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Research Article

On Picard–Krasnoselskii Hybrid Iteration Process in Banach Spaces

Thabet Abdeljawad,[1,2,3] **Kifayat Ullah**,[4] **and Junaid Ahmad**[4]

[1]*Department of Mathematics and General Sciences, Prince Sultan University, P.O.Box. 66833, Riyadh 11586, Saudi Arabia*
[2]*Department of Medical Research, China Medical University, Taichung 40402, Taiwan*
[3]*Department of Computer Sciences and Information Engineering, Asia University, Taichung, Taiwan*
[4]*Department of Mathematics, University of Science and Technology, Bannu 28100, Khyber Pakhtunkhwa, Pakistan*

Correspondence should be addressed to Junaid Ahmad; ahmadjunaid436@gmail.com

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In this research, we prove strong and weak convergence results for a class of mappings which is much more general than that of Suzuki nonexpansive mappings on Banach space through the Picard–Krasnoselskii hybrid iteration process. Using a numerical example, we prove that the Picard–Krasnoselskii hybrid iteration process converges faster than both of the Picard and Krasnoselskii iteration processes. Our results are the extension and improvement of many well-known results of the literature.

1. Introduction

A mapping F on a subset $\mathscr{G}$ of a Banach space X is called nonexpansive if

$$\|F(a) - F(b)\| \le \|a - b\|, \quad \text{for all } a, b \in \mathscr{G}. \tag{1}$$

A point $q \in \mathscr{G}$ is called a fixed point of F provided that $q = F(q)$. We denote the fixed point set of F by f_F. The set f_F is nonempty if $\mathscr{G}$ is nonempty closed convex bounded and X is uniformly convex (cf. [1–3]).

In 2008, Suzuki [4] generalized this concept as follows. The mapping F on $\mathscr{G}$ is said to satisfy condition (C) (or Suzuki mapping) if for all $a, b \in \mathscr{G}$,

$$(1/2)\|a - F(a)\| \le \|a - b\| \Longrightarrow \|F(a) - F(b)\| \le \|a - b\|. \tag{2}$$

Recently, Patir et al. [5] extended the (C) condition as follows. A mapping F on $\mathscr{G}$ is said to satisfy condition $B_{\gamma,\mu}$ if there exists $\gamma \in [0,1]$ and $\mu \in [0,(1/2)]$ satisfying $2\mu \le \gamma$ such that for each $a, b \in \mathscr{G}$,

$$\gamma\|a - F(a)\| \le \|a - b\| + \mu\|b - F(b)\|, \tag{3}$$

implying $\|F(a) - F(b)\| \le (1-\gamma)\|a - b\| + \mu(\|a - F(b)\| + \|b - F(a)\|)$.

They also showed that if a mapping satisfies the (C) condition, then it satisfies the $B_{\gamma,\mu}$ condition but the converse does not hold in general.

Let $\mathscr{G}$ be a nonempty closed convex subset of a Banach space X, $F: \mathscr{G} \longrightarrow \mathscr{G}$ be a mapping, $\lambda \in (0,1)$, and $n \ge 1$.

The well-known Picard [6] and Krasnoselskii [7] iteration processes are, respectively, defined as

$$\left.\begin{aligned} a_1 &= a \in \mathscr{G}, \\ a_{n+1} &= F(a_n), \end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned} a_1 &= a \in \mathscr{G}, \\ a_{n+1} &= (1-\lambda)a_n + \lambda F(a_n). \end{aligned}\right\} \tag{5}$$

In 2017, Okeke and Abbas [8] considered the Picard–Krasnoselskii hybrid iteration process as follows:

$$\left.\begin{aligned} a_1 &= a \in \mathscr{G}, \\ b_n &= (1-\lambda)a_n + \lambda F(a_n), \\ a_{n+1} &= F(b_n). \end{aligned}\right\} \tag{6}$$

They proved that the Picard–Krasnoselskii hybrid iteration (6) converges faster than all of Picard (4) and Krasnoselskii (5) processes for contraction mappings. We study this process in the general setting of mappings. We establish some weak and strong convergence results for mappings with condition $B_{\gamma,\mu}$. In the last section, we give an example of mapping F which satisfies condition $B_{\gamma,\mu}$ but not (C) and compare the rate of convergence of Picard–Krasnoselskii iteration, Picard iteration, and Krasnoselskii iteration.

2. Preliminaries

Let X be a Banach space. We say that X is uniformly convex [9] provided that for any $r \in (0,2]$, there is a $d > 0$ such that for any $a, b \in X$ with $\|a\| \leq 1$, $\|b\| \leq 1$, and $\|a+b\| > r$; it follows that

$$\frac{1}{2}\|a+b\| \leq (1-d). \tag{7}$$

We say that X satisfies Opial's property [10] provided that for any $\{a_n\}$ in X which weakly converges to $a \in X$ and for all $b \in X - \{a\}$, someone has

$$\limsup_{n\longrightarrow\infty}\left\|a_n - a\right\| < \limsup_{n\longrightarrow\infty}\left\|a_n - b\right\|. \tag{8}$$

Definition 1. Let $\mathscr{G}$ be a nonempty subset of a Banach space X, $\{a_n\}$ in X be bounded, and $r(a,\{a_n\}) = \limsup_{n\longrightarrow\infty}\|a_n - a\|$. The asymptotic radius of $\{a_n\}$ relative to $\mathscr{G}$ is the set $r(\mathscr{G},\{a_n\}) = \inf\{r(a,\{a_n\}) : a \in \mathscr{G}\}$. Moreover, the asymptotic center of $\{a_n\}$ relative to $\mathscr{G}$ is the set $A(\mathscr{G},\{a_n\}) = \{a \in \mathscr{G} : r(a,\{a_n\}) = r(\mathscr{G},a_n)\}$.

We know that the set $A(\mathscr{G},\{a_n\})$ is singleton whenever the underlying space is uniformly convex Banach. Also, $A(\mathscr{G},\{a_n\})$ is nonempty as well as convex whenever $\mathscr{G}$ is weakly compact and convex (see, e.g., [11, 12]).

Lemma 1 (see [5]). *Let $\mathscr{G}$ be a nonempty subset of a Banach space X and $F : \mathscr{G} \longrightarrow \mathscr{G}$ satisfies $B_{\gamma,\mu}$ condition. If q is a fixed point of $F : \mathscr{G} \longrightarrow \mathscr{G}$, then for each $a \in \mathscr{G}$,*

$$\|q - F(a)\| \leq \|q - a\|. \tag{9}$$

From Lemma 1, we obtain the following facts.

Lemma 2. *Let $\mathscr{G}$ be a nonempty subset of a Banach space X. Let $F : \mathscr{G} \longrightarrow \mathscr{G}$ satisfy condition $B_{\gamma,\mu}$. Then, the set f_F is closed. Moreover, if X is strictly convex and $\mathscr{G}$ is convex, then f_F is also convex.*

Theorem 1 (see [5]). *Let $\mathscr{G}$ be a nonempty subset of a Banach space X having Opial property. Let $F : \mathscr{G} \longrightarrow \mathscr{G}$ satisfy condition $B_{\gamma,\mu}$. If $\{a_n\}$ is sequence in X such that*

(i) $\{a_n\}$ converges weakly to a,

(ii) $\lim_{n\longrightarrow\infty}\|a_n - F(a_n)\| = 0$,

then $F(a) = a$.

Proposition 1 (see [5]). *Let $\mathscr{G}$ be a nonempty subset of a Banach space X. If $F : \mathscr{G} \longrightarrow \mathscr{G}$ satisfies condition $B_{\gamma,\mu}$ on $\mathscr{G}$, then for all $a, b \in \mathscr{G}$ and $c \in [0,1]$,*

(i) $\|F(a) - F^2(a)\| \leq \|a - F(a)\|$

(ii) At least (g) and (h) hold:

(g) $(c/2)\|a - F(a)\| \leq \|a - b\|$

(h) $(c/2)\|F(a) - F^2(a)\| \leq \|F(a) - b\|$

The Condition (g) implies $\|F(a) - F(b)\| \leq (1 - (c/2))\|a - b\| + \mu(\|a - F(b)\| + \|b - F(a)\|)$ and condition (h) implies $\|F^2(a) - F(b)\| \leq (1 - (c/2))\|F(a) - b\| + \mu(\|F(a) - F(b)\| + \|b - F^2(a)\|)$

(iii) $\|a - F(b)\| \leq (3 - c)\|a - F(a)\| + (1 - (c/2))\|a - b\| + \mu(2\|a - F(a)\| + \|a - F(b)\| + \|b - F(a)\| + 2\|F(a) - F^2(a)\|)$

We need the following useful lemma from [13].

Lemma 3. *Let X be a uniformly convex Banach space and $0 < g \leq \theta_n \leq h < 1$ for every $n \geq 1$. If $\{s_n\}$ and $\{t_n\}$ are two sequences in X such that $\limsup_{n\longrightarrow\infty}\|s_n\| \leq r$, $\limsup_{n\longrightarrow\infty}\|t_n\| \leq r$, and $\lim_{n\longrightarrow\infty}\|\theta_n s_n + (1-\theta_n)t_n\| = r$ for some $r \geq 0$, then $\lim_{n\longrightarrow\infty}\|s_n - t_n\| = 0$.*

3. Main Results

The following lemma will be used in the upcoming results.

Lemma 4. *Let $\mathscr{G}$ be a nonempty closed convex subset of a Banach space X. Suppose that $F : \mathscr{G} \longrightarrow \mathscr{G}$ satisfies condition $B_{\gamma,\mu}$ and $f_F \neq \varnothing$. If $\{a_n\}$ is a sequence defined by (6), then $\lim_{n\longrightarrow\infty}\|a_n - q\|$ exists for each $q \in f_F$.*

Proof. Suppose that $q \in f_F$. By Lemma 1, we have

$$\begin{aligned}\left\|b_n - q\right\| &= \left\|(1-\lambda)a_n + \lambda F(a_n) - q\right\| \\ &\leq (1-\lambda)\left\|a_n - q\right\| + \lambda\left\|F(a_n) - q\right\| \\ &\leq (1-\lambda)\left\|a_n - q\right\| + \lambda\left\|a_n - q\right\| \\ &\leq \left\|a_n - q\right\|,\end{aligned} \tag{10}$$

which implies that

$$\begin{aligned}\left\|a_{n+1} - q\right\| &= \left\|F(b_n) - q\right\| \\ &\leq \left\|b_n - q\right\| \\ &\leq \left\|a_n - q\right\|.\end{aligned} \tag{11}$$

Thus, $\{\|a_n - q\|\}$ is bounded as well as nonincreasing. Hence, $\lim_{n\longrightarrow\infty}\|a_n - q\|$ exists for each $q \in f_F$. □

Theorem 2. *Let $\mathscr{G}$ be a nonempty closed convex subset of a uniformly convex Banach space X. Suppose that $F : \mathscr{G} \longrightarrow \mathscr{G}$ satisfies condition $B_{\gamma,\mu}$ and let $\{a_n\}$ be a sequence defined by (6). Then, $f_F \neq \varnothing$ if and only if $\{a_n\}$ is bounded and $\lim_{n\longrightarrow\infty}\|a_n - F(a_n)\| = 0$.*

Proof. We assume $q \in A(\mathcal{G}, \{a_n\})$. By Proposition 1 (iii), for $\gamma = (c/2)$, $c \in [0,1]$,

$$\begin{aligned}\|a_n - F(q)\| &\le (3-c)\|a_n - F(a_n)\| + \left(1-\frac{c}{2}\right)\|a_n - q\| + \mu\big(2\|a_n - F(a_n)\| \\ &\quad + \|a_n - F(q)\| + \|q - F(a_n)\| + 2\|F(a_n) - F^2(a_n)\|\big) \\ &\le (3-c)\|a_n - F(a_n)\| + \left(1-\frac{c}{2}\right)\|a_n - q\| + \mu\big(2\|a_n - F(a_n)\| \\ &\quad + \|a_n - F(q)\| + \|a_n - q\| + \|a_n - F(a_n)\| + 2\|a_n - F(a_n)\|\big),\end{aligned} \tag{12}$$

(by Proposition 1 (i))

$$\begin{aligned}&\Longrightarrow (1-\mu)\limsup_{n\longrightarrow\infty}\|a_n - F(q)\| \le \left(1-\frac{c}{2}+\mu\right)\limsup_{n\longrightarrow\infty}\|a_n - q\| \\ &\Longrightarrow \limsup_{n\longrightarrow\infty}\|a_n - F(q)\| \le \left(\frac{1-(c/2)+\mu}{1-\mu}\right)\limsup_{n\longrightarrow\infty}\|a_n - q\| \\ &\qquad \le \limsup_{n\longrightarrow\infty}\|a_n - q\| \\ &\qquad \left(\text{as } \frac{1-(c/2)+\mu}{1-\mu} \le 1, \quad \text{for } 2\mu \le \gamma = \frac{c}{2}\right), \\ &\Longrightarrow r(F(q), \{a_n\}) \le r(q, \{a_n\}).\end{aligned} \tag{13}$$

So, $F(q) \in A(\mathcal{G}, \{a_n\})$. But $A(\mathcal{G}, \{a_n\})$ is singleton, and we have $F(q) = q$. Hence, $f_F \ne \varnothing$.

Conversely, let $q \in f_F$. By Lemma 4, $\lim_{n\longrightarrow\infty}\|a_n - q\|$ exists. Assume that $\lim_{n\longrightarrow\infty}\|a_n - q\| = r$. We first prove that $\lim_{n\longrightarrow\infty}\|b_n - q\| = r$. By the proof of Lemma 4, $\|a_{n+1} - q\| \le \|b_n - q\|$; therefore,

$$\liminf_{n\longrightarrow\infty}\|a_{n+1} - q\| \le \liminf_{n\longrightarrow\infty}\|b_n - q\|, \tag{14}$$

and so $r \le \liminf_{n\longrightarrow\infty}\|b_n - q\|$.

Again by the proof of Lemma 4, $\|b_n - q\| \le \|a_n - q\|$. Hence, $\limsup_{n\longrightarrow\infty}\|b_n - q\| \le r$. Therefore, we obtain $\lim_{n\longrightarrow\infty}\|b_n - q\| = r$. Also, by Lemma 1, $\|F(a_n) - q\| \le \|a_n - q\|$. It follows that $\limsup_{n\longrightarrow\infty}\|F(a_n) - q\| \le r$. By Lemma 3, we obtain

$$\lim_{n\longrightarrow\infty}\|F(a_n) - a_n\| = 0. \tag{15}$$

Now, we are in the position to establish a weak convergence of $\{a_n\}$ defined by (6) for the class of mappings with condition $B_{\gamma,\mu}$. □

Theorem 3. *Let $\mathcal{G}$ a nonempty closed convex subset of a uniformly convex Banach space X with the Opial property. Suppose that $F : \mathcal{G} \longrightarrow \mathcal{G}$ satisfies condition $B_{\gamma,\mu}$ and $f_F \ne \varnothing$. Then, $\{a_n\}$ defined by (6) converges weakly to an element of f_F.*

Proof. By Theorem 2, $\{a_n\}$ is bounded and $\lim_{n\longrightarrow\infty}\|a_n - F(a_n)\| = 0$. Since X is uniformly convex, X is reflexive. Thus, we can find a subsequence $\{a_{n_t}\}$ of $\{a_n\}$ such that $\{a_{n_t}\}$ converges weakly to some $u_1 \in \mathcal{G}$. By Theorem 1, we obtain $u_1 \in f_F$. It is sufficient to prove that $\{a_n\}$ converges weakly to u_1. Indeed, if $\{a_n\}$ does not converge weakly to u_1, then we can find a subsequence $\{a_{n_s}\}$ of $\{a_n\}$ and $u_2 \in \mathcal{G}$ such that $\{a_{n_s}\}$ converges weakly to u_2 and $u_2 \ne u_1$. Hence, $u_2 \in f_F$ by Theorem 1. Opial condition and Lemma 4 give us

$$\begin{aligned}\lim_{n\longrightarrow\infty}\|a_n - u_1\| &= \lim_{t\longrightarrow\infty}\|a_{n_t} - u_1\| < \lim_{t\longrightarrow\infty}\|a_{n_t} - u_2\| \\ &= \lim_{n\longrightarrow\infty}\|a_n - u_2\| = \lim_{s\longrightarrow\infty}\|a_{n_s} - u_2\| \\ &< \lim_{s\longrightarrow\infty}\|a_{n_s} - u_1\| = \lim_{n\longrightarrow\infty}\|a_n - u_1\|.\end{aligned} \tag{16}$$

This is a contradiction. So, $u_1 = u_2$. □

Theorem 4. *Let $\mathcal{G}$ be a nonempty closed convex subset of a uniformly convex Banach space X. Suppose that $F : \mathcal{G} \longrightarrow \mathcal{G}$ satisfies condition $B_{\gamma,\mu}$. If $f_F \ne \varnothing$ and $\liminf_{n\longrightarrow\infty}\operatorname{dist}(a_n, f_F) = 0$, where $\{a_n\}$ be the sequence defined by (6), then $\{a_n\}$ converges strongly to a fixed point of F.*

Proof. By Lemma 4, $\lim_{n\longrightarrow\infty}\|a_n - q\|$ exists, for each $q \in f_F$. Thus, $\lim_{n\longrightarrow\infty}\operatorname{dist}(a_n, f_F)$ exists. Hence,

$$\lim_{n\longrightarrow\infty}\operatorname{dist}(a_n, f_F) = 0. \tag{17}$$

We can find a subsequence $\{a_{n_t}\}$ of $\{a_n\}$ and $\{p_t\}$ in f_F with $\|a_{n_t} - p_t\| \le (1/2^t)$, $n \ge 1$. Moreover, $\{a_n\}$ is nonincreasing by the proof of Lemma 4. Hence,

$$\|a_{n_{t+1}} - p_t\| \le \|a_{n_t} - p_t\| \le \frac{1}{2^t}. \tag{18}$$

We shall prove that $\{p_t\}$ is a Cauchy in f_F.

$$\begin{aligned}\|p_{t+1} - p_t\| &\le \|p_{t+1} - a_{n_{t+1}}\| + \|a_{n_{t+1}} - p_t\| \\ &\le \frac{1}{2^{t+1}} + \frac{1}{2^t} \le \frac{1}{2^{t-1}} \longrightarrow 0, \quad \text{as } t \longrightarrow \infty.\end{aligned} \tag{19}$$

This shows that the sequence $\{p_t\}$ is Cauchy in f_F. By Lemma 2, f_F is closed. Hence, $p_t \longrightarrow \omega$ for some $\omega \in f_F$. By Lemma 4, $\lim_{n\longrightarrow\infty}\|a_n - \omega\|$ exists. So, the proof is finished.

Table 1: Convergence of Picard–Krasnoselskii hybrid, Picard, and Krasnoselskii iterations to the fixed point $q = 0$.

n	Picard–Krasnoselskii	Picard	Krasnoselskii
1	0.8	0.8	0.8
2	0.3000000000000	0.4000000000000	0.6000000000000
3	0.1125000000000	0.2000000000000	0.4500000000000
4	0.0421875000000	0.1000000000000	0.3375000000000
5	0.0158203125000	0.0500000000000	0.2531250000000
6	0.0059326171875	0.0250000000000	0.1898437500000
7	**0**	0.0125000000000	0.1423828125000
8	0	0.0062500000000	0.1067871093750
9	0	**0**	0.0800903320312
10	0	0	0.0059326171875

Finally, we prove the following strong convergence theorem for the sequence $\{a_n\}$ defined by (6) with the help of condition (I). □

Definition 2. Recall that a self-mapping F on $\mathcal{G}$ subset of a Banach space is said to satisfy condition (I) [14] if and only if there exists $\rho : [0,\infty) \longrightarrow [0,\infty)$ satisfying $\rho(0) = 0$ and $\rho(u) > 0$ for every $u > 0$ such that

$$\|a - F(a)\| \geq \rho(\operatorname{dist}(a, f_F)), \quad \text{for all } a \in \mathcal{G}. \tag{20}$$

Theorem 5. *Let $\mathcal{G}$ be a nonempty closed convex subset of a uniformly convex Banach space X. Suppose that $F : \mathcal{G} \longrightarrow \mathcal{G}$ satisfies condition $B_{\gamma,\mu}$ and $f_F \neq \varnothing$. If F satisfies condition (I), then $\{a_n\}$ defined by (6) converges strongly to a fixed point of F.*

Proof. By Theorem 2, we have

$$\lim_{n\longrightarrow\infty} \inf \|a_n - F(a_n)\| = 0. \tag{21}$$

By condition (I), we obtain

$$\lim_{n\longrightarrow\infty} \inf \operatorname{dist}(a_n, f_F) = 0. \tag{22}$$

The conclusion follows from Theorem 4. □

4. Numerical Example

In this section, we compare the rate of convergence of Picard–Krasnoselskii hybrid iteration process with Picard and Krasnoselskii iterations in general setting of mappings.

Example 1. Let $\mathcal{G} = [0,1]$ be endowed with the usual norm. Set $F(a) = 0$ if $a \in A = [0,(1/100))$ and $F(a) = (1/2)a$ if $a \in B = [(1/100),1]$. If $a = (1/160)$ and $b = (1/100)$, then $(1/2)|a - F(a)| < |a - b|$ but $|F(a) - F(b)| > |a - b|$. Hence, F does not satisfy condition (C). However, F satisfies condition $B_{1,(1/2)}$. The case when $a, b \in A$ is trivial and hence omitted. We consider the following two nontrivial cases.

When $a, b \in B$, we have

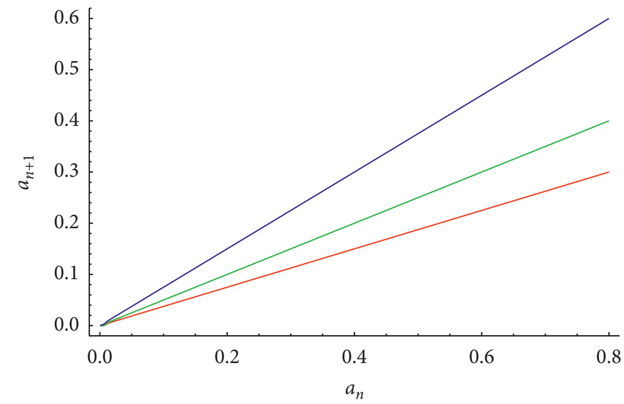

Figure 1: Convergence behavior of Picard–Krasnoselskii, Picard, and Krasnoselskii iterates for mapping F defined in Example 1 where $a_1 = 0.8$.

$$\begin{aligned}(1-\gamma)|a-b| &+ \mu(|a - F(b)| + |b - F(a)|)\\ &= \frac{1}{2}\left(\left|a - \frac{b}{2}\right| + \left|b - \frac{a}{2}\right|\right)\\ &\geq \frac{1}{2}\left|\frac{3a}{2} - \frac{3b}{2}\right|\\ &= \frac{3}{4}|a-b| \geq \frac{1}{2}|a-b| = |F(a) - F(b)|.\end{aligned} \tag{23}$$

When $a \in B$ and $b \in A$, we have

$$\begin{aligned}(1-\gamma)|a-b| &+ \mu(|a - F(b)| + |b - F(a)|)\\ &= \frac{1}{2}\left(|a| + \left|b - \frac{a}{2}\right|\right)\\ &= \frac{1}{2}|a| + \frac{1}{2}\left|b - \frac{a}{2}\right|\\ &\geq \frac{1}{2}|a| = |F(a) - F(b)|.\end{aligned} \tag{24}$$

Choose $\lambda = (1/2) \in (0,1)$; the strong convergence of the sequence $\{a_n\}$ defined by Picard–Krasnoselskii hybrid process (6) to $q = 0$ can be seen in Table 1.

Remark 1. From Table 1 and Figure 1, we observe that the Picard–Krasnoselskii hybrid iteration process converges faster than Picard and Krasnoselskii iterations in the class of mappings with condition $B_{\gamma,\mu}$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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