $r \in \mathbb{N}$, if and only if, $V_{\kappa} \in B(\ell_{\beta}^{(p_n)})_{\tau}$ is invertible.

Proof. Let the conditions be verified. Define $\gamma \in \mathbb{C}^{\mathbb{N}}$ by $\gamma_r = (1/\kappa_r)$. From Theorem 6, we have $V_{\kappa}, V_{\gamma} \in B(\ell_{\beta}^{(p_n)})_{\tau}$ and $V_{\kappa} V_{\gamma} = V_{\gamma} V_{\kappa} = I$. Then, V_{γ} is the inverse of V_{κ} . Conversely, let V_{κ} be invertible. Hence, $R(V_{\kappa}) = ((\ell_{\beta}^{(p_n)})_{\tau})_{\mathbb{N}}$. This implies $R(V_{\kappa})$ is closed. From Theorem 10, there is $b > 0$ such that $|\kappa_r| \geq b$, for all $r \in (\ker(\kappa))^c$. Now, $\ker(\kappa) = \emptyset$, else $\kappa_{r_0} = 0$, for several $r_0 \in \mathbb{N}$, we get $e_{r_0} \in \ker(V_{\kappa})$. This gives a contradiction, since $\ker(V_{\kappa})$ is trivial. So, $|\kappa_r| \geq b$, for all $r \in \mathbb{N}$. Since V_{κ} is bounded, hence from Theorem 6, there is $B > 0$ such that $|\kappa_r| \leq B$, for all $r \in \mathbb{N}$. Therefore, we have shown that $b \leq |\kappa_r| \leq B$, for all $r \in \mathbb{N}$. $\square$

Theorem 12. If $\kappa \in \mathbb{C}^{\mathbb{N}}$ and $(\ell_{\beta}^{(p_n)})_{\tau}$ is a prequasi-Banach (sss), where $\tau(\omega) = \sum_{r=0}^{\infty} \beta_r |\omega_r|^{p_r} \left(\frac{1}{H} \right)$, for all $(\omega) \in \ell_{\beta}^{(p_n)}$, then $V_{\kappa} \in B(\ell_{\beta}^{(p_n)})_{\tau}$ is Fredholm operator, if and only if, (i) $\text{card}(\ker(\kappa)) < \infty$ and (ii) $|\kappa_r| \geq \varepsilon$, for all $r \in (\ker(\kappa))^c$.

Proof. Let V_{κ} be Fredholm. If $\text{card}(\ker(\kappa)) = \infty$, hence $e_n \in \ker(V_{\kappa})$, for all $n \in \ker(\kappa)$. Since e_n 's are linearly independent, this gives $\text{card}(\ker(V_{\kappa})) = \infty$. This is a contradiction. Therefore, $\text{card}(\ker(\kappa)) < \infty$. By Theorem 10, condition (ii) is satisfied. Next, if the necessary conditions are verified, to show that V_{κ} is Fredholm. From Theorem 10, condition (ii) gives that $R(V_{\kappa})$ is closed. Condition (i) indicates that $\dim(\ker(V_{\kappa})) < \infty$ and $\dim((R(V_{\kappa}))^c) < \infty$. Therefore, V_{κ} is Fredholm. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References

- [1] A. Pietsch, "Small ideals of operators," *Studia Mathematica*, vol. 51, no. 3, pp. 265–267, 1974.
- [2] B. M. Makarov and N. Faried, "Some properties of operator ideals constructed by s numbers (In Russian)," in *Theory of Operators in Functional Spaces*, pp. 206–211, Academy of Science, Siberian section, Novosibirsk, Russia, 1977.
- [3] N. Faried and A. A. Bakery, "Mappings of type orlicz and generalized cesáro sequence space," *Journal of Inequalities and Applications*, vol. 1, 2013.
- [4] A. Pietsch, *Operator Ideals*, North-Holland Publishing Company, Amsterdam, NY, USA, 1980.
- [5] A. A. Bakery and A. R. Abou Elmatty, "Pre-quasi simple Banach operator ideal generated by s -numbers," *Journal of Function Spaces*, Article ID 9164781, 11 pages, 2020.

- [6] S. T. Chen, "Geometry of orlicz spaces," *Dissertations Mathematica*, vol. 356, pp. 1–204, 1996.
- [7] Y. A. Cui and H. Hudzik, "On the Banach-saks and weak Banach-saks properties of some Banach sequence spaces," *Acta Scientiarum Mathematicarum*, vol. 65, pp. 179–187, 1999.
- [8] W. Orlicz, "Über konjugierte Exponentenfolgen," *Studia Mathematica*, vol. 3, no. 1, pp. 200–211, 1931.
- [9] V. Klee, "Summability in $\ell(p_1, p_2, \dots)$ spaces," *Studia Mathematica*, vol. 25, pp. 277–280, 1965.
- [10] K. Sundaresan, "Uniform convexity of Banach spaces $\ell(p_i)$," *Studia Mathematica*, vol. 39, no. 3, pp. 227–231, 1971.
- [11] M. B. Abrahmse, "Multiplication operators," in *Lecture Notes in Mathematics*, pp. 17–36, Springer-Verlag, Berlin, Germany, 1978.
- [12] A. Sharma, K. Raj, and S. K. Sharma, "Products of multiplication composition and differentiation operators from H^∞ weighted Bloch spaces," *Indian Journal of Mathematics*, vol. 54, pp. 159–179, 2012.
- [13] R. K. Singh and J. S. Manhas, *Composition Operators on Function Spaces*, Elsevier, Amsterdam, Netherlands, 1993.
- [14] H. Takagi and K. Yokouchi, "Multiplication and composition operators between two $L\{P\}$ -spaces," *Function Spaces*, vol. 232, pp. 321–338, 1999.
- [15] M. İlkhani, S. Demiriz, and E. E. Kara, "Multiplication operators on Cesàro second order function spaces," *Positivity*, vol. 24, pp. 605–614, 2020.
- [16] E. E. Kara and M. Basarir, "On compact operators and some Euler $B(m)$ -difference sequence spaces," *Journal of Mathematical Analysis and Applications*, vol. 2, no. 379, pp. 499–511, 2011.
- [17] A. Pietsch, *Eigenvalues and S-Numbers*, Cambridge University Press, New York, NY, USA, 1986.
- [18] N. Faried and A. A. Bakery, "Small operator ideals formed by s numbers on generalized Cesàro and Orlicz sequence spaces," *Journal of Inequalities and Applications*, vol. 1, 2018.
- [19] M. Mursaleen and A. K. Noman, "Compactness by the Hausdorff measure of noncompactness," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 8, pp. 2541–2557, 2010.
- [20] M. Mursaleen and A. K. Noman, "Compactness of matrix operators on some new difference sequence spaces," *Linear Algebra and Its Applications*, vol. 436, no. 1, pp. 41–52, 2012.
- [21] B. S. Komal, S. Pandoh, and K. Raj, "Multiplication operators on Cesàro sequence spaces," *Demonstratio Mathematica*, vol. 4, no. 49, pp. 430–436, 2016.
- [22] T. Mrowka, *A Brief Introduction to Linear Analysis: Fredholm Operators. Geometry of Manifolds*, Massachusetts Institute of Technology: MIT OpenCourseWare, Cambridge, MA, USA, 2004.
- [23] A. A. Bakery and M. M. Mohammed, "Small pre-quasi Banach operator ideals of type orlicz- cesàro mean sequence spaces," *Journal of Function Spaces*, vol. 2019, Article ID 7265010, 9 pages, 2019.
- [24] B. Altay and F. Başar, "Generalization of the sequence space $\ell(p)$ derived by weighted means," *Journal of Mathematical Analysis and Applications*, vol. 1, no. 330, pp. 147–185, 2007.