Research Article

Fractional Integral Inequalities of Hermite–Hadamard Type for Differentiable Generalized h -Convex Functions

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In the present paper, some fractional integral inequalities of Hermite–Hadamard type for functions whose derivatives are generalized h -convex are established. Moreover, several particular cases are also discussed which can be deduced from our results. As special cases, one can obtain several new versions of the results of generalized h -convexity for other various generalizations of convex functions.

1. Introduction

The theory of inequalities is in process of continuous development state, and inequalities have become very effectual and potent tools for analyzing a large number of problems in different branches of mathematics. In last decades, the theory of inequalities has drawn the attention of most of the researchers, motivated new research directions, and affected several features of mathematical analysis and applications [1–4]. Among a variety of inequalities, some inequalities such as Jensen, Hilbert, Hadamard, Hardy, Poincare, Sobolev, Opial, and Fejér have made a significant influence on numerous branches of mathematics [5–8]. One good strategy for investigating any convexity is first to discuss main inequalities such as Hermite–Hadamard and Fejér and then to derive fractional integral inequalities for this. The fractional inequalities for convex function are also important to calculate different means. So, it is always interesting to develop fractional integral inequalities for some generalized convexity. For recent results concerning fractional Hermite–Hadamard inequalities and for different generalizations of convexity, see [9–11] and references therein.

In 2019, some new inequalities based on harmonic log-convex functions have been studied by Baloch et al. in [12]. In the same year, Sarikaya and Alp [13] studied the Hermite–Hadamard–Fejér type integral inequalities for generalized convex functions via local fractional integrals. Kwun et al. [14] studied generalized Riemann–Liouville k -fractional integrals associated with Ostrowski-type inequalities and error bounds of Hadamard inequalities. In [15], Kang et al. studied Hadamard and Fejér–Hadamard inequalities for extended generalized fractional integrals involving special functions. In [15], (h, m) -convex functions and associated fractional Hadamard and Fejér–Hadamard inequalities via an extended generalized Mittag–Leffler function were studied. Simpson’s type inequalities for strongly (s, m) -convex functions in the second sense has been studied in [16] by Kermausuor.

In the present paper, we aim to study some Hermite–Hadamard-type fractional integral inequalities for functions whose derivatives are generalized h -convex. Our results can be considered as generalization of many existing results as many existing results can be obtained directly from our results.

The paper is organized as follows. In Section 2, we give some basic definitions and known results that will help to prove our main results. In Section 3, we will develop Fejér and Hermite–Hadamard-type inequalities for generalized h -convex function. The Section 4 is devoted for fractional integral inequalities for generalized h -convex function. The results are concluded in Section 5.

2. Preliminaries

Throughout this research, assume $I \subseteq \mathbb{R}$ be an interval. Also, $\eta: A \times A \rightarrow B$ be a bifunction, where $A, B \subseteq \mathbb{R}$. Before turning to main results, we first review the basic definitions.

Definition 1 (see [17]). A function $f: I \rightarrow \mathbb{R}$ is called η -convex function if

$$f(\alpha a_1 + (1 - \alpha)a_2) \leq f(a_2) + \alpha \eta(f(a_1), f(a_2)), \quad (1)$$

for all $a_1, a_2 \in I$ and $\alpha \in [0, 1]$.

This definition was introduced in [17] and motivated by works carried out in [18, 19]; Delavar and Dragomir proved few basic results of η -convex functions.

Definition 2 (see [20]). Let $h: J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a nonnegative function. A function $f: I \rightarrow \mathbb{R}$ is known as modified h -convex if

$$f(\alpha a_1 + (1 - \alpha)a_2) \leq h(\alpha)f(a_1) + (1 - h(\alpha))f(a_2), \quad (2)$$

for all $a_1, a_2 \in I$ and $\alpha \in [0, 1]$.

We now define certain convexity which is unification of generalized and modified h -convexity.

Definition 3 (see [21]). Let h be a function defined as above. A function $f: I \rightarrow \mathbb{R}$ is called generalized h -convex function if

$$f(\alpha a_1 + (1 - \alpha)a_2) \leq f(a_2) + h(\alpha)\eta(f(a_1), f(a_2)), \quad (3)$$

for all $a_1, a_2 \in I$ and $\alpha \in [0, 1]$.

Remark 1. It is worth pointing out that the Definition 3 can ofcourse be reduced to

- (1) η -convexity for $h(\alpha) = \alpha$
- (2) η_s -convexity for $h(\alpha) = \alpha^s$
- (3) Modified h -convex function when we take $\eta(a, b) = a - b$
- (4) Convex function when we take $h(\alpha) = \alpha$ and $\eta(a, b) = a - b$

A convex function $f: I \rightarrow \mathbb{R}$, $a_1, a_2 \in I$ with $a_1 < a_2$ satisfies the following inequality:

$$f\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x) dx \leq \frac{f(a_1) + f(a_2)}{2}, \quad (4)$$

known as Hermite–Hadamard's inequality for convex function.

In [21], Fejér generalized (4) as below.

Assume a convex function $f: [a_1, a_2] \rightarrow \mathbb{R}$ and a function $g: [a_1, a_2] \rightarrow \mathbb{R}$ which is nonnegative, integrable, and symmetric with respect to $(a_1 + a_2)/2$; then,

$$\begin{aligned} f\left(\frac{a_1 + a_2}{2}\right) \int_{a_1}^{a_2} g(x) dx &\leq \int_{a_1}^{a_2} f(x)g(x) dx \\ &\leq \frac{f(a_1) + f(a_2)}{2} \int_{a_1}^{a_2} g(x) dx. \end{aligned} \quad (5)$$

Furthermore, Pachpatte developed the following inequalities for product of convex functions, see [22].

Theorem 1. Let $f, g: [a_1, a_2] \rightarrow [0, \infty)$ be convex functions on $[a_1, a_2] \subset \mathbb{R}$ and $a_1 < a_2$. Then,

$$\begin{aligned} \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x)g(x) dx &\leq \frac{1}{3}M(a_1, a_2) + \frac{1}{6}N(a_1, a_2), \\ 2f\left(\frac{a_1 + a_2}{2}\right)g\left(\frac{a_1 + a_2}{2}\right) &\leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x)g(x) dx \\ &\quad + \frac{1}{6}M(a_1, a_2) + \frac{1}{3}N(a_1, a_2), \end{aligned} \quad (6)$$

where

$$\begin{aligned} M(a_1, a_2) &= f(a_1)g(a_1) + f(a_2)g(a_2), \\ N(a_1, a_2) &= f(a_1)g(a_2) + f(a_2)g(a_1). \end{aligned} \quad (7)$$

For establishing new fractional integral inequalities, the following Lemmas are required.

Lemma 1 (see [23]). Assume a function $f: I \rightarrow \mathbb{R}$, differentiable on I° , $a_1, a_2 \in I$ with $a_1 < a_2$. If $f' \in L[a_1, a_2]$, $\zeta, \nu \in \mathbb{R}$, and $\alpha \in [0, 1]$, then

$$\begin{aligned} &\frac{\zeta f(a_1) + \nu f(a_2)}{2} + \frac{2 - \zeta - \nu}{2} f\left(\frac{a_1 + a_2}{2}\right) \\ &\quad - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x) dx \\ &= \frac{a_2 - a_1}{4} \int_0^1 \left[(1 - \zeta - t)f'\left(ta_1 + (1 - t)\frac{a_1 + a_2}{2}\right) \right. \\ &\quad \left. + (\nu - t)f'\left(t\frac{a_1 + a_2}{2} + (1 - t)a_2\right) \right] d\alpha. \end{aligned} \quad (8)$$

Lemma 2 (see [23]). For $s > 0$ and $0 \leq \epsilon \leq 1$, one has

$$\begin{aligned} \int_0^1 |\epsilon - t|^s dt &= \frac{\epsilon^{s+1} + (1 - \epsilon)^{s+1}}{s + 1}, \\ \int_0^1 t|\epsilon - t|^s dt &= \frac{\epsilon^{s+2} + (s + 1 + \epsilon)(1 - \epsilon)^{s+1}}{(s + 1)(s + 2)}. \end{aligned} \quad (9)$$

3. Hermite–Hadamard and Fejér-Type Inequalities

Here, we present our main results.

Theorem 2. Assume a generalized h -convex function $f: I \rightarrow \mathbb{R}$, $a_1, a_2 \in I$ with $a_1 < a_2$ and $f \in L[a_1, a_2]$. Then,

$$\begin{aligned} & f\left(\frac{a_1 + a_2}{2}\right) - \frac{h(1/2)}{(a_2 - a_1)} \int_{a_1}^{a_2} \eta(f(a_1 + a_2 - x), f(x)) dx \\ & \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x) dx \\ & \leq f(b) + \eta(f(a_1), f(a_2)) \int_0^1 h(\alpha) d\alpha. \end{aligned} \quad (10)$$

Proof. Since f is a generalized h -convex function, so

$$f(\alpha x + (1 - \alpha)y) \leq f(y) + h(\alpha)\eta(f(x), f(y)). \quad (11)$$

Substituting $x = ta_1 + (1 - t)a_2$, $y = (1 - t)a_1 + ta_2$, and $\alpha = 1/2$ in (11), we obtain

$$\begin{aligned} f\left(\frac{a_1 + a_2}{2}\right) & \leq f((1 - t)a_1 + ta_2) \\ & \quad + h\left(\frac{1}{2}\right)\eta(f(ta_1 + (1 - t)a_2), \\ & \quad f((1 - t)a_1 + ta_2)). \end{aligned} \quad (12)$$

By integrating the above inequality, we have

$$\begin{aligned} f\left(\frac{a_1 + a_2}{2}\right) & \leq \int_0^1 f((1 - t)a_1 + ta_2) dt \\ & \quad + h\left(\frac{1}{2}\right) \int_0^1 \eta(f(ta_1 + (1 - t)a_2), \\ & \quad f((1 - t)a_1 + ta_2)) dt \\ & \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x) dx + \frac{h(1/2)}{(a_2 - a_1)} \\ & \quad \cdot \int_{a_1}^{a_2} \eta(f(a_1 + a_2 - x), f(x)) dx. \end{aligned} \quad (13)$$

Now, finally upon recalling (11) with $x = a_1$ and $y = a_2$ and integrating over $[0, 1]$, we have

$$\frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x) dx \leq f(a_2) + \eta(f(a_1), f(a_2)) \int_0^1 h(\alpha) d\alpha. \quad (14)$$

Of course (11) and (14) yields (10). $\square$

Remark 2. By putting $\eta(a, b) = a - b$ and $h(\alpha) = \alpha$ in (10), we get classical Hermite–Hadamard inequality (4) for convex function.

Theorem 3. Assume two nonnegative generalized h -convex functions f and g , $a_1, a_2 \in I$ and $a_1 < a_2$ such that $fg \in L_1[a_1, a_2]$, then

$$\begin{aligned} & \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x)g(x) dx \\ & \leq f(a_2)g(a_2) + M'(a_1, a_2) \int_0^1 h(t) dt \\ & \quad + N'(a_1, a_2) \int_0^1 h^2(t) dt, \end{aligned} \quad (15)$$

where

$$\begin{aligned} M'(a_1, a_2) & = f(a_2)\eta(g(a_1), g(a_2)) \\ & \quad + g(a_2)\eta(f(a_1), f(a_2)), \end{aligned} \quad (16)$$

$$N'(a_1, a_2) = \eta(f(a_1), f(a_2))\eta(g(a_1), g(a_2)). \quad (17)$$

Proof. First, we note

$$\begin{aligned} f(ta_1 + (1 - t)a_2) & \leq f(a_2) + h(t)\eta(f(a_1), f(a_2)), \\ g(ta_1 + (1 - t)a_2) & \leq g(a_2) + h(t)\eta(g(a_1), g(a_2)), \end{aligned} \quad (18)$$

for all $t \in [a_1, a_2]$. By using nonnegativity of f and g , we obtain

$$\begin{aligned} & f(ta_1 + (1 - t)a_2)g(ta_1 + (1 - t)a_2) \\ & \leq f(a_2)g(a_2) + h(t)f(a_2)\eta(g(a_1), g(a_2)) \\ & \quad + h(t)g(a_2)\eta(f(a_1), f(a_2)) + h^2(t)\eta(f(a_1), \\ & \quad f(a_2))\eta(g(a_1), g(a_2)). \end{aligned} \quad (19)$$

On integrating the above inequality over $[0, 1]$, we obtain

$$\begin{aligned} & \int_0^1 f(ta_1 + (1 - t)a_2)g(ta_1 + (1 - t)a_2) dt \\ & \leq f(a_2)g(a_2) + [f(a_2)\eta(g(a_1), g(a_2)) \\ & \quad + g(a_2)\eta(f(a_1), f(a_2))] \int_0^1 h(t) dt \\ & \quad + \eta(f(a_1), f(a_2))\eta(g(a_1), g(a_2)) \int_0^1 h^2(t) dt. \end{aligned} \quad (20)$$

The above yields (15) for M' and N' as in (16) and (17). $\square$

Remark 3. If we take $\eta(a, b) = a - b$ and $h(t) = t$ in Theorem 3, then (15) reduces to (14).

Theorem 4. Assume a generalized h -convex function $f: I \rightarrow \mathbb{R}$, $a_1, a_2 \in I$ with $a_1 < a_2$ and $f \in L_1[a_1, a_2]$ and a function $g: [a_1, a_2] \rightarrow \mathbb{R}$ which is nonnegative, integrable, and symmetric, with respect to $(a_1 + a_2)/2$; then,

$$\begin{aligned} & \int_{a_1}^{a_2} f(x)g(x)dx \\ & \leq \left[f(a_2) + \frac{1}{2}\eta(f(a_1), f(a_2)) \right] \\ & \quad \cdot \int_{a_1}^{a_2} \left[h\left(\frac{a_2-x}{a_2-a_1}\right) + h\left(\frac{x-a_1}{a_2-a_1}\right) \right] g(x)dx. \end{aligned} \quad (21)$$

Proof. Since f is a generalized h -convex function and g is nonnegative, integrable, and symmetric, with respect to $(a_1 + a_2)/2$, we find that

$$\begin{aligned} & \int_{a_1}^{a_2} f(u)g(u)du \\ & = \frac{1}{2} \left[\int_{a_1}^{a_2} f(x)g(x)dx + \int_{a_1}^{a_2} f(a_1 + a_2 - x)g(a_1 + a_2 - x)dx \right] \\ & = \frac{1}{2} \int_{a_1}^{a_2} [(f(x) + f(a_1 + a_2 - u))g(x)]dx \\ & = \frac{1}{2} \int_{a_1}^{a_2} \left[f\left(\frac{a_2-x}{a_2-a_1}a_1 + \frac{x-a_1}{a_2-a_1}a_2\right) + f\left(\frac{x-a_1}{a_2-a_1}a_1 + \frac{a_2-x}{a_2-a_1}a_2\right) \right] g(x)dx \\ & \leq \frac{1}{2} \int_{a_1}^{a_2} \left[\left(f(a_2) + h\left(\frac{a_2-x}{a_2-a_1}\right)\eta(f(a_1), f(a_2)) \right) + \left(f(a_2) + h\left(\frac{x-a_1}{a_2-a_1}\right)\eta(f(a_1), f(a_2)) \right) \right] g(x)dx \\ & \leq \left[f(a_2) + \frac{1}{2}\eta(f(a_1), f(a_2)) \right] \int_{a_1}^{a_2} \left[h\left(\frac{a_2-x}{a_2-a_1}\right) + h\left(\frac{x-a_1}{a_2-a_1}\right) \right] g(x)dx. \end{aligned} \quad (22)$$

□

Remark 4. In Theorem 4, if

- (1) $\eta(a, b) = a - b$, $h(t) = t$, and $g(x) = 1$, then (21) becomes second inequality in (3)
- (2) $\eta(a, b) = a - b$ and $h(t) = t$; then, (21) becomes second inequality in (4).

4. Fractional Integral Inequalities

Theorem 5. Assume a function $f: I \rightarrow \mathbb{R}$, differentiable on I° , $a_1, a_2 \in I$ with $a_1 < a_2$, $0 \leq \zeta, \nu \leq 1$, and $f' \in L[a_1, a_2]$. If for $q \geq 1$, $|f'|^q$ is generalized h -convex on $[a_1, a_2]$, then

$$\begin{aligned} & \left| \frac{\zeta f(a_1) + \nu f(a_2)}{2} + \frac{2-\zeta-\nu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{a_2-a_1} \int_{a_1}^{a_2} f(x)dx \right| \\ & \leq \frac{a_2-a_1}{8} (2)^{1/q} \left\{ (1-2\zeta+2\zeta^2)^{1-(1/q)} \left[\frac{1}{2} |f'(a_2)|^q (1-2\zeta+2\zeta^2) + \eta(|f'(a_1)|^q, |f'(a_2)|^q) \int_0^1 h\left(\frac{1+t}{2}\right) |1-\zeta-t| dt \right]^{1/q} \right. \\ & \quad \left. + (1-2\nu+2\nu^2)^{1-(1/q)} \left[\frac{1}{2} |f'(a_2)|^q (1-2\nu+2\nu^2) + \eta(|f'(a_1)|^q, |f'(a_2)|^q) \int_0^1 h\left(\frac{t}{2}\right) |\nu-t| dt \right]^{1/q} \right\}. \end{aligned} \quad (23)$$

Proof. For $q > 1$, by using Lemma 3, generalized h -convexity of $|f'(x)|^q$ and the noted Holder's integral inequality, we obtain

$$\begin{aligned}
& \left| \frac{\zeta f(a_1) + \nu f(a_2)}{2} + \frac{2 - \zeta - \nu}{2} f\left(\frac{a_1 + a_2}{2}\right) - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x) dx \right| \\
& \leq \frac{a_2 - a_1}{4} \left[\int_0^1 |1 - \zeta - t| \left| f'\left(ta_1 + (1-t)\frac{a_1 + a_2}{2}\right) \right| dt + \int_0^1 |\nu - t| \left| f'\left(t\frac{a_1 + a_2}{2} + (1-t)a_2\right) \right| dt \right] \\
& \leq \frac{a_2 - a_1}{4} \left\{ \left(\int_0^1 |1 - \zeta - t| dt \right)^{1-(1/q)} \left[\int_0^1 |1 - \zeta - t| \left(|f'(a_2)|^q + h\left(\frac{1+t}{2}\right) \eta(|f'(a_1)|^q, |f'(a_2)|^q) \right) dt \right]^{1/q} \right. \\
& \quad \left. + \left(\int_0^1 |\nu - t| dt \right)^{1-(1/q)} \times \left[\int_0^1 |\nu - t| \left(|f'(a_2)|^q + h\left(\frac{t}{2}\right) \eta(|f'(a_1)|^q, |f'(a_2)|^q) \right) dt \right]^{1/q} \right\}. \tag{24}
\end{aligned}$$

By using Lemma 2, a direct calculation yields

$$\begin{aligned}
& \int_0^1 |1 - \zeta - t| \left(|f'(a_2)|^q + h\left(\frac{1+t}{2}\right) \eta(|f'(a_1)|^q, |f'(a_2)|^q) \right) dt \\
& = |f'(a_2)|^q \int_0^1 |1 - \zeta - t| dt + \eta(|f'(a_1)|^q, |f'(a_2)|^q) \\
& \quad \times \int_0^1 h\left(\frac{1+t}{2}\right) |1 - \zeta - t| dt \\
& = \frac{1}{2} (1 - 2\zeta + 2\zeta^2) |f'(a_2)|^q + \eta(|f'(a_1)|^q, |f'(a_2)|^q) \\
& \quad \times \int_0^1 h\left(\frac{1+t}{2}\right) |1 - \zeta - t| dt, \\
& \int_0^1 |\nu - t| \left(|f'(a_2)|^q + h\left(\frac{t}{2}\right) \eta(|f'(a_1)|^q, |f'(a_2)|^q) \right) dt \\
& = |f'(a_2)|^q \int_0^1 |\nu - t| dt + \eta(|f'(a_1)|^q, |f'(a_2)|^q) \int_0^1 h\left(\frac{t}{2}\right) |\nu - t| dt \\
& = \frac{1}{2} (1 - 2\nu + 2\nu^2) |f'(a_2)|^q + \eta(|f'(a_1)|^q, |f'(a_2)|^q) \int_0^1 h\left(\frac{t}{2}\right) |\nu - t| dt. \tag{25}
\end{aligned}$$

On putting the above two inequalities in (24) and using Lemma 2, we obtain inequality (23) for $q > 1$.

For $q = 1$, by using Lemmas 3 and 2, we obtain

$$\begin{aligned}
& \left| \frac{\zeta f(a_1) + \nu f(a_2)}{2} + \frac{2 - \zeta - \nu}{2} f\left(\frac{a_1 + a_2}{2}\right) - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x) dx \right| \\
& \leq \frac{a_2 - a_1}{4} \left\{ \int_0^1 |1 - \zeta - t| \left(|f'(a_2)| + h\left(\frac{1+t}{2}\right) \eta(|f'(a_1)|, |f'(a_2)|) \right) dt + \int_0^1 |\nu - t| \left(|f'(a_2)| + h\left(\frac{t}{2}\right) \eta(|f'(a_1)|, |f'(a_2)|) \right) dt \right\} \\
& = \frac{a_2 - a_1}{4} \left\{ \frac{1}{2} (1 - 2\zeta + 2\zeta^2) |f'(a_2)| + \eta(|f'(a_1)|, |f'(a_2)|) \times \int_0^1 h\left(\frac{1+t}{2}\right) |1 - \zeta - t| dt \right. \\
& \quad \left. + \frac{1}{2} (1 - 2\nu + 2\nu^2) |f'(a_2)| + \eta(|f'(a_1)|, |f'(a_2)|) \int_0^1 h\left(\frac{t}{2}\right) |\nu - t| dt \right\}. \tag{26}
\end{aligned}$$

□

Remark 5. By putting $\eta(a, b) = a - b$ and $h(t) = t$ in Theorem 5, we obtain inequality (10) in [23].

We can derive the following corollary by taking $\zeta = \nu$ in Theorem 5.

Corollary 1. Assume a function $f: I \rightarrow \mathbb{R}$, differentiable on I° , $a_1, a_2 \in I$ with $a_1 < a_2$, $0 \leq \zeta \leq 1$, and $f' \in L[a_1, a_2]$. If for $q \geq 1$, $|f'|^q$ is generalized h -convex on $[a_1, a_2]$, then

$$\begin{aligned} & \left| \frac{\zeta}{2} [f(a_1) + f(a_2)] + (1 - \zeta) f\left(\frac{a_1 + a_2}{2}\right) - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x) dx \right| \\ & \leq \frac{a_2 - a_1}{8} (2)^{1/q} (1 - 2\zeta + 2\zeta^2)^{1-(1/q)} \left\{ \left[\frac{1}{2} (1 - 2\zeta + 2\zeta^2) |f'(a_2)|^q + \eta(|f'(a_1)|^q, |f'(a_2)|^q) \int_0^1 h\left(\frac{1+t}{2}\right) |1 - \zeta - t| dt \right]^{1/q} \right. \\ & \quad \left. + \left[\frac{1}{2} (1 - 2\zeta + 2\zeta^2) |f'(a_2)|^q + \eta(|f'(a_1)|^q, |f'(a_2)|^q) \times \int_0^1 h\left(\frac{t}{2}\right) |\zeta - t| dt \right]^{1/q} \right\}. \end{aligned} \quad (27)$$

If taking $\zeta = \nu = (1/2), (2/3), (1/3)$, respectively, in Theorem 5, we can obtain the following corollaries.

Corollary 2. Assume a function $f: I \rightarrow \mathbb{R}$, differentiable on I° , $a_1, a_2 \in I$ with $a_1 < a_2$ and $f' \in L[a_1, a_2]$. If $|f'|^q$ for $q \geq 1$ is generalized h -convex on $[a_1, a_2]$, then

$$\begin{aligned} & \left| \frac{1}{2} \left[\frac{f(a_1) + f(a_2)}{2} + f\left(\frac{a_1 + a_2}{2}\right) \right] - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x) dx \right| \\ & \leq \frac{a_2 - a_1}{16} (2)^{2/q} \left\{ \left[\frac{1}{4} |f'(a_2)|^q + \eta(|f'(a_1)|^q, |f'(a_2)|^q) \int_0^1 h\left(\frac{1+t}{2}\right) \left| \frac{1}{2} - t \right| dt \right]^{1/q} \right. \\ & \quad \left. + \left[\frac{1}{4} |f'(a_2)|^q + \eta(|f'(a_1)|^q, |f'(a_2)|^q) \int_0^1 h\left(\frac{t}{2}\right) \left| \frac{1}{2} - t \right| dt \right]^{1/q} \right\}, \\ & \left| \frac{1}{3} \left[f(a_1) + f(a_2) + f\left(\frac{a+b}{2}\right) \right] - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x) dx \right| \\ & \leq \frac{5(a_2 - a_1)}{72} \left(\frac{18}{5}\right)^{1/q} \left\{ \left[\frac{5}{18} |f'(a_2)|^q + \eta(|f'(a_1)|^q, |f'(a_2)|^q) \int_0^1 h\left(\frac{1+t}{2}\right) \left| \frac{1}{3} - t \right| dt \right]^{1/q} \right. \\ & \quad \left. + \left[\frac{5}{18} |f'(a_2)|^q + \eta(|f'(a_1)|^q, |f'(a_2)|^q) \int_0^1 h\left(\frac{1+t}{2}\right) \left| \frac{2}{3} - t \right| dt \right]^{1/q} \right\}, \\ & \left| \frac{1}{6} \left[f(a_1) + f(a_2) + 4f\left(\frac{a_1 + a_2}{2}\right) \right] - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x) dx \right| \\ & \leq \frac{5(a_2 - a_1)}{72} \left(\frac{18}{5}\right)^{1/q} \left\{ \left[\frac{5}{18} |f'(a_2)|^q + \eta(|f'(a_1)|^q, |f'(a_2)|^q) \int_0^1 h\left(\frac{1+t}{2}\right) \left| \frac{2}{3} - t \right| dt \right]^{1/q} \right. \\ & \quad \left. + \left[\frac{5}{18} |f'(a_2)|^q + \eta(|f'(a_1)|^q, |f'(a_2)|^q) \int_0^1 h\left(\frac{t}{2}\right) \left| \frac{1}{3} - t \right| dt \right]^{1/q} \right\}. \end{aligned} \quad (28)$$

If we put $q = 1$ in Corollary 2, then we have the following inequality.

Corollary 3. Assume a function $f: I \rightarrow \mathbb{R}$, differentiable on I° , $a_1, a_2 \in I$ with $a_1 < a_2$ and $f' \in L[a_1, a_2]$. If $|f'|$ is generalized h -convex on $[a_1, a_2]$, then

$$\begin{aligned}
& \left| \frac{1}{2} \left[\frac{f(a_1) + f(a_2)}{2} + f\left(\frac{a_1 + a_2}{2}\right) \right] - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x) dx \right| \\
& \leq \frac{a_2 - a_1}{4} \left\{ \frac{1}{2} |f'(a_2)| + \eta(|f'(a_1)|, |f'(a_2)|) \int_0^1 \left[h\left(\frac{1+t}{2}\right) + h\left(\frac{t}{2}\right) \right] \left| \frac{1}{2} - t \right| dt \right\}, \\
& \left| \frac{1}{3} \left[f(a_1) + f(a_2) + f\left(\frac{a_1 + a_2}{2}\right) \right] - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x) dx \right| \\
& \leq \frac{(a_2 - a_1)}{4} \left\{ \frac{5}{9} |f'(a_2)| + \eta(|f'(a_1)|, |f'(a_2)|) \left[\int_0^1 h\left(\frac{1+t}{2}\right) \left| \frac{1}{3} - t \right| dt \right. \right. \\
& \quad \left. \left. + \int_0^1 h\left(\frac{t}{2}\right) \left| \frac{2}{3} - t \right| dt \right] \right\}, \\
& \left| \frac{1}{6} \left[f(a_1) + f(a_2) + 4f\left(\frac{a_1 + a_2}{2}\right) \right] - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x) dx \right| \\
& \leq \frac{(a_2 - a_1)}{4} \left\{ \frac{5}{9} |f'(a_2)| + \eta(|f'(a_1)|, |f'(a_2)|) \left[\int_0^1 h\left(\frac{1+t}{2}\right) \left| \frac{2}{3} - t \right| dt + \int_0^1 h\left(\frac{t}{2}\right) \left| \frac{1}{3} - t \right| dt \right] \right\}.
\end{aligned} \tag{29}$$

Remark 6. By setting $\eta(a, b) = a - b$ and $h(t) = t$, Corollaries 1–1 reduce to inequalities (15)–(17) in [23].

Theorem 6. Assume a function $f: I \rightarrow \mathbb{R}$, differentiable on I° , $a_1, a_2 \in I$ with $a_1 < a_2$, $0 \leq \zeta, \nu \leq 1$, and $f' \in L[a_1, a_2]$. If $|f'|^q$ for $q \geq 1$ is generalized h -convex on $[a_1, a_2]$, then

$$\begin{aligned}
& \left| \frac{\zeta f(a_1) + \nu f(a_2)}{2} + \frac{2 - \zeta - \nu}{2} f\left(\frac{a_1 + a_2}{2}\right) - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x) dx \right| \\
& \leq \frac{a_2 - a_1}{4} \left\{ \left[\left(\frac{(1 - \zeta)^{q+1} + \zeta^{q+1}}{q + 1} \right) |f'(a_2)|^q + \eta(|f'(a_1)|^q, |f'(a_2)|^q) \times \int_0^1 h\left(\frac{1+t}{2}\right) |1 - \zeta - t|^q dt \right]^{1/q} \right. \\
& \quad \left. + \left[\left(\frac{(1 - \nu)^{q+1} + \nu^{q+1}}{q + 1} \right) |f'(a_2)|^q \right. \right. \\
& \quad \left. \left. + \eta(|f'(a_1)|^q, |f'(a_2)|^q) \int_0^1 h\left(\frac{t}{2}\right) |\nu - t|^q dt \right]^{1/q} \right\}.
\end{aligned} \tag{30}$$

Proof. For $q > 1$, by using Lemma 3, η_h -convexity of $|f'|^q$, and Holder's integral inequality, we obtain

$$\begin{aligned}
 & \left| \frac{\zeta f(a_1) + \nu f(a_2)}{2} + \frac{2 - \zeta - \nu}{2} f\left(\frac{a_1 + a_2}{2}\right) - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x) dx \right| \\
 & \leq \frac{a_2 - a_1}{4} \left[\int_0^1 |1 - \zeta - t| \left| f'\left(ta_1 + (1-t)\frac{a_1 + a_2}{2}\right) \right| dt \right. \\
 & \quad \left. + \int_0^1 |\nu - t| \left| f'\left(t\frac{a_1 + a_2}{2} + (1-t)a_2\right) \right| dt \right] \\
 & \leq \frac{a_2 - a_1}{4} \left\{ \left(\int_0^1 dt \right)^{1-(1/q)} \left[\int_0^1 |1 - \zeta - t|^q \left(|f'(a_2)|^q + h\left(\frac{1+t}{2}\right) \right. \right. \right. \\
 & \quad \left. \left. \left. \times \eta(|f'(a_1)|^q, |f'(a_2)|^q) \right) dt \right]^{1/q} + \left(\int_0^1 dt \right)^{1-(1/q)} \right. \\
 & \quad \left. \times \left[\int_0^1 |\mu - t|^q \left(|f'(a_2)|^q + h\left(\frac{t}{2}\right) \eta(|f'(a_1)|^q, |f'(a_2)|) \right) dt \right]^{1/q} \right\} \\
 & \leq \frac{a_2 - a_1}{4} \left\{ \left(\int_0^1 dt \right)^{1-(1/q)} \left[\int_0^1 |1 - \zeta - t|^q \left(|f'(a_2)|^q + h\left(\frac{1+t}{2}\right) \right. \right. \right. \\
 & \quad \left. \left. \left. \times \eta(|f'(a_1)|^q, |f'(a_2)|^q) \right) dt \right]^{1/q} + \left(\int_0^1 dt \right)^{1-(1/q)} \right. \\
 & \quad \left. \times \left[\int_0^1 |\nu - t|^q \left(|f'(a_2)|^q + h\left(\frac{t}{2}\right) \eta(|f'(a_1)|^q, |f'(a_2)|) \right) dt \right]^{1/q} \right\} \\
 & \leq \frac{a_2 - a_1}{4} \left\{ \left[\int_0^1 |1 - \zeta - t|^q \left(|f'(a_2)|^q + h\left(\frac{1+t}{2}\right) \right. \right. \right. \right. \\
 & \quad \left. \left. \left. \times \eta(|f'(a_1)|^q, |f'(a_2)|^q) \right) dt \right]^{1/q} + \left[\int_0^1 |\nu - t|^q \right. \right. \\
 & \quad \left. \left. \times \left(|f'(a_2)|^q + h\left(\frac{t}{2}\right) \eta(|f'(a_1)|^q, |f'(a_2)|) \right) dt \right]^{1/q} \right\}.
 \end{aligned} \tag{31}$$

By Lemma 2, we obtain

$$\begin{aligned}
 & \int_0^1 |1 - \zeta - t|^q \left(|f'(a_2)|^q + h\left(\frac{1+t}{2}\right) \eta(|f'(a_1)|^q, |f'(a_2)|^q) \right) dt \\
 &= |f'(a_2)|^q \int_0^1 |1 - \zeta - t|^q dt + \eta(|f'(a_1)|^q, |f'(a_2)|^q) \\
 &\quad \times \int_0^1 h\left(\frac{1+t}{2}\right) |1 - \zeta - t|^q dt \\
 &= |f'(a_2)|^q \left(\frac{(1-\zeta)^{q+1} + \zeta^{q+1}}{q+1} \right) + \eta(|f'(a_1)|^q, |f'(a_2)|^q) \\
 &\quad \times \int_0^1 h\left(\frac{1+t}{2}\right) |1 - \zeta - t|^q dt, \tag{32} \\
 & \int_0^1 |\nu - t|^q \left(|f'(a_2)|^q + h\left(\frac{t}{2}\right) \eta(|f'(a_1)|^q, |f'(a_2)|^q) \right) dt \\
 &= |f'(a_2)|^q \int_0^1 |\nu - t|^q dt + \eta(|f'(a_1)|^q, |f'(a_2)|^q) \int_0^1 h\left(\frac{t}{2}\right) |\nu - t|^q dt \\
 &= |f'(a_2)|^q \left(\frac{\nu^{q+1} + (1-\nu)^{q+1}}{q+1} \right) + \eta(|f'(a_1)|^q, |f'(a_2)|^q) \\
 &\quad \times \int_0^1 h\left(\frac{t}{2}\right) |\nu - t|^q dt.
 \end{aligned}$$

On putting the above two inequalities in (31), we obtain inequality (30) for $q > 1$.

We can obtain the result for $q = 1$ by following the proof of (26). $\square$

Remark 7. By taking $\eta(a, b) = a - b$ and $h(t) = t$ in Theorem 6, we obtain inequality (21) of [23].

Similar to the corollaries of Theorem 5, the following corollaries of Theorem 6 can be obtained.

Corollary 4. Assume a function $f: I \rightarrow \mathbb{R}$, differentiable on I° , $a_1, a_2 \in I$ with $a_1 < a_2$, $0 \leq \zeta \leq 1$, and $f' \in L[a_1, a_2]$. If $|f'|^q$ for $q \geq 1$ is generalized h -convex on $[a_1, a_2]$, then

$$\begin{aligned}
 & \left| \frac{\zeta}{2} [f(a_1) + f(a_2)] + (1 - \zeta) f\left(\frac{a_1 + a_2}{2}\right) - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x) dx \right| \\
 & \leq \frac{a_2 - a_1}{4} \left\{ \left[\left(\frac{(1-\zeta)^{q+1} + \zeta^{q+1}}{q+1} \right) |f'(a_2)|^q + \eta(|f'(a_1)|^q, |f'(a_2)|^q) \right. \right. \\
 & \quad \times \left. \left. \int_0^1 h\left(\frac{1+t}{2}\right) |1 - \zeta - t|^q dt \right]^{1/q} + \left[\left(\frac{(1-\zeta)^{q+1} + \zeta^{q+1}}{q+1} \right) |f'(a_2)|^q \right. \right. \\
 & \quad \left. \left. + \eta(|f'(a_1)|^q, |f'(a_2)|^q) \int_0^1 h\left(\frac{t}{2}\right) |\zeta - t|^q dt \right]^{1/q} \right\}. \tag{33}
 \end{aligned}$$

Corollary 5. Assume a function $f: I \rightarrow \mathbb{R}$, differentiable on I^* , $a_1, a_2 \in I$ with $a_1 < a_2$ and $f' \in L[a_1, a_2]$. If $|f'|^q$ for $q \geq 1$ is generalized h -convex on $[a_1, a_2]$, then

$$\begin{aligned}
 & \left| \frac{1}{2} \left[\frac{f(a_1) + f(a_2)}{2} + f\left(\frac{a_1 + a_2}{2}\right) \right] - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x) dx \right| \\
 & \leq \frac{a_2 - a_1}{4} \left\{ \left[\frac{2(1/2)^{q+1}}{q+1} |f'(a_2)|^q + \eta(|f'(a_1)|^q, |f'(a_2)|^q) \int_0^1 h\left(\frac{1+t}{2}\right) \left| \frac{1}{2} - t \right|^q dt \right]^{1/q} \right. \\
 & \quad \left. + \left[\frac{2(1/2)^{q+1}}{q+1} |f'(a_2)|^q + \eta(|f'(a_1)|^q, |f'(b)|^q) \int_0^1 h\left(\frac{t}{2}\right) \left| \frac{1}{2} - t \right|^q dt \right]^{1/q} \right\}, \\
 & \left| \frac{1}{6} \left[f(a_1) + f(a_2) + 4f\left(\frac{a_1 + a_2}{2}\right) \right] - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x) dx \right| \\
 & \leq \frac{a_2 - a_1}{4} \left\{ \left[\left(\frac{(2/3)^{q+1} + (1/3)^{q+1}}{q+1} \right) |f'(a_2)|^q + \eta(|f'(a_1)|^q, |f'(a_2)|^q) \right. \right. \\
 & \quad \times \left. \int_0^1 h\left(\frac{1+t}{2}\right) \left| \frac{2}{3} - t \right|^q dt \right]^{1/q} \\
 & \quad \left. + \left[\left(\frac{(2/3)^{q+1} + (1/3)^{q+1}}{q+1} \right) |f'(a_2)|^q + \eta(|f'(a_1)|^q, |f'(a_2)|^q) \int_0^1 h\left(\frac{t}{2}\right) \left| \frac{1}{3} - t \right|^q dt \right]^{1/q} \right\}.
 \end{aligned} \tag{34}$$

If we put $q = 1$ in Corollary 5, then we have the following corollary.

Corollary 6. Assume a function $f: I \rightarrow \mathbb{R}$, differentiable on I^* , $a_1, a_2 \in I$ with $a_1 < a_2$, and $f' \in L[a_1, a_2]$. If $|f'|$ is generalized h -convex on $[a_1, a_2]$, then

$$\begin{aligned}
 & \left| \frac{1}{2} \left[\frac{f(a_1) + f(a_2)}{2} + f\left(\frac{a_1 + a_2}{2}\right) \right] - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x) dx \right| \\
 & \leq \frac{a_2 - a_1}{4} \left\{ \left[\left(\frac{1}{4} \right) |f'(a_2)| + \eta(|f'(a_1)|, |f'(a_2)|) \int_0^1 h\left(\frac{1+t}{2}\right) \left| \frac{1}{2} - t \right| dt \right] \right. \\
 & \quad \left. + \left[\left(\frac{1}{4} \right) |f'(a_2)| + \eta(|f'(a_1)|, |f'(b)|) \int_0^1 h\left(\frac{t}{2}\right) \left| \frac{1}{2} - t \right| dt \right] \right\}, \\
 & \left| \frac{1}{6} \left[f(a_1) + f(a_2) + 4f\left(\frac{a_1 + a_2}{2}\right) \right] - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x) dx \right| \\
 & \leq \frac{a_2 - a_1}{4} \left\{ \left[\left(\frac{5}{18} \right) |f'(a_2)| + \eta(|f'(a_1)|, |f'(a_2)|) \int_0^1 h\left(\frac{1+t}{2}\right) \left| \frac{2}{3} - t \right| dt \right] \right. \\
 & \quad \left. + \left[\left(\frac{5}{18} \right) |f'(a_2)| + \eta(|f'(a_1)|, |f'(a_2)|) \int_0^1 h\left(\frac{t}{2}\right) \left| \frac{1}{3} - t \right| dt \right] \right\}.
 \end{aligned} \tag{35}$$

Remark 8. Corollaries 4–6 reduce to inequalities (3.11), (3.12), and (3.13) of [23] by setting $\eta(a, b) = a - b$ and $h(t) = t$.

The following result is required for the derivation of next theorems.

Lemma 3 (see [24]). Assume a mapping $f: I \subset [0, \infty) \rightarrow \mathbb{R}$, differentiable on I° where $a_1, a_2 \in I$ with $a_1 < a_2$. If $f'' \in L[a_1, a_2]$, then

$$\begin{aligned} & \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x) dx - f\left(\frac{a_1 + a_2}{2}\right) \\ &= \frac{(a_2 - a_1)^2}{16} \left[\int_0^1 t^2 f''\left(t \frac{a_1 + a_2}{2} + (1-t)a_1\right) dt + \int_0^1 (t-1)^2 f''\left(ta_2 + (1-t)\frac{a_1 + a_2}{2}\right) dt \right]. \end{aligned} \quad (36)$$

Theorem 7. Assume a mapping $f: I \subset [0, \infty) \rightarrow \mathbb{R}$, differentiable on I° such that $f'' \in L[a_1, a_2]$, where $a_1, a_2 \in I$ with $a_1 < a_2$. If $|f''|$ is generalized h -convex on $[a_1, a_2]$, then

$$\begin{aligned} & \left| f\left(\frac{a_1 + a_2}{2}\right) - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x) dx \right| \\ & \leq \frac{(a_2 - a_1)^2}{16} \left[\frac{1}{3} \left(|f''(a_1)| + \left| f''\left(\frac{a_1 + a_2}{2}\right) \right| \right) + \eta \left(\left| f''\left(\frac{a_1 + a_2}{2}\right) \right|, |f''(a_1)| \right) \right. \\ & \quad \left. \times \int_0^1 t^2 h(t) dt + \eta \left(|f''(a_2)|, \left| f''\left(\frac{a_1 + a_2}{2}\right) \right| \right) \int_0^1 (t-1)^2 h(t) dt \right]. \end{aligned} \quad (37)$$

Proof. By using Lemma 3 and generalized h -convexity of $|f''|$, we obtain

$$\begin{aligned} & \left| f\left(\frac{a_1 + a_2}{2}\right) - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x) dx \right| \\ & \leq \frac{(a_2 - a_1)^2}{16} \left[\int_0^1 t^2 \left| f''\left(t \frac{a_1 + a_2}{2} + (1-t)a_1\right) \right| dt \right. \\ & \quad \left. + \int_0^1 (t-1)^2 \left| f''\left(ta_2 + (1-t)\frac{a_1 + a_2}{2}\right) \right| dt \right] \\ & \leq \frac{(a_2 - a_1)^2}{16} \left\{ \left[\int_0^1 t^2 \left(|f''(a_1)| + h(t) \eta \left(\left| f''\left(\frac{a_1 + a_2}{2}\right) \right|, |f''(a_1)| \right) \right) dt \right] \right. \\ & \quad \left. + \left[\int_0^1 (t-1)^2 \left(\left| f''\left(\frac{a_1 + a_2}{2}\right) \right| + h(t) \eta \left(|f''(a_2)|, \left| f''\left(\frac{a_1 + a_2}{2}\right) \right| \right) \right) dt \right] \right\} \\ & = \frac{(a_2 - a_1)^2}{16} \left\{ \left[\frac{1}{3} |f''(a_1)| + \int_0^1 t^2 h(t) \eta \left(\left| f''\left(\frac{a_1 + a_2}{2}\right) \right|, |f''(a_1)| \right) dt \right] \right. \\ & \quad \left. + \left[\frac{1}{3} \left| f''\left(\frac{a_1 + a_2}{2}\right) \right| + \int_0^1 (t-1)^2 h(t) \eta \left(|f''(a_2)|, \left| f''\left(\frac{a_1 + a_2}{2}\right) \right| \right) dt \right] \right\} \\ & = \frac{(a_2 - a_1)^2}{16} \left[\frac{1}{3} \left(|f''(a_1)| + \left| f''\left(\frac{a_1 + a_2}{2}\right) \right| \right) + \eta \left(\left| f''\left(\frac{a_1 + a_2}{2}\right) \right|, |f''(a_1)| \right) \right. \\ & \quad \left. \times \int_0^1 t^2 h(t) dt + \eta \left(|f''(a_2)|, \left| f''\left(\frac{a_1 + a_2}{2}\right) \right| \right) \int_0^1 (t-1)^2 h(t) dt \right]. \end{aligned} \quad (38)$$

□

Remark 9. By taking $\eta(a, b) = a - b$ and $h(t) = t$ in Theorem 7, we obtain inequality (9) in [24].

Theorem 8. Assume a mapping $f: I \subset [0, \infty) \rightarrow \mathbb{R}$ differentiable on I° such that $f'' \in L[a_1, a_2]$, where $a_1, a_2 \in I$ with $a_1 < a_2$. If $|f''|^q, q \geq 1$ is generalized h -convex on $[a_1, a_2]$, then the following inequality holds:

$$\begin{aligned} & \left| f\left(\frac{a_1 + a_2}{2}\right) - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x) dx \right| \\ & \leq \frac{(a_2 - a_1)^2}{16} \left(\frac{1}{3}\right)^{1/p} \left\{ \left[\frac{1}{3} |f''(a_1)|^q \right. \right. \\ & \quad \left. \left. + \eta\left(\left|f''\left(\frac{a_1 + a_2}{2}\right)\right|^q, |f''(a_1)|^q\right) \right. \right. \\ & \quad \left. \left. \times \int_0^1 t^2 h(t) dt \right]^{1/q} + \left[\frac{1}{3} |f''(a_2)|^q \right. \right. \\ & \quad \left. \left. + \eta\left(|f''(a_2)|^q, \left|f''\left(\frac{a_1 + a_2}{2}\right)\right|^q\right) \right. \right. \\ & \quad \left. \left. \times \int_0^1 (t-1)^2 h(t) dt \right]^{1/q} \right\}. \end{aligned} \quad (39)$$

Proof. Let $p \geq 1$. By using Lemma 3 and power mean inequality, we obtain

$$\begin{aligned} & \left| f\left(\frac{a_1 + a_2}{2}\right) - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x) dx \right| \\ & \leq \frac{(a_2 - a_1)^2}{16} \left[\int_0^1 t^2 \left| f''\left(t \frac{a_1 + a_2}{2} + (1-t)a_1\right) \right| dt \right. \\ & \quad \left. + \int_0^1 (t-1)^2 \left| f''\left(ta_2 + (1-t)\frac{a_1 + a_2}{2}\right) \right| dt \right] \\ & \leq \frac{(a_2 - a_1)^2}{16} \left(\int_0^1 t^2 dt \right)^{1/p} \\ & \quad \cdot \left(\int_0^1 t^2 \left| f''\left(t \frac{a_1 + a_2}{2} + (1-t)a_1\right) \right|^q dt \right)^{1/q} \\ & \quad + \frac{(b-a)^2}{16} \left(\int_0^1 (t-1)^2 dt \right)^{1/p} \\ & \quad \cdot \left(\int_0^1 (t-1)^2 \left| f''\left(ta_2 + (1-t)\frac{a_1 + a_2}{2}\right) \right|^q dt \right)^{1/q}. \end{aligned} \quad (40)$$

By using generalized h -convexity of $|f''|^q$, we obtain

$$\begin{aligned} & \int_0^1 t^2 \left| f''\left(t \frac{a_1 + a_2}{2} + (1-t)a_1\right) \right|^q dt \\ & \leq \frac{1}{3} |f''(a_1)|^q + \eta\left(\left|f''\left(\frac{a_1 + a_2}{2}\right)\right|^q, |f''(a_1)|^q\right) \int_0^1 t^2 h(t) dt, \\ & \int_0^1 (t-1)^2 \left| f''\left(ta_2 + (1-t)\frac{a_1 + a_2}{2}\right) \right|^q dt \\ & \leq \frac{1}{3} \left| f''\left(\frac{a_1 + a_2}{2}\right) \right|^q + \eta\left(|f''(a_2)|^q, \left|f''\left(\frac{a_1 + a_2}{2}\right)\right|^q\right) \\ & \quad \cdot \int_0^1 (t-1)^2 h(t) dt. \end{aligned} \quad (41)$$

Therefore, we have

$$\begin{aligned} & \left| f\left(\frac{a_1 + a_2}{2}\right) - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x) dx \right| \\ & \leq \frac{(a_2 - a_1)^2}{16} \left(\frac{1}{3}\right)^{1/p} \left\{ \left[\frac{1}{3} |f''(a_1)|^q + \eta\left(\left|f''\left(\frac{a_1 + a_2}{2}\right)\right|^q, |f''(a_1)|^q\right) \right. \right. \\ & \quad \left. \left. \cdot |f''(a_1)|^q \int_0^1 t^2 h(t) dt \right]^{1/q} \right. \\ & \quad \left. + \left[\frac{1}{3} \left| f''\left(\frac{a_1 + a_2}{2}\right) \right|^q + \eta\left(|f''(a_2)|^q, \left|f''\left(\frac{a_1 + a_2}{2}\right)\right|^q\right) \right. \right. \\ & \quad \left. \left. \cdot \int_0^1 (t-1)^2 h(t) dt \right]^{1/q} \right\}. \end{aligned} \quad (42)$$

□

Remark 10. By taking $\eta(a, b) = a - b$ in Theorem 8, we obtain inequality (2.11) in [24].

5. Conclusions

Convexity has many applications in applied sciences as well as in pure mathematics. In the present report, we studied h -convex functions. We proved some Hermite–Hadamard-type fractional integral inequalities for functions whose derivatives are generalized h -convex. Moreover, we also gave several particular cases, which can be deduced from our results. One can obtain several new versions of the results of generalized h -convexity for other various generalizations of convex functions as a special case of our results, and our results can be considered as extension of many existing results.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors of this paper declare that they have no conflicts of interest.

Authors' Contributions

All authors have equal contribution.

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